Insurance Contracts in Markets with Stochastic Volatility *or* Stochastic Interest Rates

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Toronto: the place to be in 2010 !

- Where are you gonna be in 2010 ?
- The Fields Institute is going to host a thematic program in Mathematical Finance
- Activities from January to July 2010 include: 3 major workshops, 4-6 weekend meetings, 3 graduate courses.
- Main themes are: *mathematical foundations*, *computational finance*, and *emerging applications*.
- Opportunities for long and short term visitors, postdocs, graduate students, etc.
- Contact the organizers: T. Hurd and MRG (McMaster), M. Rindesbacher (U of T), V. Henderson (Warwick), Y. Ait-Sahalia (Princeton).

Market Model

- We consider two factor stochastic volatility models of the form:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma(t, Y_t) S_t dW_t^1 \\
dY_t &= a(t, Y_t) dt + b(t, Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2]
\end{aligned}
$$

- Here μ and $-1 < \rho < 1$ are constants, $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ are deterministic functions, and W_t^1 and W_t^2 are independent one dimensional P–Brownian motions.
- In addition, we assume the existence of a risk-free bank account paying a constant interest rate $r = 0$.

Optimal Hedging and Investment

- We assume that, after selling an insurance contract B_T maturing at a future time T, the insurance company tries to solve the stochastic control problem

$$u^s(x, S, y, t) = \sup_{H \in \mathcal{A}} E^{x,S,y,t} \left[U \left(X_T^{H,x} - B_T \right) \right],$$

  where $X_t^{H,x}$ is value of a selffinancing portfolio with initial wealth x and H_t units of the stock, with the remaining value invested in the bank account.
- When $B = 0$, this reduces to the Merton problem:

$$u^0(x, y, t) = \sup_{H \in \mathcal{A}} E^{x,y,t} \left[U \left(X_T^{H,x} \right) \right]$$

Utility based pricing

- The sellers indifference price for the claim B is the solution π^s to the equation

$$u^0(x, y, t) = u^s(x + \pi^s(x, S, y, t), S, y, t)$$

- From now on, we consider an exponential utility function of the form:

$$U(x) = -e^{-\gamma x}, \quad \gamma > 0.$$

- We can then write

$$\begin{aligned} u^s(x, S, y, t) &= -e^{-\gamma x} G(S, y, t) = -e^{-\gamma x} e^{\phi(S,y,t)} \\ u^0(x, y, t) &= -e^{\gamma x} F(y, t) = -e^{-\gamma x} e^{\psi(y,t)} \end{aligned}$$

- The indifference price is then given by

$$\pi^s(S, y, t) = \frac{1}{\gamma} \log\left(\frac{G(S, y, t)}{F(y, t)}\right) = \frac{1}{\gamma}(\phi(S, y, t) - \psi(y, t)).$$

The solution to Merton's problem

- It is well-known that the power transformation $F(y,t) = f(y,t)^{1/1-\rho^2}$ leads to the linear equation

$$f_t + \left[a - \frac{b\rho\mu}{\sigma}\right] f_y + \frac{1}{2} b^2 f_{yy} = \frac{(1-\rho^2)\mu^2}{2\sigma^2} f,$$

  subject to $f(y,T) = 1$.
- Using Feynman–Kac, we obtain

$$f(t,y) = \widetilde{E}^{y,t}\left[e^{-\int_0^T \frac{(1-\rho^2)\mu^2}{2\sigma^2(s,Y_s)} ds}\right]$$

  where

$$dY_s = \left(a - \frac{b\mu\rho}{\sigma}\right) ds + b(\rho d\widetilde{W}_s^1 + \sqrt{1-\rho^2} d\widetilde{W}_s^2)$$

  with $d\widetilde{W}_t^1 = dW_t^1 + \frac{\mu}{\sigma} dt$ and $d\widetilde{W}_t^2 = dW_t^2$
- Therefore, the solution to Mertons problem can be calculated explicitly for a large class of processes Y_t.

