MATH 702, Winter 2015, Homework 1, Due Monday, January 26

Total: 20 marks.

(1) Let k be a field, A be the polynomial ring $k[x_1, \ldots, x_n]$, and $\mathbb{A}^n$ be the affine n-space over k. For each $f \in A$ we denote by

(i) $Z(f)$ the affine algebraic set of f, i.e., the zero locus of f;

(ii) U_f the complement of $Z(f)$ in $\mathbb{A}^n$.

Show that the collection of sets $\{U_f | f \in A\}$ form a *basis of open sets* in $\mathbb{A}^n$ (for the Zariski topology), i.e., every open subset in $\mathbb{A}^n$ is a union of members in $\{U_f | f \in A\}$. (2 marks)

Answer: For any open subset U of $\mathbb{A}^n$, write $U = \mathbb{A}^n - Z(I)$ for some ideal I of A. Then we can just write the tautology $I = \langle f \rangle_{f \in I}$. Since $Z(I) = Z(\langle f \rangle_{f \in I}) = \cap_{f \in I} Z(f)$, we have

$$U = \mathbb{A}^n - \left(\bigcap_{f \in I} Z(f) \right) = \bigcup_{f \in I} (\mathbb{A}^n - Z(f)) = \bigcup_{f \in I} U_f.$$

(Remark: We don't need to use the fact that A is Noetherian.) □

Remark: We usually call the open sets in $\{U_f | f \in A\}$ *basic open sets* of $\mathbb{A}^n$, and say that the basic open sets *generate* arbitrary open sets. When we prove statements concerning open sets, it is usually enough to assume that the open sets involved in the questions are basic open sets.

(2) Some topological properties of $\mathbb{A}^n$.

(i) A topological space X is called

(a) *T1* if for every pair of distinct points $P, Q \in X$, there is an open subset U containing P but not Q, and another open subset V containing Q but not P;

(b) *T2*, or *Hausdorff*, if it satisfies the same condition in T1 with U and V disjoint.

Show that if the base field k is infinite, then $\mathbb{A}^n$ is T1 but not T2. (3 marks)

Answer: In $\mathbb{A}^n$, every point $P = (P_1, \ldots, P_n)$ is closed, because $P = Z(\langle x_1 - P_1, \ldots, x_n - P_n \rangle)$. (1 mark. Note: I stress that this fact is important. Later we will see some topology of which a point may not be closed.) If $P \neq Q$, then $P \in U = \mathbb{A}^n - Q$, where U is open. Similarly, $Q \in V = \mathbb{A}^n - P$, where V is open. Therefore, $\mathbb{A}^n$ is T1.

To show that $\mathbb{A}^n$ is not T2, we show that two open subsets in $\mathbb{A}^n$ must have non-empty intersection. It is enough to show that for every non-zero $f, g \in A = k[\mathbb{A}^n]$, the open sets U_f and U_g have non-empty intersection. Indeed we can check that

$$U_f \cap U_g = (\mathbb{A}^n - Z(f)) \cap (\mathbb{A}^n - Z(g)) = \mathbb{A}^n - (Z(f) \cup Z(g)) = \mathbb{A}^n - Z(fg) = U_{fg}$$

Since fg is also non-zero, U_{fg} is non-empty.
(2 marks. It is important to assume that k is infinite, since otherwise there exists polynomial f

whose corresponding open set is empty, e.g., when $k = \mathbb{F}_p$ and $f = x^p - x \in k[x]$, then $U_f = \emptyset$.) $\square$

(ii) A topological space X is called *quasi-compact* if every open cover of X has a finite subcover, i.e. if there exists a collection (possibly infinite) $\{U_i\}$ of open subsets of X such that $X = \cup U_i$, then there exists a finite sub-collection $\{U_{i_1}, \ldots, U_{i_k}\}$ of $\{U_i\}$ such that $X = U_{i_1} \cup \cdots \cup U_{i_k}$.
Show that $\mathbb{A}^n$ is quasi-compact. (2 marks)

