S4D03/S6D03 2019/2020: Assignment Five

1. Let X be a Poisson random variable with parameter $\lambda > 0$. Show that $\frac{X-\lambda}{\sqrt{\lambda}}$ converges in distribution to the standard normal random variable Z as λ converges to infinity.

Solution: Let $Y_\lambda = \frac{X-\lambda}{\sqrt{\lambda}}$. Then by direct calculation,

$$\begin{aligned} E[e^{itY_\lambda}] &= E[e^{itX/\sqrt{\lambda}}]e^{-it\sqrt{\lambda}} \\ &= e^{\lambda[e^{it/\sqrt{\lambda}}-1]}e^{-it\sqrt{\lambda}} \\ &= \exp\{\lambda[e^{it/\sqrt{\lambda}} - 1 - \frac{it}{\sqrt{\lambda}}]\} \\ &= exp\{-\frac{t^2}{2} + O(\frac{1}{\sqrt{\lambda}})\} \end{aligned}$$

which implies the result.

2. Let $\{X_n : n \geq 1\}$ be a sequence of independent random variables with

$$\mathbb{P}\{X_n = n\} = \mathbb{P}\{X_n = -n\} = \frac{1}{2n},\ \mathbb{P}\{X_n = 0\} = 1 - \frac{1}{n}.$$

Set

$$S_n = \sum_{k=1}^{n} X_k,\ B_n^2 = \sum_{k=1}^{n} \mathrm{Var}[X_k].$$

Show that $\frac{S_n}{B_n}$ converges in distribution to a random variable W which has a characteristic function of the form

$$exp\{-\int_0^{\sqrt{2}} x^{-1}(1 - \cos xt)dx\}.$$

Solution: By direct calculation, we have

$$\begin{aligned} E[e^{it\frac{S_n}{B_n}}] &= \exp\left\{\sum_{k=1}^{n} \ln\left(1 + \frac{\cos(t\frac{k}{B_n}) - 1}{k}\right)\right\} \\ &= \exp\left\{\sum_{k=1}^{n} \frac{\cos(t\frac{k}{B_n}) - 1}{k} + o(1)\right\} \\ &= \exp\left\{\sum_{k=1}^{n} \frac{1}{B_n}\frac{\cos(tx_k^n) - 1}{x_k^n} + o(1)\right\} \\ &\longrightarrow \exp\left\{\int_0^{\sqrt{2}} \frac{\cos tx - 1}{x} dx\right\} \end{aligned}$$

where $x_k^n = \frac{k}{B_n}$.

3. Assume that the random variables X and Y are independent, and $X+Y$ and X have the same distribution. Show that $Y=0$ almost surely.

Solution: Since $X+Y$ and X have the same distribution and X,Y are independent, we get

$$E[e^{it(X+Y)}] = E[e^{itX}]E[e^{itY}] = E[e^{itX}]$$

which implies that $E[e^{itY}] = 1$ for t in a neighbourhood of zero. This implies that both the mean and the variance of Y are zero. Thus $Y=0$ almost surely.