The macrodynamics of household debt, growth, and inequality

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Abstract

How do inequality and growth evolve in the long run and why? We address this question by analyzing the interplay between household debt, growth and inequality within a monetary, stock-flow consistent framework. We first consider a Goodwin-Keen model where household consumption, rather than investment by firms, is the key behavioural driver for the dynamics of the economy. Whenever consumption exceeds current income, households can borrow from the banking sector. The resulting three dimensional dynamical system for wage share, employment rate, and household debt exhibits the characteristic asymptotic equilibria of the original Keen model, namely the analogue of Solovian balanced growth path with a stable NAIRU in addition to deflationary equilibria with explosive debt and collapsing employment. We then extend this set-up by separating the household sector into workers and investors, obtaining a four-dimensional system with analogous types of asymptotic behaviour. Our main result is that long-run increasing inequality between these two classes of households occurs if and only if the system approaches one of the equilibria with unbounded debt ratios. More specifically, we find that one essential channel of increased inequality is the wealth transfer from workers to investors due to interest paid on debt from the former to the latter. Finally, when properly rewritten, the celebrated inequality $r > g$ turns out to be a necessary condition for the asymptotic stability of long-run debt-deflation. Our findings shed new light on the relationships between fairness and efficiency, and have implications for public economic policy.

JEL codes: C61, E20, E32, D63.

Keywords: Stock-flow consistency, Goodwin, Keen, household debt, NAIRU, inequality, stagnation.

1 Introduction

This article addresses what is arguably one of the most basic economic questions: how do inequality and growth evolve in the long run and what are the determinants of this evolution?

In his influential book Piketty (2014), Piketty uses an extensive dataset to document the marked rise in income and wealth inequality observed in most of the developed world since the 1980s, as measured by the increase in the share of income and wealth in the top quantiles of their

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corresponding distributions. Although the patterns of capital accumulation and accompanying inequality discussed in the book are unequivocally present in the data, the theoretical underpinnings used by Piketty to extrapolate these data regularities for the distant future have been fiercely contested (for example, see Acemoglu and Robinson (2015) and Stiglitz (2015)). In a parallel development, Summers (see for example Summers (2016)) rekindled the interest of the economic profession in the theory of secular stagnation first proposed in Hansen (1939) (see also Gordon (2014)). Although Summers (2016) mentions inequality as one of many factors contributing to secular stagnation, the explicit mechanism connecting them is not discussed. The missing link can be found, however, in the work of Steindl and his followers. In Steindl (1952), it is posited that monopoly power allows firms to increase their markup, leading to an increase of the share of profits, and therefore savings, while at the same time having less pressure to invest as they enjoy captive markets and reduced competition. In later work, Steindl takes into account the distribution of income between workers and capitalists, rather then just wages and profits (see Dutt (2006) and references therein). The model in Dutt (2006) builds on this Steindlian analysis by explicitly considering the effects of growing levels of consumer debt observed since the 1950s, a phenomenon conspicuously unexplored by both Piketty and Summers. The conclusion in Dutt (2006) is that consumer debt is expansionary at first, as borrowing provides an additional source of income for workers, but might lead to stagnation in the long run, as interest payments are a transfer of income from workers, who have higher propensity to spend, to capitalists, who tend to spend less, similarly to a result also obtained in Palley (1994).

In this paper we address the link between inequality and long term economic growth but using Goodwin (1967), Keen (1995) and Akerlof and Stiglitz (1969) as starting points instead. We extend the Lotka-Volterra dynamics between employment and wages introduced in the papers just cited in three ways. First, we consider in Section 2 a stock-flow consistent, monetary, dual version of the standard Keen model, where households borrow money to finance their expenditures, while firms adjust their production so as to satisfy aggregate demand. We recover results analogous to those obtained in the original model, namely there exist essentially two long-run equilibria: an interior one corresponding to finite debt-ratio and nonzero wage share and employment rate and an explosive one, characterized by an infinite debt-to-income ratio and collapsing wages and employment.

