S4D03/S6D03 2019/2020: Test One Solution

QUESTION 1

PART A For any

$$\omega \in \overline{\lim_{n\to\infty}} (A_m \cup B_m)$$

which means there exists a subsequence $\{A_{n_i} \cup B_{n_i}\}$ such that $\omega \in A_{n_i} \cup B_{n_i}$ for any i, i.e. ω belongs to an infinite number of $A_{n_i} \cup B_{n_i}$.

Suppose ω only belongs to a finite number of A_{n_i} and a finite number of B_{n_i}, then ω would only belong to a finite number of $A_{n_i} \cup B_{n_i}$, contradiction.

Then at least exits a subsequence $\{A_{n_i}\}$ or $\{B_{n_i}\}$ that ω belongs to every set in this subsequence, i.e. $\omega \in \overline{\lim_{n\to\infty}} A_n$ or $\omega \in \overline{\lim_{n\to\infty}} B_n$.

$$\omega \in \left(\overline{\lim_{n\to\infty}} A_n\right) \cup \left(\overline{\lim_{n\to\infty}} B_n\right)$$

$$\Rightarrow \left(\overline{\lim_{n\to\infty}} A_n\right) \cup \left(\overline{\lim_{n\to\infty}} B_n\right) \supseteq \overline{\lim_{n\to\infty}} C_n$$

PART B For any

$$\omega \in \left(\overline{\lim_{n\to\infty}} A_n\right) \cup \left(\overline{\lim_{n\to\infty}} B_n\right)$$

a) $\omega \in \left(\overline{\lim_{n\to\infty}} A_n\right)$, which means that there exists a subsequence $\{A_{n_i}\}$ such that $\omega \in A_i$ for any i, $A_{n_i} \subset A_{n_i} \cup B_{n_i}$, so there exists a subsequence $\{A_{n_i} \cup B_{n_i}\}$ such that $\omega \in A_{n_i} \cup B_{n_i}$ for any i, i.e. $\omega \in \overline{\lim_{n\to\infty}} (A_m \cup B_m)$.

b) $\omega \in \left(\overline{\lim_{n\to\infty}} A_n\right)^c \cap \left(\overline{\lim_{n\to\infty}} B_n\right) \subset \overline{\lim_{n\to\infty}} B_n$ similarly, $\omega \in \overline{\lim_{n\to\infty}} (A_m \cup B_m)$.

$$\Rightarrow \left(\overline{\lim_{n\to\infty}} A_n\right) \cup \left(\overline{\lim_{n\to\infty}} B_n\right) \subseteq \overline{\lim_{n\to\infty}} C_n$$

In conclusion,

$$\left(\overline{\lim_{n\to\infty}} A_n\right) \cup \left(\overline{\lim_{n\to\infty}} B_n\right) = \overline{\lim_{n\to\infty}} (A_m \cup B_m).$$

QUESTION 2

$$\text{LHS} = I_{A\cup B}(\omega) = \begin{cases} 1 & \text{if } \omega \in A \cup B \\ 0 & \text{if } \omega \in (A\cup B)^c \end{cases}$$

$$\begin{aligned} \text{RHS} =& I_A(\omega) + I_B(\omega) = I_A(\omega)I_B(\omega) \\ =& \begin{cases} 1+0-1\cdot 0 = 1 & \text{if } \omega \in A \cap B^c \\ 0+1-0\cdot 1 = 1 & \text{if } \omega \in A^c \cap B \\ 1+1-1\cdot 1 = 1 & \text{if } \omega \in A \cap B \\ 0+0-0\cdot 0 = 0 & \text{if } \omega \in A^c \cap B^c \end{cases} \\ =& \begin{cases} 1 & \text{if } \omega \in (A\cap B^c)\cup(A^c\cap B)\cup(A\cap B) \\ 0 & \text{if } \omega \in (A\cup B)^c \end{cases} \\ =& \begin{cases} 1 & \text{if } \omega \in A\cup B \\ 0 & \text{if } \omega \in (A\cup B)^c \end{cases} \end{aligned}$$

LHS=RHS for any $\omega \in \Omega$. Proved.

