MATH 702, Winter 2015, Homework 2,
Due Monday, February 16

Total: 16 marks.

(1) (Dummit-Foote, Sec 15.2, Ex. 54) Given two ideals I, J in a ring R, define the *ideal quotient*

$$(I : J) = \{x \in R | xJ \subseteq I\}.$$

Note that $(I : J)$ is also an ideal of R, and $I \subseteq (I : J)$. Now let $R = k[\mathbb{A}^n]$.

(a) Show that $Z(I) - Z(J)$, the set of points in $Z(I)$ not lying in $Z(J)$, is contained in $Z((I : J))$. (2 marks)

Answer: Since $J(I : J) \subseteq I$, we have $Z(I) \subseteq Z(J(I : J)) = Z((I : J)) \cup Z(J)$, which is equivalent to say that $Z(I) - Z(J) \subseteq Z((I : J))$. □

(b) Show that if V and W are affine algebraic sets, then $(I(V) : I(W)) = I(V - W)$. (1 mark)

Answer: We can check that

$$\begin{aligned} f \in (I(V) : I(W)) &\Leftrightarrow fI(W) \subseteq I(V) \\ &\Leftrightarrow (fg)|_V \equiv 0 \text{ for all } g \text{ such that } g|_W \equiv 0 \\ &\Leftrightarrow_{(*)} f|_{V-W} \equiv 0 \\ &\Leftrightarrow f \in I(V - W). \end{aligned}$$

((*) is not completely trivial, only $\Leftarrow$ is easy. For $\Rightarrow$, Suppose $(fg)|_V \equiv 0$ for all g such that $g|_W \equiv 0$, then $f(P)g(P) = 0$ for all $P \in V - W$. Now take g such that $g|W \equiv 0$ but $g(P) \neq 0$, then it forces $f(P) = 0$. Such a function g exists because W is a closed subset AND our sets are T1 (i.e., a point is a closed subset). For if W is not closed, then we can take $P \in \bar{W} - W$, then $g|W \equiv 0$ forces $g|_{\bar{W}} \equiv 0$ because

$$\begin{aligned} g|W \equiv 0 &\Rightarrow W \subseteq g^{-1}\{0\} \\ &\Rightarrow \bar{W} \subseteq g^{-1}\{0\} \qquad \text{(because } g^{-1}\{0\} \text{ is a closed subset.)} \\ &\Rightarrow g|_{\bar{W}} \equiv 0. \end{aligned}$$

I have not deducted marks if you did not mention this.) □

(c) Suppose that k is algebraically closed and I is a radical ideal. Prove that $Z((I : J)) = \overline{Z(I) - Z(J)}$, the closure of $Z(I) - Z(J)$. (2 marks) (Hint: J is not necessarily a radical ideal.)

Answer: Note that J may not a radical ideal. But we will prove that $(I : J) = (I : \sqrt{J})$. Assuming this, the statement we want to prove becomes

$$Z((I : \sqrt{J})) = Z((I : J)) = \overline{Z(I) - Z(J)} = \overline{Z(I) - Z(\sqrt{J})}.$$

Therefore, we can assume that J is a radical ideal. By Nullstellensatz, we can write $I = I(V)$ and $J = I(W)$, where $V = Z(I)$ and $W = Z(J)$. By (ii), we have

$$(I : J) = (I(V) : I(W)) = I(V - W) = I(Z(I) - Z(J)).$$

Taking zero sets, we have

$$Z((I:J)) = ZI(Z(I) - Z(J)) = \overline{Z(I) - Z(J)}.$$

We now prove that $(I:J) = (I:\sqrt{J})$. It is easy to check that $(I:\sqrt{J}) = \{x \in R | x\sqrt{J} \subseteq I\} \subseteq \{x \in R | xJ \subseteq I\} = (I:J)$. Conversely, if $x \in (I:J)$, then $xJ \subseteq I$. If $k \in \sqrt{J}$, then $k^m \in J$ and $(xk)^m = x^{m-1}(xk^m) \in x^{m-1}(xk^m) \in x^{m-1}(xJ) = x^{m-1}I \subseteq I$. Hence $xk \in \sqrt{I} = I$ and so $x\sqrt{J} \subseteq I$. □

Remark: Usually, we impose the algebraically-closed condition for k because we want to use the Hilbert's Nullstellensatz (that $IZ(J) = \sqrt{J}$ for every ideal J) and its consequences.