Second, we extend this dual model in Section 3 by dividing the household sector into workers, whose income arises solely from wages, and investors, whose income arises solely from capital and the profits of the banking system. Our main result is that income inequality explodes in the neighbourhood of the explosive equilibrium, while it stabilizes around the interior one. As in Dutt (2006) and Palley (1994), this allows us to analyze inequality beyond the conventional trade-off between efficiency and equity. According to the latter, indeed, efficiency is concerned with the issue of increasing the size of the economic pie, while equity or fairness deals with its distribution. In case of conflict, which should have priority? A familiar argument in favour of efficiency goes as follows: it might well be the case that emphasizing the growth of the pie's size leads those with lower income to a better state than by focusing on fairness, as their absolute income might be larger in the first case than in the second. We challenge this traditional wisdom by showing that the trade-off between efficiency and equity does not necessarily hold: in our set-up, increasing inequality is an unequivocal characterization of paths leading to the collapse, hence to a radical shrink of the pie. Fighting against inequality improves long-run efficiency.

The third contribution of this paper consists in a reinterpretation of the relationship (in real terms) $r > g$ that has been recently emphasized by Piketty (2014). As shown in Acemoglu and Robinson (2015), and contrary to Piketty's claim, this inequality does not imply *per se* a rise in inequality. On the other hand, Piketty conflates in the variable r two parameters: the interest rate on debt and the return on capital. The recent empirical evolution of these two variables forces us to disentangle these two interpretations of r. Taking this distinction into account, we show that, in our set-up, $r > g + i$ (where r is the nominal interest rate and i is inflation, as we deal with a monetary setting) is a necessary condition for the local stability of the explosive equilibrium. Therefore, an economy may converge towards the interior equilibrium, hence stabilizing inequality, even though $r > g+i$ is satisfied (in accordance with the result in Acemoglu and Robinson (2015)).

But, if $r \leq g + i$, the bad equilibrium is no longer locally stable, which considerably reduces its chance to be reached.

Section 4 summarizes the main results and discusses ways in which our framework can be extended in future work.

2 The dual Keen model

We first set the scene by considering an economy with a single type of households, who can borrow money from the banking sector to fulfill their consumption plans, and firms that adjust their investment so that total output matches aggregate demand.

2.1 Preliminaries

On the production side, we adopt the same Leontieff function with total capital utilization adopted in the Goodwin (1967) and Keen (1995) models, namely

$$Y = \frac{K}{\nu} = a\ell, \tag{1}$$

where Y and K denote, respectively, output and capital in real terms, $\nu > 0$ is a constant capital-output ratio, $a \geq 0$ is productivity per worker and ℓ is the number of employed workers. Capital accumulates according to

$$\dot{K} = I - \delta K, \tag{2}$$

where I is real gross investment and $\delta \geq 0$ is a constant depreciation rate. It follows that the growth rates of the economy and of capital are the same and given by

$$\frac{\dot{Y}}{Y} = \frac{\dot{K}}{K} = \frac{\kappa(\cdot)}{\nu} - \delta, \tag{3}$$

where $\kappa(\cdot) := I/Y$ is a real-valued function determining investment per unit of output, whose argument we leave unspecified for now. We denote the wage rate by $\mathrm{w} = W/\ell$, where W is the total nominal wage bill. Consequently, the wage share of nominal output is given

$$\omega := \frac{W}{pY} = \frac{\mathrm{w}}{pa}.$$

If N denotes total workforce, which we assume to be exponentially growing at rate $\beta > 0$, then $\lambda = \ell/N$ is the employment rate and evolves according to:

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} = \frac{\kappa(\cdot)}{\nu} - \delta - \alpha - \beta. \tag{4}$$