Answer: Let $\mathbb{A}^n = \cup_{f \in S} U_f$ for some subset $S \subseteq A = k[\mathbb{A}^n]$. Taking complement, we have $\cap_{f\in S} Z(f) = Z(\langle f \rangle_{f \in S}) = \emptyset$. It means that there is a function in $\langle f \rangle_{f\in S}$ vanishing nowhere in $\mathbb{A}^n$. Such a function must be of the form $g = \sum_{i=1}^k a_i f_i$ (a finite sum!) for some $f_1, \ldots, f_k \in S$. Then we have $Z(\langle f_1, \ldots, f_k \rangle) = \emptyset$ because $g \in \langle f_1, \ldots, f_k \rangle$ and is nowhere vanishing. We have found a finite subcover $\{U_{f_1}, \ldots, U_{f_k}\}$ of the cover $\{U_f\}_{f \in S}$ such that $\mathbb{A}^n = \cup_{i=1}^k U_{f_i}$. Therefore, $\mathbb{A}^n$ is quasi-compact.
(Remark: Just like Question 1, We don't need to use the fact that A is Noetherian.) $\square$

(iii) Given two topological space X and Y, we define the *product topology* on $X \times Y$ by taking the collection of open subsets $U \times V$ as a basis, where $U \subseteq X$ and $V \subseteq Y$ are open.
Is the topology of $\mathbb{A}^2$ equal to the product topology of $\mathbb{A}^1 \times \mathbb{A}^1$? (2 marks)

Answer: Many arguments work in this case. For example, we can argue that the open subset $\mathbb{A}^2 - Z(x - y)$ (i.e., a plane without the diagonal) is not a union of

$$(\mathbb{A}^1 - \{\text{finite many points}\}) \times (\mathbb{A}^1 - \{\text{finite many points}\}).$$

It is because, say, the above product of open subsets (and also their unions) can only exclude finitely many points in the diagonal of $\mathbb{A}^2$. $\square$

(3) (c.f. Dummit-Foote, Sec.15.1, Ex.5) Let M be an Noetherian R-module and $\phi : M \to M$ is an R-module endomorphism. Prove that

(i) $\ker(\phi^n) \cap \text{image}(\phi^n) = 0$ if n is large enough; (2 marks) (Hint: Observe that $\ker \phi \subseteq \ker \phi^2 \subseteq \cdots$.)

(ii) if ϕ is surjective, then it is an isomorphism. (2 marks)

(iii) If ϕ is injective, is it necessarily an isomorphism? (1 mark)

Answer:

(i) Observe that we have the increasing chain of R-submodules $\ker \phi \subseteq \ker \phi^2 \subseteq \cdots$ in M, and so we have $\ker(\phi^n) = \ker(\phi^{n+1}) = \cdots$ for large enough n, by that M is Noetherian. (1 mark) Now if $x \in \ker(\phi^n) \cap \text{image}(\phi^n)$, then $x = \phi^n(y)$ and $\phi^n(x) = \phi^{2n}(y) = 0$. Then $y \in \ker(\phi^{2n}) = \ker(\phi^n)$ and so $x = \phi^n(y) = 0$. (1 mark)

(ii) If ϕ is surjective, then ϕ^n is also surjective, or $\text{image}(\phi^n) = M$. Then $\ker(\phi^n) \cap \text{image}(\phi^n) = \ker(\phi^n) = 0$. Since $\ker(\phi) \subseteq \ker(\phi^n)$, we have $\ker(\phi) = 0$, and so ϕ is injective.

(iii) If ϕ is injective, then it is not necessarily an isomorphism. For example, take $M = R = \mathbb{Z}$, and $\phi : \mathbb{Z} \to \mathbb{Z}$, $x \mapsto 2x$.

$\square$

(4) (Dummit-Foote, Sec.15.2, Ex.2) Let R be a ring and I, J be ideals of R. Prove that

(a) $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. (2 marks)

Answer: We can check easily, by definition of $\sqrt{I}$, that if $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$. Therefore, since $IJ \subseteq I \cap J$, it is then easy to check that

$$\sqrt{IJ} \subseteq \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J}.$$

Suppose that $x \in \sqrt{I} \cap \sqrt{J}$, then $x^m \in I$ and $x^n \in J$ for some m, n, then $x^{m+n} \in IJ$ and so $x \in \sqrt{IJ}$. Therefore

$$\sqrt{IJ} \subseteq \sqrt{I \cap J} \subseteq \sqrt{I} \cap \sqrt{J} \subseteq \sqrt{IJ},$$

forcing all these ideals to be equal. □

(b) $\sqrt{I+J} = \sqrt{\sqrt{I} + \sqrt{J}}$. (2 marks)

