MATH 702, Winter 2015, Homework 4, Solutions

Total: 20 marks.

(1) (Atiyah-MacDoald, Ex. 3.20) I proved in class that

$$\varphi : R \to S \text{ is LO (Lying-Over) if and only if } \varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R) \text{ is surjective.}$$

Now given a ring morphism $\varphi : R \to S$ in general. We say that φ is LB if it satisfies:

$$\text{(LB)} \qquad \text{every prime ideal of } S \text{ is extended from a prime ideal of } R.$$

(a) Prove that

$$\varphi : R \to S \text{ is LB} \qquad \Rightarrow \qquad \varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R) \text{ is injective.}$$

(2 marks) (Hint: Denote by $e(\mathfrak{p})$ the extension of $\mathfrak{p}$ and $c(\mathfrak{q})$ the contraction of $\mathfrak{q}$, and think about them as maps between sets of ideals of R and S, then apply the general idea on saturation: ec is the identity map on the subsets of extended ideals.)

Answer: If $\varphi^*(\mathfrak{q}_1) = \varphi^*(\mathfrak{q}_2)$, then by definition $\varphi^{-1}\mathfrak{q}_1 = \varphi^{-1}\mathfrak{q}_2$. Think about this statement using contraction, we have $c(\mathfrak{q}_1) = c(\mathfrak{q}_2)$. Given (LB), we know that each $\mathfrak{q}_i$ is extended from a prime ideal $\mathfrak{p}_i$ of R, i.e. $e(p_i) = p_1 S = q_i$, so $c(\mathfrak{q}_1) = ce(\mathfrak{p}_1) = c(\mathfrak{q}_2) = ce(\mathfrak{p}_2)$. Apply e we have $ece(\mathfrak{p}_1) = ece(\mathfrak{p}_2)$. By the saturation property, this is equal to $e(\mathfrak{p}_1) = e(\mathfrak{p}_2)$, which is just $\mathfrak{q}_1 = \mathfrak{q}_2$. □

(Remark: I could not find a name of this fact in the literature. Perhaps we may call it "Lying Below Theorem".)

(b) Is the converse true? (1 mark) (Hint: It is enough to think of some rings, each of which has a one-point spectrum.)

Answer: No, for example $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$, $1 \mapsto 2$. The only prime ideals on both sides are 0 and $2\mathbb{Z}/4\mathbb{Z}$ respectively, and clearly $2\mathbb{Z}/4\mathbb{Z}$ cannot be extended from 0. □

Other example from students.

- $k \hookrightarrow k[x]/\langle x^2 \rangle$. The only prime ideals on both sides are 0 and $\langle x \rangle / \langle x^2 \rangle$ respectively.
- $R \to (R/p) \times K_R$, for a fixed prime ideal p.

(2) Let R be a ring.

(a) If $f \in R$, write down a canonical morphism $\mathrm{Spec}(R_f) \to \mathrm{Spec}(R)$ and show that the image is the basic open set

$$U_f = \{\mathfrak{p} \in \mathrm{Spec}(R), \text{ where } f \notin \mathfrak{p}\}.$$

(3 marks)

Answer: (The complete proof below is actually similar to Prop 38 on p.709 of the textbook. It is okay if you state some of the facts there, as we covered them in class.)

We denote by $i : R \to R_f$ the natural map $r \mapsto r/1$, then the corresponding map on spectrums is just the contraction map

$$Spec(R_f) \to Spec(R),\ c(q) = i^{-1}q.$$

We claim that the image is U_f defined above. First we must have $f \notin i^{-1}q$ for all $q \in Spec(R_f)$, otherwise if $f/1 \in e(i^{-1}q) = ec(q) = q$ (similar to the proof of Prop 38(1) on p.709, you may provide details). But $f/1$ is invertible in R_f, so q contains an invertible element, which is the whole ring R_f and is not a prime ideal.

Conversely, for every prime ideal $p \in Spec(R)$ with $f \notin p$, we show that it is coming from a $q \in Spec(R_f)$. Similar to above, we can show that the extended ideal $e(p)$ is a proper ideal in R_f (i.e., $e(p)$ cannot be the whole ring R_f). Moreover, $e(p)$ is a prime ideal (similar to the proof of Prop 38(3) on p.709, you may provide details). Therefore, $q = e(p) \in Spec(R_f)$ has contraction $p \in Spec(R)$.