(2) (i) (Dummit-Foote, Sec 15.2, Ex. 20) Suppose $\varphi : V \to W$ is a surjective morphism of affine algebraic sets. Prove that if V is a variety, then W is a variety. (1 mark)

Answer: If $W = W_1 \cup W_2$ where W_1, W_2 are closed proper subsets of W, then we define $V_i = \varphi^{-1}W_i$ for $i = 1, 2$, which are clearly closed subsets. Since W_i is a proper subset and φ is surjective, V_i is also proper. Moreover, we must have $V = V_1 \cup V_2$, which contradicts that V is irreducible. □

(ii) (Dummit-Foote, Sec 15.2, Ex. 26) Let

$$V = Z(xz - y^2, yz - x^3, z^2 - x^2y) \subset \mathbb{A}^3.$$

Show that if k is infinite, then V is an affine variety. (3 marks) (Hint: Consider $\mathbb{A}^1 \to V$, $t \mapsto (t^3, t^4, t^5)$.)

Answer: We define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$, $t \mapsto (t^3, t^4, t^5)$. This is a morphism because it is defined by polynomials. The image lies in V: we can check $xz - y^2 = (t^3)(t^5) - (t^4)^2 = 0$ and similarly for $yz - x^3$ and $z^2 - x^2y$. To show that φ is surjective, notice that if $(x, y, z) \in V$ with $x = 0$, then all x, y, z are zero, and so $\varphi(0) = (0, 0, 0)$. If $x \neq 0$, then we take $t = y/x$ and so $t \mapsto (t^3, t^4, t^5) = ((y/x)^3, (y/x)^4, (y/x)^5)$. We can check that $(y/x)^3 = x$ since

$$\frac{y^3}{x^3} = \frac{(xz)y}{x^3} = \frac{x(yz)}{x^3} = \frac{x(x^3)}{x^3} = x.$$

Similarly, we can check $(y/x)^4 = y$ and $(y/x)^5 = z$. Finally, if we apply (a), then we know that V is irreducible, so it is an affine variety. □

(3) (Dummit-Foote, Sec 15.3, Ex. 2) Let k be a field and $R = k[x,y]/\langle x^2 - y^3\rangle$. For $\bar{x}, \bar{y} \in R$ the natural projections of x, y, denote $t = \bar{x}/\bar{y}$ and $K = k(t)$. Prove that $k[t]$ is the integral closure of R in K. (3 marks) (Hint: It is enough to show that K is the field of fraction of R. Why?)

Answer: Recall that we have a morphism $\mathbb{A}^1 \to V = Z(x^2 - y^3)$, $t \mapsto (t^3 - t^2)$ which is bijective (but not an isomorphism, as the inverse map $(\bar{x}, \bar{y}) \mapsto \bar{x}/\bar{y}$ is not a polynomial). This induces a morphism of rings $\Phi : k[\bar{x}, \bar{y}] \to k[t]$, which is injective (one way to check this is to show that Φ maps a k-basis $\{\bar{y}^i, \bar{x}\bar{y}^i, \bar{x}^2\bar{y}^i\}_{i\in\mathbb{Z}}$ of R into $\{t^{3i}, t^{3i+2}, t^{3i+4}\}_{i\in\mathbb{Z}}$, which is k-linearly independent in $k[t]$.)

We then show that $K = k(t)$ is the field of fraction K_R of $R = k[\bar{x}, \bar{y}]$. Clearly $K \subseteq K_R$ since $t = \bar{x}/\bar{y}$ as a quotient of $\bar{x} \in R$ by $\bar{y} \in R$. Conversely, every $f(\bar{x}, \bar{y}) \in R$, if non-zero, can be written as $f(t^3, t^2)$, which is also non-zero since the ring morphiasm Φ is injective. Hence we have $(f(\bar{x}, \bar{y}))^{-1} = (f(t^3, t^2))^{-1} \in K = k(t)$.

Finally, we know that $k[t]$ is a PID, since t is a genuine variable of $\mathbb{A}^1$. (In a more technical term, we say that t is transcendental over k.). Hence it is a UFD. We checked in class (or from Example 3, p.693 of the book) that a UFD is integrally closed (in its field of fractions). Moreover t is integral over R as $t^2 - \bar{y} = (\bar{x}/\bar{y})^2 - \bar{y} = (\bar{x}^2 - \bar{y}^3)/\bar{y} = 0$. Hence $k[t]$ is integral over R, and is the integral closure of

R in $K_R = K = k(t)$.

$\square$

(4) First recall some basic notions.