Answer: We have $I + J \subseteq \sqrt{I} + \sqrt{J}$ implies that $\sqrt{I+J} \subseteq \sqrt{\sqrt{I} + \sqrt{J}}$ Conversely if $x \in \sqrt{\sqrt{I} + \sqrt{J}}$, then $x^m = y + z$, for some $y \in \sqrt{I}$, some $z \in \sqrt{J}$, and some integer m. Hence $y^n \in I$ and $z^p \in J$ for some integers n, p. Then $x^{mnp} \in I + J$ and $x \in \sqrt{I+J}$. □

(5) A map $f : X \to Y$ between topological spaces is a *homeomorphism* (note the spelling!) if f is bijective and both f and f^{-1} are continuous.

Let $\phi : V \to W$ be a morphism of affine algebraic sets and $\tilde{\phi} : k[W] \to k[V]$ be the associated k-algebra morphism of coordinate rings. Prove that if $\tilde{\phi}$ is surjective, then ϕ is a homeomorphism of V onto a closed subset of W. (2 marks) (Hint: First describe the image of ϕ as a closed subset of W.)

Answer: Consider the ideal $\ker \tilde{\phi}$ of $k[W]$, which provides a closed subset $Z(\ker \tilde{\phi})$ of W. We claim that ϕ is a homeomorphism of V onto $Z(\ker \tilde{\phi})$.

We first show that ϕ maps V into $Z(\ker \tilde{\phi})$ and is injective. The first claim is easy: for all $f \in \ker \tilde{\phi}$, we have $\tilde{\phi}(f) \equiv 0_V$ the zero function of V. Hence if $Q = \phi(P) \in \phi(V)$, then $f(Q) = f(\phi(P)) = (\tilde{\phi}(f))(P) = 0_V(P) = 0$. This implies that $Q \in Z(\ker \tilde{\phi})$. To show that $\phi : V \to Z(\ker \tilde{\phi})$ is injective, we check if $\phi(P) = \phi(P')$, then $(\tilde{\phi}(f))(P) = f(\phi(P)) = f(\phi(P')) = (\tilde{\phi}(f))(P')$ for all $f \in k[W]$. Since $\tilde{\phi}$ is surjective, we have $g(P) = g(P')$ for all $g \in k[V]$, and so $P = P'$.

We then show that $\ker \tilde{\phi} = IZ(\ker \tilde{\phi})$. (Note we always have $J \subseteq IZ(J)$ by definition, but in general $J = IZ(J)$ is not true. The equality only holds for some special ideals J.) If $f \in IZ(\ker \tilde{\phi})$, then $f|_{Z(\ker \tilde{\phi})} \equiv 0$. Since $\phi(V) \subseteq Z(\ker \tilde{\phi})$, we have $f \circ \phi(V) \equiv 0$. This implies that $\tilde{\phi}(f) \equiv 0$ as a function of V. Therefore $f \in \ker \tilde{\phi}$.

That $\ker \tilde{\phi} = IZ(\ker \tilde{\phi})$ implies that $k[W]/\ker \tilde{\phi} \cong k[W]/IZ(\ker \tilde{\phi}) = k[Z(\ker \tilde{\phi})]$, which then implies that $\tilde{\phi} : k[Z(\ker \tilde{\phi})] \to k[V]$ is an isomorphism of k-algebra. Denote the inverse k-algebra morphism of $\tilde{\phi}$ by $\tilde{\psi} : k[V] \to k[Z(\ker \tilde{\phi})]$. There is a corresponding morphism of affine algebraic sets $\psi : Z(\ker \tilde{\phi}) \to V$, by the equivalence of categories shown in class. Since $\tilde{\psi} \circ \tilde{\phi} = \mathrm{id}_{k[Z(\ker \tilde{\phi})]}$ and $\tilde{\phi} \circ \tilde{\psi} = \mathrm{id}_{k[V]}$, they induce the equalities $\phi \circ \psi = \mathrm{id}_{Z(\ker \tilde{\phi})}$ and $\psi \circ \phi = \mathrm{id}_V$, we have indeed V and $Z(\ker \tilde{\phi})$ are isomorphic as an affine algebraic set. In particular, ϕ is a homeomorphism onto its image (whose inverse is ψ). □