The evolution of the wage rate w is provided by a bargaining equation of the form

$$\frac{\dot{\mathrm{w}}}{\mathrm{w}} = \Phi(\lambda) + \gamma i(\omega), \tag{5}$$

where $\Phi : [0, 1) \to \mathbb{R}$ is the short-run Phillips curve (see, e.g., Gordon (2011), Gordon (2013), Gregory (2001), and Gregory (2014)) and $\gamma \in [0, 1]$ is a parameter measuring the extent to which inflation $i(\omega)$ is incorporated in the bargaining for nominal wages, with $\gamma = 1$ corresponding to no money illusion, in which case (5) is equivalent to the bargaining in terms of real wages assumed in Goodwin (1967) and Akerlof and Stiglitz (1969). Here inflation is given by

$$i(\omega) := \frac{\dot{p}}{p} = \eta_p(m\omega - 1), \tag{6}$$

where $m \geq 1$ is target markup towards which the price level p adjusts in the (imperfectly competitive) goods market with a relaxation time is $1/\eta_p$. Labour productivity $a = Y/\ell$ is assumed to grow exponentially at an exogenous rate $\alpha \geq 0$, so that the dynamics of the wage share is given by

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot{a}}{a} - \frac{\dot{p}}{p} = \Phi(\lambda) - \alpha - (1 - \gamma)i(\omega). \tag{7}$$

2.2 Accounting structure

The structure of the three-sector economy considered in this section is the same as in the Keen (1995) model and is described in Table 1.

We assume that firms and banks are privately owned by a subset of the households, so that the assets of the aggregate household sector consist of firm equity E_f, bank equity E_b, and deposits M_h, whereas their liabilities consist of loans L_h, resulting in net worth (i.e., wealth) equal to $X_h = E + M_h - L_h$, where $E := E_f + E_b$. The assets of the aggregate firm sector are the capital stock pK and deposits M_f, whereas their liabilities are the loans L_f and equity $E_f = pK + M_f - L_f$, here treated as a balancing variable so that the net worth of firms is $X_f = 0$. Similarly, the loans are the assets of the banking sector, and its liabilities are the deposits, whereas bank equity, also treated as a balancing variable, is $E_b = (L_h + L_f) - (M_h + M_f)$. Hence, the net worth of banks is $X_b = 0$. We therefore obtain that

$$\begin{aligned} X = X_h &= E_f + E_b + M_h - L_h \\ &= pK + M_f - L_f + (L_h + L_f) - (M_h + M_f) + M_h - L_h \\ &= pK, \end{aligned}$$

that is to say, the total wealth in the economy equals the wealth of the households, which in turn corresponds to the non financial assets of the firm sector. Observe that we do not consider the case where firms and banks are owned by shareholders through publicly traded stocks. In particular, firms and banks in this model do not issue or buy back any shares.

The budget constraint for the household sector implies that whenever nominal household consumption exceeds household disposable income, the difference needs to be financed by an increased in household net debt $D_h = L_h - M_h$. Conversely, if household disposable income exceeds nominal household consumption, the difference, which consists of household savings S_h, is used to decrease household net debt. In other words, we have that

$$\dot{D}_h = \dot{L}_h - \dot{M}_h = -S_h = pC_h - (\mathrm{w}\ell - rD_h + \Delta_b). \tag{8}$$

Here, C_h denotes real household consumption, whereas household disposable income consists of $(\mathrm{w}\ell - rD_h + \Delta_b)$, where $r > 0$ is a constant nominal short-run interest rate paid to banks on net debt D_h and Δ_b denotes dividends received from banks, which are assumed to be privately owned by a subset of households.

As in the original Keen (1995) model, we assume in this section that firms retain all their profits in order to finance investment. That is why Table 1 does not have a row representing dividends paid by firms. When the amount to be invested exceeds profits, then firms finance the difference by increasing their loans from the banking sector. Conversely, when profits exceed investment, the difference is used to either repay existing loans or accumulate deposits. Denoting the net debt of firms by $D_f := L_f - M_f$, we see from Table 1 that net profits for firms, after paying wages, interest on net debt, and accounting for depreciation (i.e., consumption of fixed capital) are given by

$$\Pi = pY - \mathrm{w}\ell - rD_f - p\delta K, \tag{9}$$

where we assumed, for simplicity, that firms pay the same constant interest rate r on net debt D_f as households. In the absence of distributed profits paid to shareholders in the form of dividends, the savings of the firm sector are given by $S_f = \Pi$. It therefore follows from the budget constraint of the firm sector that

$$\dot{D}_f = p(I - \delta K) - S_f = pI - (pY - \mathrm{w}\ell - rD_f). \tag{10}$$