Finally, we check that the contraction map is 1-1. Given $c(q) = c(q')$, then using $q = ec(q)$ above we obtain $q = ec(q) = ec(q') = q'$. □

(b) If $\mathfrak{p} \in \mathrm{Spec}(R)$, write down a canonical morphism $\mathrm{Spec}(R_\mathfrak{p}) \to \mathrm{Spec}(R)$ and show that the image is the intersection of all open neighborhood of $\mathfrak{p}$ in $\mathrm{Spec}(R)$. (2 marks)

Answer: We denote by $j : R \to R_p$ the natural map $r \mapsto r/1$, then the corresponding map on spectrums is just the contraction map

$$Spec(R_p) \to Spec(R),\ c(q) = j^{-1}q.$$

From the correspondence of prime ideals (Prop 38(3) on p.709), this map can be expressed as a bijection,

$$Spec(R_p) \xrightarrow{c} \{q \in Spec(R),\ q \cap (R - p) = \emptyset\}.$$

We then rewrite this as

$$\begin{aligned}
\mathrm{Spec}(R_\mathfrak{p}) &= \{q \in Spec(R),\ q \subseteq p\} \\
&= \{q \in Spec(R),\ f \notin q \text{ for all } f \in R - p\} \\
&= \bigcap_{f \in R - p} \{q \in Spec(R),\ f \notin q\} \\
&= \bigcap_{f \in R - p} U_f.
\end{aligned}$$

We know that $\{U_f\}_{f \in R-p}$ for a fundamental basis of open neighborhood of p, so (from basic topology) the above intersection is equal to the intersection of all open neighborhood of $\mathfrak{p}$. □

(3) Recall some basic notions.

(i) Let $\varphi : R \to S$ be a ring homomorphism. We say that φ has the *going-down property*, or that φ is GD, if it satisfies the following condition: Let $\mathfrak{p}_1 \supseteq \cdots \supseteq \mathfrak{p}_n$ be a chain of prime ideals of R and $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_m$ (with $m < n$) be a chain of prime ideals of S such that $\varphi^{-1}\mathfrak{q}_i = \mathfrak{p}_i$ for all $i = 1, \ldots, m$, then the chain $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_m$ can be extended to a chain $\mathfrak{q}_1 \supseteq \cdots \supseteq \mathfrak{q}_n$ of prime ideals of S such that $\varphi^{-1}\mathfrak{q}_i = \mathfrak{p}_i$ for all $i = 1, \ldots, n$.
For example, if φ is the inclusion map of $R \subseteq S$, and S is integral over R, then (with some extra conditions) φ is GD, by Theorem 26 of the textbook.

(ii) Let $\varphi : X \to Y$ be a continuous map of topological space. We say that φ is *open* if it maps every open subset of X onto a open subset of Y.

Prove that

$$\varphi^* : \text{Spec}(S) \to \text{Spec}(R) \text{ is open} \qquad \Rightarrow \qquad \varphi : R \to S \text{ is GD (Going-Down)}.$$

(5 marks) (Hint: Following Atiyah-MacDoald, Ex. 5.10, you may first try to show that

$$\begin{aligned} & \varphi^* : \text{Spec}(S_{\mathfrak{q}}) \to \text{Spec}(R_{\mathfrak{p}}) \text{ is surjective whenever } \varphi^{-1}\mathfrak{q} = \mathfrak{p} \\ \Rightarrow \qquad & \varphi : R \to S \text{ is GD (Going-Down)}. \end{aligned}$$

Then prove the another implication using Question 2(b).)

Answer: Notice that we regard $Spec(S_q)$ as a subset of $Spec(S)$ using Ex 2(b) above, so the notation $\varphi^* : \text{Spec}(S_{\mathfrak{q}}) \to \text{Spec}(R_{\mathfrak{p}})$ is the restriction of the map $\varphi^* : \text{Spec}(S) \to \text{Spec}(R)$.

