S4D03/S6D03 2019/2020: Assignment Two Solution

1 [5']. Let f be a real-valued measurable function on the probability space $(\Omega, \mathcal{F}, P)$. Assume that $f(\omega) \geq 1$ almost surely under P and $\int f(\omega)P(d\omega) = 1$. Show that $f(\omega) = 1$ almost surely under P.

SOLUTION:

Let $A_n = \{\omega : 1 \leq f(\omega) \leq 1 + \frac{1}{n}\}$, since $f(\omega) \geq 1$ almost surely, $A_n^c = \{\omega : f(\omega) > 1 + \frac{1}{n}\}$, $\bigcup\limits_{n=1}^{\infty} A_n^c = \{\omega : f(\omega) > 1\}$. To show that $f(\omega) = 1$ almost surely under P, it is sufficient to show $P\left(\bigcup\limits_{n=1}^{\infty} A_n^c\right) = 0$

$$\begin{aligned}
1 = \int f(\omega)P(d\omega) &= \int_{A_n} f(\omega)P(d\omega) + \int_{A_n^c} f(\omega)P(d\omega) \\
&\geq \int_{A_n} 1\, P(d\omega) + \int_{A_n^c} (1 + \frac{1}{n})P(d\omega) \quad \text{by the property of integration} \\
&= \int (1\, I_{A_n}(\omega) + (1 + \frac{1}{n})I_{A_n^c}(\omega))P(d\omega) \\
&= P(A_n) + (1 + \frac{1}{n})P(A_n^c) \\
&= P(A_n) + P(A_n^c) + \frac{1}{n}P(A_n^c) \\
&= 1 + \frac{1}{n}P(A_n^c)
\end{aligned}$$

$1 \geq 1 + \frac{1}{n}P(A_n^c)$ for any $n \in \mathbb{Z}^+$ and $P(A_n^c) \geq 0$, this implies $P(A_n^c) = 0$ for any n.

$$P\left(\bigcup_{n=1}^{\infty} A_n^c\right) \leq \sum_{n=1}^{\infty} P(A_n^c) = 0$$

$$P\left(\bigcup_{n=1}^{\infty} A_n^c\right) = 0 \quad \text{as required}$$

2. Let $\{A_n\}_{n\geq 1}$ and $\{B_n\}_{n\geq 1}$ be two sequences of measurable sets in the measurable space $(\Omega, \mathcal{F})$. Set $C_n = A_n \cap B_n, D_n = A_n \cup B_n$.

(1) [4] Show that

$$\left(\overline{\lim_{n\to\infty}} A_n\right) \cap \left(\overline{\lim_{n\to\infty}} B_n\right) \supset \overline{\lim_{n\to\infty}} C_n$$

and

$$\left(\underline{\lim_{n\to\infty}} A_n\right) \cup \left(\underline{\lim_{n\to\infty}} B_n\right) \subset \underline{\lim_{n\to\infty}} D_n.$$

(2) [2] Show by example the two inclusions in (1) can be strict.

SOLUTION:
(1) **PART A** For any

$$\omega \in \overline{\lim_{n\to\infty}} C_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} (A_m \cap B_m)$$

which means for any $N \geq 1$, exists $m \geq N$ such that $\omega \in A_m \cap B_m$, then $\omega \in A_m$ and $\omega \in B_m$.

$$\omega \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \quad \text{and} \quad \omega \in \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B_m$$

$$\omega \in \left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m \right) \cap \left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} B_m \right) = \left(\overline{\lim_{n\to\infty}} A_n \right) \cap \left(\overline{\lim_{n\to\infty}} B_n \right)$$

therefore

$$\left(\overline{\lim_{n\to\infty}} A_n \right) \cap \left(\overline{\lim_{n\to\infty}} B_n \right) \supset \overline{\lim_{n\to\infty}} C_n$$

PART B For any

$$\omega \in \left(\underline{\lim}_{n\to\infty} A_n \right) \cup \left(\underline{\lim}_{n\to\infty} B_n \right)$$

a) $\omega \in \left(\underline{\lim}_{n\to\infty} A_n \right)$ means that there exists $N_1 \geq 1$, such that for any $m \geq N_1$, $\omega \in A_m$, or
b) $\omega \in \left(\underline{\lim}_{n\to\infty} A_n \right)^c \cap \left(\underline{\lim}_{n\to\infty} B_n \right) \subset \underline{\lim}_{n\to\infty} B_n$ means that there exists $N_2 \geq 1$, such that for any $m \geq N_2$, $\omega \in B_m$.

Then there exists $N = \max\{N_1, N_2\} \geq 1$, such that for any $m \geq N$, $\omega \in A_m$ or $\omega \in B_m$, i.e., $\omega \in A_m \cup B_m$.

$$\omega \in \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} (A_m \cup B_m) = \underline{\lim}_{n\to\infty} D_n$$

therefore

$$\left(\underline{\lim}_{n\to\infty} A_n \right) \cup \left(\underline{\lim}_{n\to\infty} B_n \right) \subset \underline{\lim}_{n\to\infty} D_n.$$

(2) Let

$$A_n = \begin{cases} \{0\} & \text{if } n \text{ is odd} \\ \{1\} & \text{if } n \text{ is even} \end{cases} \qquad B_n = \begin{cases} \{1\} & \text{if } n \text{ is odd} \\ \{0\} & \text{if } n \text{ is even} \end{cases}$$

then $C_n = A_n \cap B_n = \varnothing$ for all n, $\overline{\lim_{n\to\infty}} C_n = \varnothing$.

