MATH 702, Winter 2015, Homework 3,
Due Monday, March 9

Total: 22 marks.

(1) If V and W are affine algebraic sets, define the *Cartesian product*

$$V \times W = \{(P, Q) | P \in V \text{ and } Q \in W\}.$$

Prove that

(a) $V \times W$ is also an affine algebraic set (1 mark);

Answer: Suppose that $V = Z(\langle f_i \rangle_i)$ and $W = Z(\langle g_j \rangle_j)$, where $f_i \in k[\mathbb{A}^m] = k[x_1, \dots, x_m]$ and $g_j \in k[\mathbb{A}^n] = k[y_1, \dots, y_n]$. Then

$$V \times W = Z(\left\langle \tilde{f}_i, \tilde{g}_j \right\rangle_{i,j}),$$

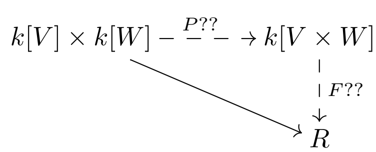
where $\tilde{f}_i(P, Q) = f_i(P)$, $\tilde{g}_i(P, Q) = g_i(Q)$, and $\langle f_i, g_j \rangle$ is now an ideal in $k[\mathbb{A}^{m+n}] = k[x_1, \dots, x_m, y_1, \dots, y_n]$. $\square$

(b) $k[V \times W] \cong k[V] \otimes_k k[W]$ as k-algebras (4 marks). (Hint: It may be hard to prove that $k[V] \otimes_k k[W] \to k[V \times W]$, whatever defined, is injective and surjective. Use Universal Property instead.)

Answer: As mentioned, it may be hard (or just tedious) to prove that $k[V] \otimes_k k[W] \to k[V \times W]$, which is explicitly

$$k[x_1, \dots, x_m]/\langle f_i \rangle \otimes_k k[y_1, \dots, y_n]/\langle g_j \rangle \to k[x_1, \dots, x_m, y_1, \dots, y_n]/\left\langle \tilde{f}_i, \tilde{g}_j \right\rangle,$$
$$\sum_k F_k \otimes G_k \mapsto \sum_k F_k G_k,$$

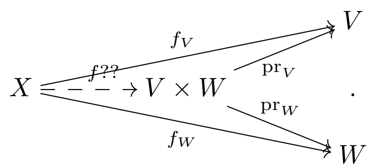
is injective and surjective. We use Universal Property instead. This means we want to establish a commutative diagram



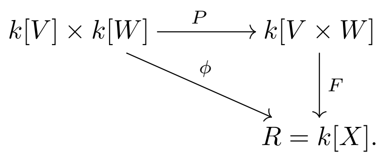
for every given k-algebra R and k-bilinear morphism $k[V] \times k[W] \to R$.

We first establish the k-bilinear morphism $P : k[V] \times k[W] \to k[V \times W]$. We can define an obvious morphism of affine algebraic sets $\text{pr}_V : V \times W \to V$, the usual projection, and let $\text{pr}_V^* : k[V] \to k[V \times W]$ be the corresponding k-algebra morphism. Similarly, we have $\text{pr}_W^* : k[W] \to k[V \times W]$. Putting the two morphisms together, we obtain a k-bilinear morphism $P = \text{pr}_V^* \times \text{pr}_W^* : k[V] \times k[W] \to k[V \times W]$.

Now given a k-bilinear morphism $\phi : k[V] \times k[W] \to R$. We first make a reduction: since the image of ϕ is finitely generated, by replacing R by this image, we can assume that R is finite generated. Hence R is a coordinate ring $k[X]$ for some affine algebraic set X. To establish the diagram above, we establish the dual diagram: given morphisms $f_V : X \to V$ and $f_W : X \to W$, we want to define $f : X \to V \times W$ such that we have the commutative diagram



Clearly, we can define $f : X \to V \times W$ by $x \mapsto (f_V(x), f_W(x))$. The corresponding k-algebra morphism $F : k[V \times W] \to k[X]$ is the one that fits in the diagram



This setup holds for every finitely generated k-algebra morphism R. By the Universal Property, $k[V \times W]$ must be isomorphic to $k[V] \otimes k[W]$. □

Remark: Some of you use the example I mentioned in class

$$R/I \otimes_R R/J \cong R/I + J.$$

Take $R = k[\mathbb{A}^{m+n}] = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$, $I = \left\langle \tilde{f}_i \right\rangle_i$ and $J = \langle \tilde{g}_j \rangle_j$, so that

$$R/I = k[x_1, \ldots, x_m, y_1, \ldots, y_n]/\left\langle \tilde{f}_i \right\rangle_i \cong k[x_1, \ldots, x_m]/\langle f_i \rangle_i = k[V]$$

(the isomorphism is given by assigning all y_j to 0),

$$R/J = k[x_1, \ldots, x_m, y_1, \ldots, y_n]/\langle \tilde{g}_j \rangle_j \cong k[y_1, \ldots, y_n]/\langle g_j \rangle_j = k[W],$$

and $R/I + J \cong k[V \times W]/\left\langle \tilde{f}_i, \tilde{g}_j \right\rangle_{i,j}$ by (a).