Life insurance

- Consider now a claim of the form $B_T = \mathbf{1}_{\{\tau \leq T\}}$.
- Here τ is the arrival time of the first jump of an inhomogeneous Poisson process with intensity $\lambda(t)$, that is

$$P(\tau > t) = e^{-\int_0^t \lambda(s) ds}.$$

- Crucially, we assume that τ is independent of (W^1, W^2).
- In this case, we have

$$\begin{aligned}
u^s(x + \pi^s, S, y, t) &= \sup_{H \in \mathcal{A}} E\left[-e^{\gamma(x + \pi^s + \int_0^T H_s dS_s - B_T)}\right] \\
&= e^{-\gamma \pi^s} E\left[e^{\gamma B_T}\right] \sup_{H \in \mathcal{A}} E\left[-e^{\gamma(x + \int_0^T H_s dS_s)}\right] \\
&= e^{-\gamma \pi^s} E\left[e^{\gamma B_T}\right] u^0(x, S, y, t).
\end{aligned}$$

- Therefore, the indifference price in this case is given by

$$\pi^s = \frac{1}{\gamma} \log E\left[e^{\gamma B_T}\right].$$

Random horizon

- To obtain a nontrivial indifference price for contracts that are independent of the financial market, we need to consider the following modified problem: where

$$
\begin{aligned}
u^0(x, y, t) &= \sup_{H \in \mathcal{A}} E[U(X_{\tau \wedge T})] \\
&= \sup_{H \in \mathcal{A}} E\left[\int_0^\infty U(X_{\tau \wedge T}) d\Phi(t)\right] \\
&= E\left[U(X_T)(1 - \Phi(T)) + \int_0^T U(X_u) d\Phi(u)\right]
\end{aligned}
$$

$$
\Phi(t) = P[\tau \leq t] = 1 - e^{\int_0^t \lambda(s) ds}
$$

Solution to Merton's problem for random horizon

- Using dynamic programming, we find that the value function $u^0(x, y, t) = -e^{-\gamma x} F(y, t)$ for the random horizon satisfies the HJB equation

$$\begin{aligned} F_t \quad + \quad & \left[a - \frac{b\mu}{\sigma}\right] F_y + \frac{1}{2} b^2 F_{yy} \\ & - \left(\frac{\mu^2}{2\sigma^2} + \lambda(t)\right) F + \lambda(t) = \frac{1}{2} b^2 \rho^2 \frac{F_y^2}{F}, \end{aligned}$$

subject to $F(y, T) = e^{\int_0^T \lambda(t) dt}$.

- Unfortunately, the power transformation used before does not lead to a linear equation. To proceed, we take $\rho = 0$ and obtain

$$\begin{aligned} F(y, t) \quad = \quad & e^{-\int_0^T \lambda(s) ds} \widetilde{E}^{y,t} \left[e^{-\int_t^T \left(\frac{\mu^2}{2\sigma^2(s, Y_s)} + \lambda(s) \right) ds} \right] \\ & + \int_t^T \widetilde{E}^{y,t} \left[\lambda(s) e^{-\int_t^s \left(\frac{\mu^2}{2\sigma^2(u, Y_u)} + \lambda(u) \right) du} \right] ds \end{aligned}$$

Continuous life annuity - random horizon

- In the setting of the previous two slides, consider an insurance contract that pays a continuous annuity at a rate of 1 unit per period of time until $\tau \wedge T$.
- It turns out that the value function $u^s(x, y, t)$ in this case satisfies the same HJB equation satisfied by $u^0(x, y, t)$, expect for an extra term of the form $\gamma G(y, t)$.
- Therefore, still in the case $\rho = 0$, we have

$$
\begin{aligned}
G(y,t) \quad = \quad & e^{-\int_0^T \lambda(s)ds} \widetilde{E}^{y,t} \left[e^{-\int_t^T \left(\frac{\mu^2}{2\sigma^2(s,Y_s)} + \lambda(s) - \gamma \right) ds} \right] \\
& + \int_t^T \widetilde{E}^{y,t} \left[\lambda(s) e^{-\int_t^s \left(\frac{\mu^2}{2\sigma^2(u,Y_u)} + \lambda(u) - \gamma \right) du} \right] ds
\end{aligned}
$$

Equity–linked contracts

- For an indifference price that genuinely depends on the underlying market, consider the an insurance contract that pays $g(S_\tau, \tau)$ at time τ.
- In this case, inserting $u^s(x, S, y, t) = -e^{-\gamma x} e^{\phi(S,y,t)}$ into the corresponding HJB equation leads to