Finally, savings of the banking sector are given by

$$S_b = r(L_h + L_f) - r(M_h + M_f) - \Delta_b - pC_b = r(D_f + D_h) - \Delta_b - pC_b, \tag{11}$$

where C_b denotes real consumption by banks. Using the fact that

$$Y = C + I = (C_h + C_b) + I, \tag{12}$$

	Households	Firms		Banks	Row Sum
Balance sheet					
Capital stock		$+pK$			pK
Deposits	$+M_h$	$+M_f$		$-M$	0
Loans	$-L_h$	$-L_f$		L	0
Equity	E	$-E_f$		$-E_b$	0
Column sum (Net worth)	X_h	0		0	pK
Transactions		Current	Capital		
Consumption	$-pC_h$	$+pC$		$-pC_b$	0
Investment		$+pI$	$-pI$		0
Accounting memo [GDP]		$[pY]$			
Depreciation		$-p\delta K$	$+p\delta K$		0
Wages	$+\mathrm{w}\ell$	$-\mathrm{w}\ell$			0
Interest on loans	$-rL_h$	$-rL_f$		$+rL$	0
Interest on deposits	$+rM_h$	$+rM_f$		$-rM$	0
Bank dividends	$+\Delta_b$			$-\Delta_b$	0
Column sum (balances)	S_h	S_f	$-pI+p\delta K$	S_b	0
Flows of funds					
Change in capital stock		$+p\dot{K}$			$p\dot{K}$
Change in deposits	$+\dot{M}_h$	$+\dot{M}_f$		$-\dot{M}$	0
Change in loans	$-\dot{L}_h$	$-\dot{L}_f$		$+\dot{L}$	0
Column sum (savings)	S_h	S_f		S_b	$p\dot{K}$
Change in firm equity	$+\dot{E}_f$	$-(S_f+\dot{p}K)$			0
Change in bank equity	$+\dot{E}_b$			$-S_b$	0
Change in net worth	$\dot{E}+S_h$	0		0	$\dot{p}K+p\dot{K}$

Table 1: Balance sheet, transactions and flow of funds for a three-sector economy.

we can then verify that

$$\begin{aligned} S_h+S_f+S_b = {} & (\mathrm{w}\ell - rD_h+\Delta_b-pC_h)+(pY-\mathrm{w}\ell-rD_f-p\delta K) && (13)\\ & +r(D_f+D_h)-\Delta_b-pC_b = p(I-\delta K), && (14)\end{aligned}$$

so that savings always equal net investment in the economy.

2.3 Aggregate behavioural rules

We briefly recall that the original Keen (1995) model is based on the assumption that investment is given by

$$I=\kappa(\pi)Y, \tag{15}$$

where $\kappa:\mathbb{R}\to\mathbb{R}$ is an increasing function of the pre-depreciation profit share

$$\pi=\frac{pY-\mathrm{w}\ell-rD_f}{pY}=1-\omega-rd_f, \qquad d_f=\frac{D_f}{pY}. \tag{16}$$

Using (3) we find that

$$\frac{\dot{Y}}{Y}=\frac{\kappa(1-\omega-rd_f)}{\nu}-\delta, \tag{17}$$

from which we obtain that

$$\begin{aligned}\frac{\dot{d}_f}{d_f} &= \frac{\dot{D}_f}{D_f} - \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} \\ &= \frac{\kappa(1-\omega-rd_f)-(1-\omega-rd_f)}{d_f} - \frac{\kappa(1-\omega-rd_f)}{\nu} + \delta - i(\omega) \\ &= \left[r - \frac{\kappa(1-\omega-rd_f)}{\nu} + \delta - i(\omega)\right] + \frac{\kappa(1-\omega-rd_f)-(1-\omega)}{d_f} \end{aligned} \tag{18}$$