(2) Let R be a ring. Denote $R^m = R \times \cdots \times R$ (m times).

(a) Show that if a morphism $R^m \to R^n$ is surjective, then $m \geq n$. (3 marks) (Hint: Note that it is true if R is a vector space. How can we reduce this problem to a vector space problem? Hint: tensor product.)

Answer: Take a maximal ideal M of R, so that $K = R/M$ is a field. We apply the tensor product $- \otimes_R K$ to the morphism $R^m \to R^n$ and get

$$R^m \otimes_R K \to R^n \otimes_R K,$$

which is again surjective (as tensor product preserves subjectivity). Notice that, by the 'distributive law' of tensor product,

$$R^m \otimes_R K \cong (R \otimes_R K)^m \cong K^m,$$

so that the above morphism is a K-morphism $K^m \to K^n$. This is a K-linear morphism of vector spaces. If it is surjective, then by counting dimensions we have $m \geq n$. (This dimension counting property does not hold in general for R-modules, if R is not a field.) □

Remark: Some of you checked some of the basic properties like:

- $R \otimes_R R/I \cong R/I$, or in general $R \otimes_R M \cong M$,
- $(M \oplus N) \otimes_R P \cong (M \otimes_R P) \oplus (N \otimes_R P)$.

Checking these facts is a good practice for you to get more familiar with tensor product.

(b) If a morphism $R^m \to R^n$ is injective, is it always true that $m \leq n$? (?? marks, you are not required to submit this problem.)

Remarks:

i. You cannot use the same method as in the first part, because tensor product does not preserve injectivity.

ii. This is Q.11 in Ch.2 of Atiyah-MacDonald, and is perhaps one of the hardest problems in the book. Actually I couldn't prove it, and so far I cannot find anyone proved it: all the so-called 'solutions' in the internet are either false or contain some unexplainable vague arguments. If you think you have a convincing solution, please let me know.

iii. In fact, I believe the answer is false. If R is non-commutative, there is an example in Ex.13, p.190 of Hungerford's Algebra textbook (GTM 73). But I couldn't find an example for commutative R.

(3) Prove that "being integrally closed" (i.e. being integrally closed in its field of fractions) is a local property for an integral domain, in the sense of the following: given an integral domain R, prove that the following are equivalent.

(a) R is integrally closed (i.e. integrally closed in its field of fractions $K = K_R$);

(b) R_P is integrally closed for each prime ideal P of R;

(c) R_M is integrally closed for each maximal ideal M of R.

(3 marks) (Hint: Let S be the integral closure of R in K. Then consider the inclusion morphism $f : R \to S$.)

Answer: We first show that if S is the integral closure of R, then S_P is the integral closure of R_P. We already knew that S_P is integral over R_P, so it is enough to show that if $x \in K$ is integral over R_P, then $x \in S_P$. There is an equation for x as

$$x^n + (a_{n-1}/b_{n-1})x^{n-1} + \cdots + (a_1/b_1)x^1 + a_0/b_0 = 0,$$

where $a_i \in R$, $b_i \in R - P$. By clearing the denominator (with detail skipped), we can reduce the above equation to another one of the form

$$(ax)^n + c_{n-1}(ax)^{n-1} + \cdots + c_1(ax)^1 + c_0 = 0,$$

where $a \in R - P$, $c_i \in R$. By integrally we have $ax \in S$, and so $x = ax/a \in S_P$.

We therefore have

R is integrally closed

$\Leftrightarrow$ $R = S$

$\Leftrightarrow$ $R_P = S_P$, for all prime ideal P (by local property)

$\Leftrightarrow$ $R_P =$ the integral closore of R_P in K, for all prime ideal P (proved avove)

$\Leftrightarrow$ R_P is integrally closed, for all prime ideal P.

□

(4) (You may assume that k is algebraically closed.) Let V be an affine variety.

(a) Prove that the subset of singular points of V is a closed subset of V. (2 marks) (Hint: You may use the following fact: the rank r of an $m \times n$ matrix A is the maximal number satisfies the following: there exists a $r \times r$ sub-matrix B in A such that $\det(B) \neq 0$.)