We now prove the statement, following the hint. Given $\varphi^* : \text{Spec}(S_{\mathfrak{q}}) \to \text{Spec}(R_{\mathfrak{p}})$ is surjective if $\varphi^{-1}\mathfrak{q} = \mathfrak{p}$, and given $p_1 \subset p$, we take the prime ideal $p' = p_1 R_p \in \text{Spec}(R_{\mathfrak{p}})$, and take $q' \in \text{Spec}(S_{\mathfrak{q}})$ be a point in the preimage $(\varphi^*)^{-1}(\{p'\})$. Such a prime ideal q' must be of the form $q_1 S_q$ for some prime ideal $q_1 \subseteq q$, by Prop 38(3) of p.709. The fact that $\varphi^*(\{q'\}) = p'$ implies that $\varphi^{-1}q_1 = p_1$. Therefore $\varphi : R \to S$ is GD.

We then show that

$$\begin{aligned} & \varphi^* : \text{Spec}(S) \to \text{Spec}(R) \text{ is open} \\ \Rightarrow \qquad & \varphi^* : \text{Spec}(S_{\mathfrak{q}}) \to \text{Spec}(R_{\mathfrak{p}}) \text{ is surjective whenever } \varphi^{-1}\mathfrak{q} = \mathfrak{p}. \end{aligned}$$

The following proof is similar to the hint given in Atiyah-MacDoald. Remember from Ex.2(b) above that we can write $Spec(S_q) = \cap_{f \notin q} U_f$, where U_f is a basic open subset of $Spec(S)$. If we can show that

$$(*) \qquad \varphi^*(Spec(S_q)) = \bigcap_{f \notin q} \varphi^*(U_f),$$

then since $\varphi^*(U_f)$ is an open subset containing $\varphi^*(q) = p$, the right side of (*) is the intersection of a subcovering of the full covering of all open neighborhoods of p, which certainly contains the intersection of the full covering, which is $Spec(R_p)$.

It remains to prove (*). Many of you only stated this fact without proof. I assume that you are thinking this statement is trivial, but in fact it is not (only one direction is trivial from set-theory). However, by tracing back the exercises in Atiyah-MacDoald, I realize that the proof is too technical to be a 5-mark homework exercise. I try to rephrase the arguments to the level of our course and summarize them in the following proof. For the homework, I don't require you to give the full detail, but at least you have to mention that you realized there is some problem happened from (*).

We now prove (*). Remember that $Spec(S_q) = \cap_{f \notin q} U_f$, so the direction

$$\varphi^*(Spec(S_q)) \subset \bigcap_{f \notin q} \varphi^*(U_f)$$

is just coming from arguments of point-set. It remains to prove

$$\varphi^*(Spec(S_q)) \supset \bigcap_{f \notin q} \varphi^*(U_f).$$

The trick is to prove the contra-positive statement

$$p \notin \varphi^*(Spec(S_q)) \qquad \Rightarrow \qquad p \notin \bigcap_{f \notin q} \varphi^*(U_f), \text{ i.e., } p \notin \varphi^*(U_f) \text{ for some } f \notin q.$$

Notice that $p \notin \varphi^*(Spec(S_q))$ if and only if the preimage $(\varphi^*)^{-1}(\{p\}) \cap Spec(S_q) = \emptyset$. The trick to proceed is to use Ex 4(i) below: the preimage above is actually a spectrum, which is that of the ring $S_q \otimes_R (R_p/pR_p)$. We then apply the following easy fact which is not often used,

$$Spec(S_q \otimes_R (R_p/pR_p)) \text{ is an emptyset if and only if } S_q \otimes_R (R_p/pR_p) \text{ is a zero ring.}$$

