**S4D03/S6D03 2019/2020: Assignment One Solution** [5 marks each]

1. Construct an example showing the union of two $\sigma$-fields is not a $\sigma$-field. Verify your result.

**SOLUTION:**

Let $\Omega = \{1, 2, 3, 4\}$, $\mathcal{F}_1 = \{\varnothing, \{1, 2\}, \{3, 4\}, \Omega\}$ and $\mathcal{F}_2 = \{\varnothing, \{1, 3\}, \{2, 4\}, \Omega\}$ are two $\sigma$-fields.

Let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 = \{\varnothing, \{1, 2\}, \{3, 4\}, \{1, 3\}, \{2, 4\}, \Omega\}$

Suppose $\mathcal{F}$ is a $\sigma$-field, then for any $A, B \in \mathcal{F}$, $A \cap B \in \mathcal{F}$.

Pick $A = \{1, 2\}$, $B = \{1, 3\}$, $A \cap B = \{1\} \notin \mathcal{F}$, which is a contradiction.

$\therefore$ $\mathcal{F}$ as a union of two $\sigma$-field is not a $\sigma$-field.

2. Consider the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Set $\mathcal{A} = \{\{1\}, \{3, 4\}, \{2, 4, 5\}\}$. Find the $\sigma$-field $\mathcal{F}$ generated by $\mathcal{A}$.

**SOLUTION:**

Claim: $\mathcal{F} = \sigma(\mathcal{A}) = \sigma(\{\{1\}, \{3\}, \{4\}, \{6\}, \{2, 5\}\})$.

To prove this claim, it is required to show that i) $\mathcal{A} \subseteq \mathcal{F}$, ii) For any $\sigma$-field $\mathcal{G}$ on $\Omega$ containing $\mathcal{A}$, $\mathcal{F} \subseteq \mathcal{G}$

Part i) $\{1\} \in \mathcal{F}$, $\{3, 4\} = \{3\} \cup \{4\} \in \mathcal{F}$, $\{2, 4, 5\} = \{4\} \cup \{2, 5\} \in \mathcal{F}$. i.e. every elements in $\mathcal{A}$ is also in $\mathcal{F}$, $\mathcal{A} \subseteq \mathcal{F}$.

Part ii) Suppose $\mathcal{G}$ is a $\sigma$-field on $\Omega$ containing $\mathcal{A}$, $\{1\} \in \mathcal{A}$, $\{3\} = \{3, 4\} \cap \{2, 4, 5\}^c$,
$\{4\} = \{3, 4\} \cap \{2, 4, 5\}$, $\{6\} = (\{1\} \cup \{3, 4\} \cup \{2, 4, 5\})^c$, $\{2, 5\} = \{3, 4\}^c \cap \{2, 4, 5\}$,
therefore, $\sigma(\{\{1\}, \{3\}, \{4\}, \{6\}, \{2, 5\}\}) \subseteq \mathcal{G}$

$\mathcal{F} = \sigma(\{\{1\}, \{3\}, \{4\}, \{6\}, \{2, 5\}\})$
$= \{\varnothing, \{1\}, \{3\}, \{4\}, \{6\}, \{2, 5\},$
$\{1, 3\}, \{1, 4\}, \{1, 6\}, \{1, 2, 5\}, \{3, 4\}, \{3, 6\}, \{3, 2, 5\}, \{4, 6\}, \{4, 2, 5\}, \{6, 2, 5\},$
$\{1, 3, 4\}, \{1, 3, 6\}, \{1, 3, 2, 5\}, \{1, 4, 6\}, \{1, 4, 2, 5\}, \{1, 6, 2, 5\}, \{3, 4, 6\}, \{3, 4, 2, 5\}, \{3, 6, 2, 5\}, \{4, 6, 2, 5\},$
$\{1, 3, 4, 6\}, \{1, 3, 4, 2, 5\}, \{1, 3, 6, 2, 5\}, \{1, 4, 6, 2, 5\}, \{3, 4, 6, 2, 5\}, \Omega\}$

3. Let $E_1, E_2, \ldots$ be a sequence of disjoint measurable sets in the measurable space $(\Omega, \mathcal{F})$. Given a sequence of real numbers $a_1, a_2, \ldots$, define

$$f_n(\omega) = \sum_{i=1}^{n} a_i I_{E_i}(\omega)$$

and

$$f(\omega) = \sum_{i=1}^{\infty} a_i I_{E_i}(\omega).$$

Show that $f_n$ converges pointwise to $f$ as $n$ tends to infinity.

**SOLUTION:**

We want to show that $\lim_{n\to\infty} f_n(\omega) = f(\omega)$ for any $\omega \in \Omega$.

Let $E^c = (\bigcup\limits_{i=1}^{\infty} E_i)^c$.

If $\omega \in E^c$, $f_n(\omega) = 0$ for any $n$, and $f(\omega) = 0$, $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

If $\omega \notin E^c$, then $\omega$ belongs to one and only one $E_i$ since $E_i$'s are disjoint, say $\omega \in E_k$, then $I_{E_k}(\omega) = 1$ and $I_{E_i}(\omega) = 0$ for any $i \neq k$. $f_n(\omega) = a_k$ for any $n \geq k$, and $f(\omega) = a_k$, that is $\lim_{n\to\infty} f_n(\omega) = f(\omega)$.

4. Let $\Omega$ be the set of all rational numbers in $[0, 1]$. Set

$$\mathcal{C} = \{A_{a,b} : 0 \leq a \leq b \leq 1, A_{a,b} = \{\omega \in \Omega : a \leq \omega \leq b\}\}$$

and define the set function

$$\mu(A_{a,b}) = b - a.$$

Show that $\mu$ is not a probability.

**SOLUTION:**

$\Omega$ is a countable set.

$$\mu(\Omega) = \mu\left(\bigcup_{r\in\Omega} A_{r,r}\right) = \sum_{r\in\Omega} \mu(A_{r,r}) = \sum_{r\in\Omega} (r - r) = \sum_{r\in\Omega} 0 = 0 \neq 1$$

$\mu$ is not a probability.