The dynamical system consisting of (7), (4) and (18) then corresponds to what we call the primal Keen model, which is driven by investment decisions and has the debt ratio of firms as its key state variable, in addition to the wage share and employment. The primal model implicitly assumes that total consumption adjusts to investment according to

$$C = C_h + C_b = Y - I = (1-\kappa(1-\omega-rd_f))Y. \tag{19}$$

We can then see that several alternative specifications of bank and household behaviour are compatible with the general structure of the model. For example, one can assume that

$$\Delta_b = r(D_h + D_f), \quad C_b = 0, \quad C_h = C, \tag{20}$$

that is, all bank profits are paid to households in the form of dividends and there is no consumption by banks, in which case $S_b = 0$ and the equity of banks remain constant. Moreover, it follows from (8) and (10) that

$$\begin{aligned}\dot{D}_h &= pC_h - \mathrm{w}\ell + rD_h - r(D_h + D_f) \\ &= pY - pI - \mathrm{w}\ell - rD_f = -\dot{D}_f. \end{aligned} \tag{21}$$

Another alternative with $S_b = 0$ consists of setting

$$\Delta_b = 0, \quad C_b = r(D_h + D_f), \quad C_h = C - r(D_h + D_f), \tag{22}$$

that is, banks retain and consume all profits, in which case it is easy to see that (21) still holds. For either (20) or (22), or any other specification of bank and household behaviour leading to $S_b = 0$, we see that the equity E_b of the banking sector remains constant. It should be clear that more general specifications of Δ_b, C_b and C_h would also be compatible with the primal Keen model, provided that total consumption satisfies (19).

We are now ready to complete the specification of the dual Keen model. It consists of assuming that (20) holds and that, in addition, total consumption is given by

$$C = C_h := c(\omega - rd_h)Y, \tag{23}$$

where $c : \mathbb{R} \to \mathbb{R}_+$ is an increasing function determining the household consumption to output ratio and $d_h = D_h/(pY)$ is the ratio of net debt of households to output. Observe that the disposable income of the aggregate household sector consists of

$$\mathrm{w}\ell - rD_h + \Delta_b = \mathrm{w}\ell + rD_f = \mathrm{w}\ell - rD_h + rE_b, \tag{24}$$

where we have used that $E_b = D_h + D_f$. Consequently, the variable $(\omega - rd_h)$ measures the share of disposable income of households, apart from the constant rE_b.

Notice that we do not strive for micro-foundations of the aggregate consumption function, $C(\cdot)$. This is consistent with the celebrated Sonnenschein-Mantel-Debreu theorem (see, e.g. Debreu (1974)) which essentially says that "everything is possible" at the aggregate level, even though individual households might conform to the standard utility-maximization programme under some intertemporal budget constraint with rational expectations. Thus we content ourselves with taking

$C(\cdot)$ as given. For practical applications, $C(\cdot)$ should be empirically estimated as, e.g., in Bastidas et al. (017a) and Bastidas et al. (017b).

Differently from the primal Keen model, we now assume that investment is the adjusting variable instead, so that the capital accumulation dynamic now reads

$$\dot{K} = Y - C - \delta K = (1 - c(\omega - rd_h))Y - \delta K. \tag{25}$$

Combining (25) and (1), we obtain that the growth rate of real output is

$$g(\omega, d) := \frac{\dot{Y}}{Y} = \frac{1 - c(\omega - rd_h)}{\nu} - \delta. \tag{26}$$

We then have the following dynamics for the household debt ratio $d_h = D_h/pY$:

$$\begin{aligned}
\frac{\dot{d}_h}{d_h} &= \frac{\dot{D}_h}{D_h} - \frac{\dot{Y}}{Y} - \frac{\dot{p}}{p} = \frac{\dot{D}_h}{D_h} - g(\omega, d) - i(\omega) \\
&= \left[r - \frac{1 - c(\omega - rd_h)}{\nu} + \delta - i(\omega)\right] + \frac{c(\omega - rd_h) - \omega - re_b}{d_h},
\end{aligned} \tag{27}$$