QUESTION 3

Known that $X_n \sim \text{Unif}\ (3 - \frac{1}{n^2}, 3 + \frac{1}{n})$, $n \geq 1$.
Want to show that $\lim\limits_{n\to\infty} P(|X_n - 3| > \epsilon) = 0$ for any $\epsilon > 0$.

$$\begin{aligned} & P(|X_n - 3| > \epsilon) \\ =& P(X_n > 3+\epsilon \text{ or } X_n < 3-\epsilon) \\ =& P(X_n > 3+\epsilon) + P(X_n < 3-\epsilon) \\ =& \begin{cases} 0 & \text{if } n \geq \frac{1}{\epsilon} \\ \dfrac{\frac{1}{n} - \epsilon}{\frac{1}{n} + \frac{1}{n^2}} & \text{if } \frac{1}{\sqrt{\epsilon}} < n < \frac{1}{\epsilon} \\ \dfrac{\frac{1}{n} + \frac{1}{n^2} - 2\epsilon}{\frac{1}{n} + \frac{1}{n^2}} & \text{if } n \leq \frac{1}{\sqrt{\epsilon}} \end{cases} \end{aligned}$$

For any $\epsilon > 0$, exists $N_\epsilon = [\frac{1}{\epsilon}] + 1$ such that for any $n \geq N_\epsilon$, $P(|X_n - 3| > \epsilon) = 0$. Therefore, $\lim\limits_{n\to\infty} P(|X_n - 3| > \epsilon) = 0$.

QUESTION 4

Want to show that $P(\{\omega : \lim_{n\to\infty} X_n(\omega) = 2\}) = 1 \iff \sum_{i=1}^{\infty} p_i < \infty$.

Let $A = \bigcap_{r\in\mathbb{Q}+} \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} \{\omega : |X_n(\omega) - 2| \le r\}$, it is equivalent to show that

$$P(A) = 1 \iff \sum_{i=1}^{\infty} p_i < \infty$$

$A^c = \bigcup_{r\in\mathbb{Q}+} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{\omega : |X_n(\omega) - 2| > r\}$ and let $B_{nr} = \{\omega : |X_n(\omega) - 2| > r\}$.

$$A^c = \bigcup_{r\in\mathbb{Q}+} \overline{\lim_{n\to\infty}} B_{nr}$$

"$\Leftarrow$"

$$\sum_{n=1}^{\infty} P(B_{nr}) = \sum_{n=1}^{\infty} P(|X_n - 2| > r) \le \sum_{n=1}^{\infty} P(|X_n - 2| > 0) = \sum_{n=1}^{\infty} p_i < \infty$$

By Borel-Cantelli Lemma, $P(\overline{\lim_{n\to\infty}} B_{nr}) = 0$

$$P(A^c) = P(\bigcup_{r\in\mathbb{Q}+} \overline{\lim_{n\to\infty}} B_{nr}) \le \sum_{r\in\mathbb{Q}+} P(\overline{\lim_{n\to\infty}} B_{nr}) = 0$$

$\therefore P(A^c) = 0 \quad P(A) = 1$

"$\Rightarrow$"

To show $P(A) = 1$ implies $\sum_{i=1}^{\infty} p_i < \infty$, it is equivalent to show $\sum_{i=1}^{\infty} p_i = \infty$ implies $P(A) \neq 1$.

Since $X_1, X_2, \ldots$ are independent, $B_{1r}, B_{2r}, \ldots$ are independent for any r. Pick $r = 1$,

$$\sum_{n=1}^{\infty} P(B_{n1}) = \sum_{n=1}^{\infty} P(|X_n - 2| > 1) = \sum_{n=1}^{\infty} P(X_n = 1 - \frac{1}{n}) = \sum_{n=1}^{\infty} p_i = \infty$$

By Borel-Cantelli Lemma, $P(\overline{\lim_{n\to\infty}} B_{n1}) = 1$

$$P(A^c) = P(\bigcup_{r\in\mathbb{Q}+} \overline{\lim_{n\to\infty}} B_{nr}) \ge P(\overline{\lim_{n\to\infty}} B_{n1}) = 1$$

$\therefore P(A^c) = 1 \quad P(A) = 0 \neq 1$

QED