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 S^2 \phi_{SS} + \rho\sigma b S \phi_{yS} + \frac{1}{2} b^2 \phi_{yy} + \left(a - \frac{\mu b \rho}{\sigma}\right)\phi_y \\ +\frac{1}{2} b^2 (1-\rho^2)\phi_y^2 + \lambda(t)\left[e^{\gamma g + \psi - \phi} - 1\right] = \frac{\mu^2}{2\sigma^2} \\ \phi(y, S, T) = 0 \end{cases} \tag{1}$$

Fast-mean reversion asymptotics

- Let us now take

$$dY_t = \alpha(m - Y_t)dt + \beta(\rho dW_t + \sqrt{1-\rho^2}dZ_t)$$

and consider the regime $\frac{1}{\alpha} = \varepsilon << 1$, with $\beta = \sqrt{2\nu}/\sqrt{\varepsilon}$ where ν^2 is a fixed variance for the invariant distribution of Y_t.

- We then look for expansion of the form

$$\phi^{\varepsilon} = \phi^{(0)}(y, S, t) + \sqrt{\varepsilon}\phi^{(1)}(y, S, t) + \varepsilon\phi^{(2)}(y, S, t) + ...$$

Operators

- The previous PDE can be rewritten in compact notation as

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\phi + NL^{\phi} = \frac{\mu^2}{2\sigma^2} \qquad (2)$$

where $NL^{\phi} = \lambda(t)\left[e^{\gamma g+\psi-\phi} - 1\right] + \frac{\mu^2}{\varepsilon}(1-\rho^2)\phi_y^2.$

- Here

$$\mathcal{L}_0 = \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}$$

$$\mathcal{L}_1 = \sqrt{2}\rho\nu\left(\sigma(y)S\frac{\partial^2}{\partial y\partial S} - \frac{\mu}{\sigma(y)}\frac{\partial}{\partial y}\right)$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2}$$

Main result

- The insurer's indifference price satisfy:

$$|P(y,S,t) - P^{(0)}(S,t) - \widetilde{P1}(y,S,t)| = \mathcal{O}(\varepsilon) \qquad (3)$$

  where

$$\widetilde{P1}(y,S,t) = -(T-t)(V_3 S^3 P^{(0)}_{SSS} + V_2 S^2 P^{(0)}_{SS})$$

- Here $P^{(0)}$ satisfies

$$\begin{cases} P^{(0)}_t + \frac{1}{2}\sigma_\star^2 P^{(0)}_{SS} + \frac{\lambda(t)}{\gamma}\left[e^{\gamma(g-P^{(0)})} - 1\right] = 0 \\ P^{(0)}(S,T) = 0 \end{cases} \qquad (4)$$

  where $\sigma_\star^2 = \langle \sigma^2 \rangle$.

Stochastic Interest Rates

- Consider now

$$\begin{cases} dS_s = \gamma(r_s, S_s, s)S_s ds + \sigma_1(r_s, S_s, s)S_s dW_s^1 + \sigma_2(r_s, S_s, s)S_s dW_s^2 \\ S_t = S \end{cases}$$

- We model the short rate as

$$\begin{cases} dr_s = (a_0(s)r_s + b_0(s))ds + \sqrt{c(s)r_s + d(s)}dW_s^1 \\ r_t = r \end{cases}$$

- We consider path–dependent claims of the form

$$C(S_s, r_s, s) = \int_t^\tau c_1(S_s, r_s, s)ds + c_2(S_\tau, r_\tau, \tau), \quad t \leq s \leq \tau \leq T$$

The pricing equation

- Therefore, P satisfies the following nonlinear PDE:

$$\begin{cases} P_t + (ar+b)P_r + \frac{1}{2}(cr+d)P_{rr} + \sqrt{cr+d}\sigma_1 S P_{Sr} \\ +\frac{1}{2}(\sigma_1^2+\sigma_2^2)S^2 P_{SS} - \frac{1}{2}\left(\theta^2 + \frac{(\gamma-\theta\sigma_1)^2}{\sigma_2^2}\right) \\ -\frac{\lambda(t)}{\gamma}\left(1 - e^{\gamma c_2 e^{-\int_0^t r(s)ds} - \gamma P}\right) = 0 \\ P(S,r,T) = 0 \end{cases}$$

- Here λ is the market price of risk for the bond market and has the form

$$\lambda(r_s, s) = \frac{(a_0(s) - a(s))r_s + (b_0(s) - b(s))}{\sqrt{c(s)r_s + d(s)}} \tag{5}$$

for some deterministic functions a and b.