Remember I showed in class that we can write S_q as a direct limit $S_q = \lim_{\rightarrow U_f \ni q} S_f$, where S_f is the localization of S at f. Now

$$\begin{aligned} & S_q \otimes_R (R_p/pR_p) = \lim_{\rightarrow U_f \ni q} S_f \otimes_R (R_p/pR_p) = 0 \\ \Rightarrow \quad & S_f \otimes_R (R_p/pR_p) = 0 \text{ for some } f \notin q, \end{aligned}$$

which implies (by using Ex 4(i) again) that $p \notin \varphi^*(U_f)$. □

(4) (i) Let $\varphi : R \to S$ be a ring morphism, and $\varphi^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be the corresponding map of schemes. If $\mathfrak{p} \in \mathrm{Spec}(R)$, take $D = R - \mathfrak{p}$ which is a multiplicatively closed subset and denote $S_{\mathfrak{p}} = D^{-1}S$, the localization of S at $\mathfrak{p}$ (if we view S as an R-module). Prove that the fibre of φ^* over $\mathfrak{p} \in \mathrm{Spec}(R)$, i.e., the preimage $(\varphi^*)^{-1}(\mathfrak{p})$ as a subset of $\mathrm{Spec}(S)$, is homeomorphic to

$$\mathrm{Spec}(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}) = \mathrm{Spec}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \otimes_R S).$$

(4 marks) (Hint: Define the morphisms in the commutative diagram

$$\begin{array}{ccc} S & \longrightarrow & S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \\ \uparrow & & \uparrow \\ R & \longrightarrow & R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \end{array} .$$

Take Spec and check how the point in $\mathrm{Spec}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ goes through the diagram.)

Answer: It is better to consider the diagram in two stages

$$\begin{array}{ccccc} S & \longrightarrow & S_p & \longrightarrow & S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \\ \uparrow & & \uparrow & & \uparrow \\ R & \longrightarrow & R_p & \longrightarrow & R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} \end{array}$$

and the corresponding diagram of spectrums

$$(**) \qquad \begin{array}{ccccc} Spec(S) & \xleftarrow[c]{\quad} & Spec(S_p) & \xleftarrow[c]{\quad} & Spec(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}) \\ \downarrow{\scriptstyle \varphi^*} & & \downarrow{\scriptstyle \varphi_p^*} & & \downarrow{\scriptstyle \bar{\varphi}_p^*} \\ Spec(R) & \xleftarrow[c]{\quad} & Spec(R_p) & \xleftarrow[c]{\quad} & Spec(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}) \end{array} .$$

We first show that

$$\text{the image of } (Spec(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}) \to Spec(S)) = \{q \in Spec(S) \text{ such that } \varphi^* q = \varphi^{-1} q = p\}.$$

Firstly, we know that

$$\text{the image of } (Spec(S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}) \to Spec(S_p)) = \{q' \in Spec(S_p) \text{ such that } q' \supseteq pS_p\}.$$

We know that the image of $Spec(S_p)$ in $Spec(S)$ consists of prime ideals q such that

$$q \cap \varphi(R - p) = \emptyset \qquad \Leftrightarrow \qquad \varphi^{-1}(q) \subseteq p.$$

Hence

$$\begin{aligned}&\text{the image of } Spec(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}) \to Spec(S)\\ =&\{q \in Spec(S) \text{ such that } \varphi^{-1}(q) \subseteq p \text{ and is of the form } c(q') \text{ where } q' \supseteq pS_p\}.\end{aligned}$$

But using the commutative diagram we have

$$\begin{aligned} q' \supseteq pS_p \quad &\Leftrightarrow \quad \varphi_p^*(q) \supseteq pR_p \\ &\Leftrightarrow \quad \varphi^*(\{q\}) = \varphi^*(\{c(q')\}) = c(\varphi_p^*(\{q'\})) \supseteq p. \end{aligned}$$

Therefore, we must have $\varphi^{-1}(q) = p$. Note that the two contractions $Spec(S) \overset{c}{\leftarrow} Spec(S_p) \overset{c}{\leftarrow} Spec(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$ are injective, so $Spec(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$ is bijective to its image in $Spec(S)$.

Finally, $Spec(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$ is homeomorphic to its image in $Spec(S)$, because for every closed subset $Z(\bar{J}_p)$ of $Spec(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$, where J is an ideal of S and $\bar{J}_p$ is the corresponding ideal in $S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p}$, we have

$$cc(Z(\bar{J}_p)) = Z(J) \cap \{q \in Spec(S),\ \varphi^{-1}q = p\}.$$

This can be shown using the correspondence of prime ideals in the commutative diagram. The idea is very similar to the arguments in the previous part of the proof, so I skip the details.