Answer: Suppose that $V = Z(\langle f_1, \dots, f_m \rangle) \subseteq \mathbb{A}^n$. Recall that

$$\begin{aligned} & P \in V \text{ is a singular point} \\ \Leftrightarrow \quad & \text{the Jacobian matrix } A = \left[\frac{\partial f_i}{\partial x_j}(P)\right] \text{ has rank strictly smaller than } r = n - \dim V \\ \Leftrightarrow \quad & \text{all } r \times r \text{ sub-matrices } B \text{ in } A \text{ has } \det(B) = 0. \end{aligned}$$

Let $B_1, \dots, B_M$ be all $r \times r$ sub-matrices. For each B_j, let g_j be the polynomial $\det(B_j)$, then the set of singular points of V is given by $Z(\langle f_1, \dots, f_m, g_1, \dots, g_M \rangle)$ which is clearly a closed subset of V. □ □

(b) (Sard's Theorem) Suppose now V is a hypersurface, i.e. V is of the form $Z(f)$, a variety cut off by a single polynomial. Prove that the subset of singular points of V is a *proper* closed subset of V. (2 marks) (Note: you may have to distinguish the cases when the characteristic of k is 0 or a prime number.)

Answer: If $V = Z(\langle f \rangle)$, then the singular locus of V is given by $Z(\left\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right\rangle)$. To show that this is a proper subset of V, it is enough to show that at least one $\frac{\partial f}{\partial x_j} \notin \langle f \rangle$. We separate into two cases.

i. When $\text{char}(k) = 0$, if f is a non-constant polynomial in a variable x_j, then $\frac{\partial f}{\partial x_j}$ is a non-zero polynomial and has degree $\lneq \deg(f)$ (this degree is the highest power of x_j), but every non-zero polynomial in $\langle f \rangle$ has degree $\geq \deg(f)$. Therefore $\frac{\partial f}{\partial x_j}$ cannot lie in $\langle f \rangle$.

ii. When $\text{char}(k) > 0$, then it may happen that all $\frac{\partial f}{\partial x_j} \equiv 0$, in which case each term of f in x_j is a p-power, i.e. $f = \sum_{i_1,\dots,i_n} a_{i_1,\dots,i_n} x_1^{pi_1} \cdots x_n^{pi_n}$, where each coefficient $a_{i_1,\dots,i_n} \in k$. Since k is algebraically closed, each $a_{i_1,\dots,i_n}$ is a p-power, so write $a_{i_1,\dots,i_n} = b^p_{i_1,\dots,i_n}$ for some $b_{i_1,\dots,i_n} \in k$, then

$$f = \sum_{i_1,\dots,i_n} b^p_{i_1,\dots,i_n} x_1^{pi_1} \cdots x_n^{pi_n} = \left(\sum_{i_1,\dots,i_n} b_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n} \right)^p,$$

which contradicts that f is irreducible (remember that $V = Z(f)$ is a variety).

□

Remark: The Theorem still holds if Y is a general affine variety, but we need a special trick beyond this course to reduce it to the hypersurface case. See I. Theorem 5.3 in Hartshrone for detail.)

(5) (Dummit-Foote, Sec 15.4, Ex. 27) Recall the curve $V = Z(xz - y^2, yz - x^3, z^2 - x^2 y) \subset \mathbb{A}^3$ in HW2, Q.2(ii). Let $\varphi : \mathbb{A}^1 \to V$ be the map $t \mapsto (t^3, t^4, t^5)$.

(a) Describe the differential $d\varphi_t : \mathbb{T}_{t,\mathbb{A}^1} \to \mathbb{T}_{\varphi(t),V}$ explicitly, for each $t \in \mathbb{A}^1$. (2 marks)

Remark: The notation $d\varphi_t$ is the same as $D_t \varphi$ I used in class.

Answer: We checked in class that if $V \subseteq \mathbb{A}^m$, $W \subseteq \mathbb{A}^n$, and if $\varphi : V \to W$ is given by polynomials $\varphi = (\varphi_1, \dots, \varphi_n)$, then the differential map at $P \in V$ is given by the Jacobian matrix $\left[\frac{\partial \varphi_i}{\partial x_j}(P)\right]$. In the question, the differential map is just $(\frac{d}{dt} t^3, \frac{d}{dt} t^4, \frac{d}{dt} t^5) = (3t^2, 4t^3, 5t^4)$. □

(b) Prove that φ is *not* an isomorphism of affine algebraic sets. (2 marks)