(i) Let $\Phi : A \to B$ be a ring homomorphism. We say that Φ has the *going-up property*, or that Φ is GU, if it satisfies the following condition: Let $P_1 \subseteq \cdots \subseteq P_n$ be a chain of prime ideals of A and $Q_1 \subseteq \cdots \subseteq Q_m$ (with $m < n$) be a chain of prime ideals of B such that $\Phi^{-1}Q_i = P_i$ for all $i = 1, \ldots, m$. Then the chain $Q_1 \subseteq \cdots \subseteq Q_m$ can be extended to a chain $Q_1 \subseteq \cdots \subseteq Q_n$ of prime ideals such that $\Phi^{-1}Q_i = P_i$ for all $i = 1, \ldots, n$.
For example, if Φ is the inclusion map of $A \subseteq B$, and B is integral over A, then Φ is GU, by Theorem 26 of the textbook.

(ii) Let $\varphi : X \to Y$ be a continuous map of topological space. We say that φ is *closed* if it maps every closed subset of X onto a closed subset of Y.

Let the base field k be algebraically closed. Suppose that $\varphi : V \to W$ is a morphism of affine algebraic sets, and $\tilde{\varphi} : k[W] \to k[V]$ be the corresponding morphism of rings. Prove that

$$\varphi : V \to W \text{ is closed if and only if } \tilde{\varphi} \text{ is GU.}$$

(4 marks)

Answer:

- In the proof below, I will denote a prime idaal by $\mathfrak{p}$ or $\mathfrak{q}$ instead of P or Q, which should be used for denoting points. Sorry for the confusions.
- Some of the techniques in the proof are similar to those in the proof of

$$\varphi : V \to W \text{ is surjective if and only if } \tilde{\varphi} \text{ is LO (Lying-Over).}$$

done in class.

($\Rightarrow$) Given $\varphi : V \to W$ is closed, and let $\mathfrak{p} \subseteq \mathfrak{p}'$ be two prime ideals in $k[W]$, and $\mathfrak{q}$ be a prime ideal in $k[V]$ such that $\tilde{\varphi}^{-1}\mathfrak{q} = \mathfrak{p}$. (The general statement for GU follows by applying induction to the above statement.)
Applying Z to $\mathfrak{p} \subseteq \mathfrak{p}'$, we have

$$Z(\mathfrak{p}') \subseteq Z(\mathfrak{p}) = Z(\tilde{\varphi}^{-1}\mathfrak{q}) = \overline{\varphi(Z(\mathfrak{q}))} = \varphi(Z(\mathfrak{q})),$$

where the last equality holds because φ is closed. So $Z(\mathfrak{p}')$ lies in the image of φ. Let J be an ideal containing $\mathfrak{q}$ in $k[V]$ such that $Z(J) = \varphi^{-1}Z(\mathfrak{p}')$, so that $\varphi(Z(J)) = Z(\mathfrak{p}')$. We claim that we can choose J to be a prime ideal. Otherwise, let $Z(J) = \cup_i Z(\mathfrak{q}_i)$ be a decomposition into varieties, so that every $\mathfrak{q}_i$ is a prime ideal. Then we have a decomposition of $Z(\mathfrak{p}')$ as

$$Z(\mathfrak{p}') = \varphi(Z(J)) = \bigcup_i \varphi(Z(\mathfrak{q}_i)) = \bigcup_i \overline{\varphi(Z(\mathfrak{q}_i))},$$

again because φ is closed. As $Z(\mathfrak{p}')$ is irreducible, we have $Z(\mathfrak{p}') = \overline{\varphi(Z(\mathfrak{q}_i))} = \varphi(Z(\mathfrak{q}_i))$ for some $\mathfrak{q}_i$. We can take J to be $\mathfrak{q}_i$ so that $J \supseteq \mathfrak{q}$ and $\tilde{\varphi}^{-1}J = \mathfrak{p}'$.

($\Leftarrow$) Given $\tilde{\varphi}$ is GU. Given every closed subset $Z(J)$ in V, by decomposition into varieties we can assume that $J = \mathfrak{p}$ is a prime ideal. To show that $\varphi(Z(\mathfrak{p}))$ is equal to its closure, suppose that $Q \in \overline{\varphi(Z(\mathfrak{p}))} = Z(\varphi^{-1}\mathfrak{p})$, then the maximal ideal m_Q corresponding to Q contains $\varphi^{-1}\mathfrak{p}$. By GU there exists an ideal $\mathfrak{n}$ containing $\mathfrak{p}$ such that $\varphi^{-1}\mathfrak{n} = \mathfrak{m}_Q$. We can assume that $\mathfrak{n}$ is maximal, as its preimage is maximal. Let P be the point corresponding to $\mathfrak{n}$ by Nullstellensatz. This implies $Q = \varphi(P) \in \varphi(Z(\mathfrak{p}))$. Therefore, $\overline{\varphi(Z(\mathfrak{p}))} \subseteq \varphi(Z(\mathfrak{p}))$, or in other words, $\varphi(Z(\mathfrak{p}))$ is closed. $\square$