S4D03/S6D03 2019/2020: Test Two Solution

QUESTION 1

$$\mathrm{E}[X_n] = \frac{1}{2n}$$

$$Y_n = X_n - \mathrm{E}[X_n] = \begin{cases} 1 - \frac{1}{2n}, & -\frac{1}{2n} \\ -\frac{1}{2n}, & 1 - \frac{1}{2n} \end{cases}$$

$$\mathrm{E}[X_n] = \mathrm{E}[X_n - \mathrm{E}[X_n]] = 0$$

$$\mathrm{Var}(Y_n) = \mathrm{Var}(X_n) = \mathrm{E}[X_n^2] - (\mathrm{E}[X_n])^2 = \frac{1}{2n} - \frac{1}{4n^2}$$

Let $\sigma_k^2 = \mathrm{Var}(Y_k) = \frac{1}{2k} - \frac{1}{4k^2}$, then

$$B_n^2 = \sum_{k=1}^n \sigma_k^2 = \sum_{k=1}^n \frac{1}{2k} - \sum_{k=1}^n \frac{1}{4k^2} \approx \frac{1}{2} \ln n$$

since $\sum_{k=1}^n \frac{1}{4k^2}$ converges.

Let $S_n = \sum_{k=1}^n Y_n$,

$$\lim_{n\to\infty} \frac{S_n}{B_n} = \lim_{n\to\infty} \frac{\sqrt{2}(Y_1 + Y_2 + \cdots + Y_n)}{\sqrt{\ln n}}$$

First, let's check the Feller's condition:

$$0 < \frac{\max_k\{\frac{1}{2k} - \frac{1}{4k^2}\}}{B_n^2} \leq \frac{\max_k\{\frac{1}{2k}\}}{\ln n} \to 0$$

The Feller's condition holds, then if the Lindeberg Condition holds, then CLT holds.

$$\frac{1}{B_n^2} \sum_{k=1}^n \mathrm{E}[Y_k^2 \mathbb{I}_{\{|Y_k| > \varepsilon B_n\}}]$$

$|Y_k| < 1$ for any k, and B_n diverges, $|Y_k| \leq \varepsilon B_n$ is true for any $\epsilon > 0$ as $n \to \infty$, i.e. $\mathbb{I}_{\{|Y_k| > \varepsilon B_n\}} = 0$ for all k as $n \to \infty$.

$$\lim_{n\to\infty} \frac{1}{B_n^2} \sum_{k=1}^n \mathrm{E}[Y_k^2 \mathbb{I}_{\{|Y_k| > \varepsilon B_n\}}] = 0$$

In conclusion,

$$\frac{\sqrt{2}(Y_1 + Y_2 + \cdots + Y_n)}{\sqrt{\ln n}} \xrightarrow{D} Z \sim N(0,1)$$

QUESTION 2

Please refer to the lecture notes on November 4th.

Any other example satisfying the required conditions is also acceptable.

QUESTION 3

Let $p_{nk} = \frac{k}{n^2}$

$$\max_k\{p_{nk}\} = \frac{n}{n^2} = \frac{1}{n} \to 0$$

$$\sum_{k=1}^{n} p_{nk} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{n(n+1)}{2n^2} \to \frac{1}{2}$$

By Law of Small Number (Poisson Approximation),

$$\sum_{k=1}^{n} X_{nk} \xrightarrow{D} Y \sim \text{Pois}(1/2)$$