where $e_b = E_b/(pY)$. To summarize our set-up so far, we end up with the following 3-dimensional non-linear system:

$$\begin{array}{rcl}
\dot{\omega} & = & \omega[\Phi(\lambda) - \alpha - (1 - \gamma)i(\omega)] \\
\dot{\lambda} & = & \lambda\left[\frac{1 - c(\omega - rd_h)}{\nu} - (\alpha + \beta + \delta)\right] \\
\dot{d}_h & = & d_h\left[r - \frac{1 - c(\omega - rd_h)}{\nu} + \delta - i(\omega)\right] + c\big(\omega - rd_h\big) - \omega - re_b.
\end{array} \tag{28}$$

2.4 Long-run equilibria

Being the dual to Keen's seminal model, our model exhibits a similar pattern of dynamics. Let us make the following set of assumptions similar to Grasselli and Nguyen Huu (2015).

Assumption 2.1. 1. The consumption function $c : \mathbb{R} \to \mathbb{R}_+$ is $\mathcal{C}^1$, strictly increasing over $\mathbb{R}_+$, and verifies

(i) $y \leq 0 \Rightarrow c(y) = c_- > 0$

(ii) $\lim_{z \to +\infty} c(y) := c_+ \leq 1$.

(iii) $c_- < 1 - \nu(\alpha + \beta + \delta) < c_+$

2. The Philips curve $\Phi : [0, 1) \to \mathbb{R}$ is $\mathcal{C}^1$, strictly increasing and admits a vertical asymptote at $\lambda = 1$. Moreover, $\Phi'(0) = 0$ and

$$\Phi(0) < \min\{\alpha, \gamma(\alpha + \beta) - \beta\} \tag{29}$$

The first condition on the consumption function expresses the fact that households need a minimum level of subsistence consumption even at negative income, whereas the second condition corresponds to the fact that total consumption cannot exceed total output. The third condition, on the other hand, is related to the existence of an interior equilibrium.

We first verify that the dynamical system (28) admits a class of trivial equilibria of the form $(\overline{\omega}_0, \overline{\lambda}_0, \overline{d}_0) = (0, 0, \overline{d}_0)$ for arbitrary $\overline{d}_0 \geq 0$, provided that the identity

$$r + \delta - \frac{1 - c_-}{\nu} + \eta_p = 0$$

is satisfied (recall that $c(-r\overline{d}_0) = c_-$). Since this is structurally unstable, we can disregard this family of equilibria.

2.4.1 The balanced-growth path

We now consider the existence of an interior equilibrium for (28). Begin by defining $\eta_1 := c^{-1}\big(1-\nu(\alpha+\beta+\delta)\big)$, which exists because of condition (a)-(ii) in Assumption 2.1, and take $\overline{\omega}_1 - r\overline{d}_1 = \eta_1$ so that $\dot{\lambda} = 0$ in the second equation of (28). It follows that the growth rate of real output (26) at this equilibrium is

$$g(\overline{\omega}_1, \overline{d}_1) = \frac{1 - c(\overline{\omega}_1 - r\overline{d}_1)}{\nu} - \delta = \alpha + \beta, \tag{30}$$

so that $e_b = E_b/(pY) \to 0$, since E_b is constant[1]. Since $g(\overline{\omega}_1, \overline{d}_1)$ also equals the growth rate of capital (recall (3)), it follows that an equilibrium with this growth rate describes the analog of the balanced-growth path in Solow (1956), albeit with different stability properties. In fact, our whole setup may be seen as an extension of Solow's capital accumulation dynamics, obtained by adding debt and a Phillips curve, together with a Leontieff production function. Using (30), we can now verify that

$$\overline{\omega}_1 = \eta_1 + r\Big[\frac{1-\nu(\alpha+\beta+\delta)-\eta_1}{\alpha+\beta+i(\overline{\omega}_1)}\Big]. \tag{31}$$

$$\overline{\lambda}_1 = \Phi^{-1}\big(\alpha + (1-\gamma)i(\overline{\omega}_1)\big). \tag{32}$$

$$\overline{d}_1 = \frac{1-\nu(\alpha+\beta+\delta)-\eta_1}{\alpha+\beta+i(\overline{\omega}_1)} \tag{33}$$

is an interior equilibrium for (28).