Answer: We know that if φ is an isomorphism of affine algebraic sets, then $D_t\varphi$ is an isomorphism of k-vector spaces for each $t \in \mathbb{A}^1$, which means that $(3t^2, 4t^3, 5t^4)$ has to be non-zero for each $t \in \mathbb{A}^1$. However, at $t = 0$ we have $(3t^2, 4t^3, 5t^4) = (0, 0, 0)$. Therefore φ cannot be an isomorphism. □

(6) (Dummit-Foote, Sec 15.4, Ex. 28) If k is an algebraically closed field, the quotient $k[x]/\langle x^2\rangle$ is called *the ring of dual numbers over* k. If V is an affine algebraic set over k, prove that the set

$$\{k\text{-algebra morphism } k[V] \to k[x]/\langle x^2\rangle\}$$

is bijective to

$$\{(P, \mathbf{v}), \text{ where } P \in V \text{ and } \mathbf{v} \in \mathbb{T}_{P,V}\}.$$

(This means giving a k-algebra morphism $k[V] \to k[x]/\langle x^2\rangle$ is equivalent to specifying a point $P \in V$ with a tangent vector $\mathbf{v} \in \mathbb{T}_{P,V}$.) (3 marks)

Answer: Let $\mathcal{S}$ be the first set and $\mathcal{T}$ be the second set above. We first define a map $\mathcal{S} \to \mathcal{T}$. Given $\varphi \in \mathcal{S}$, which is a k-algebra morphism $\varphi : k[V] \to k[x]/\langle x^2\rangle$.

- Consider the natural surjective morphism $\pi_0 : k[x]/\langle x^2\rangle \to k$, $a + bx \mapsto a$ with kernel $\langle x\rangle$ and the composition
$$\varphi_0 : k[V] \xrightarrow{\varphi} k[x]/\langle x^2\rangle \xrightarrow{\pi_0} k,$$
which is a k-algebra morphism. The kernel of φ_0 is a maximal ideal, hence by Nullstellensatz it corresponds to a point $P \in V$. (1 mark)
- Since $k[x]/\langle x^2\rangle = k \oplus kx$ as a k-vector space, we define $\pi_1 : k \oplus kx \to k$, $a + bx \mapsto b$ and the composition
$$\varphi_1 : m_{P,V} \hookrightarrow k[V] \xrightarrow{\varphi} k[x]/\langle x^2\rangle = k \oplus kx \xrightarrow{\pi_1} k.$$
(Note that π_1 is a k-linear but not a k-algebra morphism.) As $\varphi(m_P) \subseteq \langle x\rangle$, we have $\varphi(m_P^2) \subseteq \langle x^2\rangle$. Hence the above composition induces a k-linear map $m_{P,V}/m_{P,V}^2 \cong m_P/(m_P^2 + I(V)) \to k$, which is an element in $\mathbb{T}_{P,V}$. (1 mark)

  We remark that ϕ being a k-algebra morphism is equivalent to ϕ_0 being a k-algebra morphism and ϕ_1 satisfying
$$\phi_1(fg) = \phi_0(f)\phi_1(g) + \phi_0(g)\phi_1(f).$$

We then show that the map $\mathcal{S} \to \mathcal{T}$ is bijective. The above construction implies that $\mathcal{S}$ is bijective to $\mathcal{U} = \{\phi_0, \phi_1\}$ satisfying the above properties. To show that $\mathcal{U}$ is bijective to $\mathcal{T}$, we can show that ϕ_0 is bijective to $P \in V$ by Nullstellensatz, and if given $\Phi : m_P/(m_P^2 + I(V)) \to k$, we define $\Phi_1(f) = \Phi(f - f(P)1)$ (where 1 is the constant function 1). We then check that Φ_1 satisfies the property of φ_1 above, i.e. to check

$$(*) \qquad \Phi_1(fg) = \Phi_0(f)\Phi_1(g) + \Phi_0(g)\Phi_1(f),$$

where we already know that $\Phi_0(f) = f(P)$. Notice that

$$\Phi_1(fg) = \Phi(fg - f(P)g(P)1) = \Phi((f - f(P)1)g) + \Phi((g - g(P)1)f(P)1).$$

(Note you cannot naively write $\Phi(fg - f(P)g(P)1) = \Phi(fg) - \Phi(f(P)g(P)1)$, as Φ is defined on m_P only but not on the whole $k[V]$. However, note that $f - f(P)1$, $g - g(P)1 \in m_P$.) After expansions and some cancelations, (*) is changed into

$$\Phi((f - f(P)1)(g - g(P)1)) = 0$$

which is true as $(f - f(P)1)(g - g(P)1) \in m_P^2 \subseteq \ker\Phi$. (1 mark) □