$0 \in \overline{\lim_{n\to\infty}} A_n$ since 0 exists in a subsequence of A_n. For the same reason, $0 \in \overline{\lim_{n\to\infty}} B_n$.

$$0 \in \left(\overline{\lim_{n\to\infty}} A_n \right) \cap \left(\overline{\lim_{n\to\infty}} B_n \right) \quad \text{but} \quad 0 \notin \overline{\lim_{n\to\infty}} C_n$$

The first inclusion can be strict.

$D_n = A_n \cup B_n = \{0,1\}$ for all n, $\varliminf_{n\to\infty} D_n = \{0,1\}$. However, neither 0 nor 1 exists in a tail of A_n (or B_n). $\varliminf_{n\to\infty} A_n = \varliminf_{n\to\infty} B_n = \varnothing$. The second inclusion can also be strict.

3 [4']. Consider the following two simple functions on a probability space $(\Omega, \mathcal{F}, P)$

$$f(\omega) = \sum_{i=1}^{3} a_i I_{A_i}(\omega),$$
$$g(\omega) = \sum_{j=1}^{4} b_j I_{B_j}(\omega).$$

Find $\int (f(\omega)+g(\omega))^2 P(d\omega)$.

SOLUTION:

$$\begin{aligned}
&(f(\omega)+g(\omega))^2\\
=&\Big(\sum_{i=1}^{3} a_i I_{A_i}(\omega) + \sum_{j=1}^{4} b_j I_{B_j}(\omega)\Big)^2\\
=&\sum_{i=1}^{3}\sum_{k=1}^{3} a_i I_{A_i}(\omega) a_k I_{A_k}(\omega) + \sum_{j=1}^{4}\sum_{l=1}^{4} b_i I_{B_i}(\omega) b_l I_{B_l}(\omega) + 2\sum_{i=1}^{3}\sum_{j=1}^{4} a_i I_{A_i}(\omega) b_j I_{B_j}(\omega)\\
=&\sum_{i=1}^{3} a_i^2 I_{A_i}(\omega) + \sum_{i\neq k}\sum_{i,k=1}^{3} a_i a_k I_{A_i\cap A_k}(\omega) + 2\sum_{i=1}^{3}\sum_{j=1}^{4} a_i b_j I_{A_i\cap B_j}(\omega) + \sum_{j\neq l}\sum_{j,l=1}^{4} b_j b_l I_{B_j\cap B_l}(\omega) + \sum_{j=1}^{4} b_j^2 I_{B_j}(\omega)\\
=&\sum_{i=1}^{3} a_i^2 I_{A_i}(\omega) + 2\sum_{i=1}^{3}\sum_{j=1}^{4} a_i b_j I_{A_i\cap B_j}(\omega) + \sum_{j=1}^{4} b_j^2 I_{B_j}(\omega)
\end{aligned}$$

is a simple function.

$$\begin{aligned}
&\int (f(\omega)+g(\omega))^2 P(d\omega)\\
=&\sum_{i=1}^{3} a_i^2 P(A_i) + 2\sum_{i=1}^{3}\sum_{j=1}^{4} a_i b_j P(A_i\cap B_j) + \sum_{j=1}^{4} b_j^2 P(B_j)
\end{aligned}$$

4 [5']. Let $X_n, n \geq 2$ be a sequence of random variables such that

$$P\{X_n = 0\} = 1 - \frac{2}{n^2},$$
$$P\{X_n = n\} = P\{X_n = -n\} = \frac{1}{n^2}.$$

Show that $\{X_n\}_{n\geq 2}$ converges to 0 almost surely.

SOLUTION:

Lemma 10.2 Let $\{a_n\}$ be a sequence with limit a, Y_n is a sequence of random variable satisfying $\sum_{n=1}^{\infty} P(|Y_n - a_n| \geq \epsilon) \leq \infty$ for any $\epsilon > 0$, then Y_n converges to a almost surely.

Choose $\{a_n\} = 0$ for all n, then $a = \lim_{n\to\infty} = 0$. Without loss of generality, let $0 < \epsilon < 1$.

$$\begin{aligned} &\sum_{n=2}^{\infty} P(|X_n - a_n| \geq \epsilon) \\ =&\sum_{n=2}^{\infty} (P(X_n \geq \epsilon) + P(X_n \leq -\epsilon)) \\ =&\sum_{n=2}^{\infty} (P(X_n = n) + P(X_n = -n)) \\ =&\sum_{n=2}^{\infty} \frac{2}{n^2} \\ <&\infty \end{aligned}$$

By Lemma 10.2, $\{X_n\}_{n\geq 2}$ converges to 0 almost surely.