Lemma 2.1. *A sufficient condition for the existence of $\overline{\lambda}_1 > 0$ in (32) is a strictly positive nominal growth rate, that is, $\alpha+\beta+i(\overline{\omega}_1) > 0$. Moreover, in this case, we have that $\overline{d}_1 \geq 0$ if, and only if, $c(\overline{\omega}_1 - r\overline{d}_1) \geq \overline{\omega}_1 - r\overline{d}_1$.*

Proof. If $i(\overline{\omega}_1) \geq 0$, then the inequality $\Phi(0) < \alpha \leq \alpha + (1-\gamma)i(\overline{\omega}_1)$ implies that (32) admits a solution $\overline{\lambda}_1 > 0$. On the other hand, if $i(\overline{\omega}_1) < 0$, then (29) implies that

$$\alpha + (1-\gamma)i(\overline{\omega}_1) > \gamma(\alpha+\beta) - \beta > \Phi(0),$$

provided $\alpha+\beta+i(\overline{\omega}_1) > 0$, from which it follows again that $\overline{\lambda}_1 > 0$ exists. The second statement in the lemma, namely that equilibrium household debt is positive if, and only if, equilibrium consumption exceeds equilibrium disposable income, follows directly from the facts that $c(\overline{\omega}_1 - r\overline{d}_1) = 1 - \nu(\alpha+\beta+\delta)$ and $\overline{\omega}_1 - r\overline{d}_1 = \eta_1$. □

Observe that $i(\overline{\omega}_1) = \eta_p(m\overline{\omega}_1 - 1)$, so (31) is a quadratic equation. In what follows, we assume that this equation has at least one solution with $\overline{\omega}_1 > 0$. As in Grasselli and Nguyen-Huu (2015), whenever the markup m is relatively high, namely $m \geq 1/\overline{\omega}_1$, the economy is asymptotically inflationary and we observe a trade-off between long-run inflation and employment according to (32). Notice that this holds even when $r = 0$. Indeed, according to (31), the latter simply means that a change in inflation does not affect the wage share $\overline{\omega}_1$.

Regarding the speed of convergence towards the interior equilibrium, the eigenvalues of the Jacobian matrix of system (28) allow us to provide a "quick and dirty" estimation. In the vicinity of the long-run steady state, indeed, and considering, for example, the rate λ of employment, a Taylor expansion yields:

$$\lambda(t) = \overline{\lambda}_1 + e^{-\varepsilon_\lambda t}(\lambda(0) - \overline{\lambda}_1). \tag{34}$$

Since $\varepsilon_\lambda = \frac{1-c}{\nu} - (\alpha+\beta+\delta)$ (see Appendix A1), we can calibrate (34) to check how quickly actual economies are likely to approach this balanced-growth path, provided they are already

[1] Observe that, had we assumed that banks pay proportional dividends $\Delta_b = \delta_b(D_f + D_h)$, in which case the equity E_b grows at constant rate $(r - \delta_b)$, then $e_b = E_b/(pY) \to 0$ would still hold provided $r - \delta_b < \alpha + \beta + i(\overline{\omega}_1)$.

in its neighbourhood, . Typically, ε_λ is about 3% per annum (this would arise with 1 to 2% population growth, 1 to 2% growth in productivity, and 3 to 4% depreciation, while $\nu \equiv 3$ and $c \equiv 0.7$). Therefore, λ moves 3% of the remaining distance towards $\overline{\lambda}_1$ each year, and takes approximately 23 years to get halfway of its steady-state level.[2] What is more important, an increase of ν or c actually slows down the speed at which the economy converges towards its structural (un)employment rate.

We end this section with two remarks comparing properties of the interior equilibrium with well-known concepts in macroeconomics.