Sadly, almost none of you used the correspondence of prime ideals between $Spec(S_p/pS_p)$ and $Spec(S)$, but instead constructed the map $Spec(S_p/pS_p) \to Spec(S)$ directly. □

(ii) Now let $\varphi : \mathbb{Z} \to \mathbb{Z}[x]$ be the inclusion map, and $\langle f \rangle \in \mathrm{Spec}(\mathbb{Z}[x])$ where $f(x) = x^4 + 1$. For each prime number p, describe

$$\overline{\{\langle f \rangle\}} \cap (\varphi^*)^{-1}(\{\langle p \rangle\}),$$

i.e., describe the prime ideals in $\mathbb{Z}[x]$ lying over $p \in \mathbb{Z}$ and containing f. (3 marks) (Hint: Following the hint from the book, you may need to separate into three cases: $p = 2$, $p \equiv 1 \mod 8$, and $p \not\equiv 1 \mod 8$. More hint: The reason of considering $p \mod 8$ is related to the roots of X^4+1.)

Answer: The prime ideals of $\mathbb{Z}[x]$ containing $\langle f \rangle$ corresponds bijectively to the prime ideals in $S = \mathbb{Z}[x]/\langle f \rangle$, so those lying over the prime ideal $\langle p \rangle$ of $R = \mathbb{Z}$ are exactly the prime ideals in

$$S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R S = \frac{\mathbb{Z}/p\mathbb{Z})[x]}{\langle x^4+1 \rangle}.$$

Therefore, we switch the problem to check how $\frac{(\mathbb{Z}/p\mathbb{Z})[x]}{\langle x^4+1 \rangle}$ decomposes, or in other words how x^4+1 decomposes mod p. We separate into 3 cases.

i. When $p = 2$, we have $x^4+1 = x^4-1 = (x-1)^4 \mod 2$. Hence the only prime ideal in $\frac{(\mathbb{Z}/2\mathbb{Z})[x]}{\langle x-1 \rangle^4}$ is the one generated by $x - 1$.

ii. When $p \equiv 1 \mod 8$, notice that the roots of x^4+1 are primitive 8th roots of unity, so these four roots $z_1, \ldots, z_4$ all lie in $(\mathbb{Z}/p\mathbb{Z})^\times$. We have

$$\frac{(\mathbb{Z}/p\mathbb{Z})[x]}{\langle x^4+1 \rangle} \cong \frac{(\mathbb{Z}/p\mathbb{Z})[x]}{\langle x - z_1 \rangle} \oplus \cdots \oplus \frac{(\mathbb{Z}/p\mathbb{Z})[x]}{\langle x - z_4 \rangle},$$

and the prime ideals are $\langle x - z_1 \rangle, \cdots, \langle x - z_4 \rangle$. Notice that the right side is a direct sum of four fields, so its spectrum is a 4-point space.

iii. When $p \not\equiv 1 \mod 8$, then the four roots $z_1, \ldots, z_4$ do not lie in $(\mathbb{Z}/p\mathbb{Z})^\times$. However, since $p^2 \equiv 1 \mod 8$, the four roots lie in the quadratic extension $\mathbb{F}_{p^2}$ of $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. If we denote the four roots by $z_1, \bar{z}_1, z_2, \bar{z}_2$, where $\bar{z}_i$ is the $\mathbb{F}_{p^2}/\mathbb{F}_p$-conjugate of z_i, and denote $f_i \in (\mathbb{Z}/p\mathbb{Z})[x]$ the quadratic irreducible polynomial with roots $z_i, \bar{z}_i$, then

$$\frac{\mathbb{Z}/p\mathbb{Z})[x]}{\langle x^4+1 \rangle} \cong \frac{(\mathbb{Z}/p\mathbb{Z})[x]}{\langle f_1 \rangle} \oplus \frac{(\mathbb{Z}/p\mathbb{Z})[x]}{\langle f_2 \rangle},$$

and the two prime ideals are $\langle f_1 \rangle$ and $\langle f_2 \rangle$. Notice that the right side has a 2-point spectrum.

□