Remark 2.1. By normalizing capital through the effective labor force, $k := K/aN$, one can ask the question as to whether the balanced-growth path in our model does satisfy Solow's celebrated golden rule. Namely, we ask whether the equilibrium normalize capital $\overline{k}_1$ maximizes welfare, defined here as the flow of consumption per unit of effective labor force, $\mathcal{C} := C/aN$. It is easy to see that the answer is positive, since the dynamics of normalized capital is

$$\dot{k} = \frac{\kappa(1-\omega-rd_f)}{\nu}k - (\alpha+\beta+\delta)k,$$

while, at the interior equilibrium, $\dot{k} = \dot{\mathcal{C}} = 0$. The latter equality follows in a straightforward way from the fact that $C = Y - I$ and $\dot{\kappa}(1-\omega-rd_f) = 0$ at the steady state.

Remark 2.2. Because of the non-trivial trade-off between long-run inflation and employment embodied in (32), the analog of a "long-run Phillips curve" cannot be vertical in our setting. Hence, the structural long-run unemployment rate $1-\overline{\lambda}_1$ is not to be confused with the "natural rate of unemployment" introduced in Friedman (1968) and Phelps (1968). Moreover, as will be clear from our stability analysis of the balanced-growth path, monetary policy (captured here through the setting of r, the short-run nominal interest rate) does play a role, even in the long run, since it influences the asymptotic local stability of the interior steady state. Similarly, if the markup m is linked to the monopoly power of firms (as suggested, among others, by Kalecki (1971)), the institutional rules governing the consumption good market will have an influence on employment (via the impact of m on $\overline{\lambda}_1$). This contrasts with the very concept of "natural unemployment", which embodies the idea that public policy, whatsoever, is ineffective in reducing underemployment. Actually, our structural unemployment rate, $1-\overline{\lambda}_1$, is closer to the NAIRU ("Non Acccelerating Inflation Rate of Unemployment") introduced in Tobin (1980), since, at the balanced growth path, inflation, $i(\overline{\omega}_1)$, remains constant. As for the NIRU ("Non Inflationary Rate of Unemployment") of Modigliani and Papademos (1975), it is given by $\lambda_1^* := \Phi^{-1}(\alpha)$ and corresponds to the special case $m = 1/\overline{\omega}_1$.

2.4.2 Equilibria with collapsing employment

As in Grasselli and Costa Lima (2012), it is straightforward to verify that an explosive equilibrium of the form $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, \pm\infty)$ arises from the change of variable $u := 1/d_h$ in (28). In addition, as in Grasselli and Nguyen Huu (2015), when $\gamma < 1$ the introduction of inflation gives rise to another class of economically undesirable equilibria of the form $(\overline{\omega}_3, \overline{\lambda}_3, \overline{d}_3) = (\overline{\omega}_3, 0, \overline{d}_3)$ where

$$\overline{\omega}_3 = \frac{\Phi(0)-\alpha}{m(1-\gamma)\eta_p} + \frac{1}{m}, \tag{35}$$

and $\overline{d}_3$ is either a finite solution of

$$d\Big[r+\delta-\frac{1-c(\overline{\omega}^3-rd)}{\nu} - i(\overline{\omega}^3)\Big] = \overline{\omega}^3 - c(\overline{\omega}^3-rd) \tag{36}$$

or else $\overline{d}_3 = \pm\infty$. As observed in Grasselli and Nguyen Huu (2015), it follows from the second equation in (28) that

$$i(\overline{\omega}_1) = \frac{\Phi(\overline{\lambda}_1)-\alpha}{1-\gamma} > \frac{\Phi(0)-\alpha}{1-\gamma} = i(\overline{\omega}_3), \tag{37}$$

[2]The half-life, t^*, is the solution of $e^{-\varepsilon t^*} = 1/2$, which yields: $t^* = -\ln(1/2)/\varepsilon_\lambda \equiv 0.69/\varepsilon$.

so that inflation at an equilibrium of this form is necessarily lower than inflation at the interior equilibrium. Moreover, in view of (29), we have the $i(\overline{\omega}_3) < 0$, so that any equilibrium of the form $(\overline{\omega}_3, 0, \overline{d}_3)$ is, in fact, strictly deflationary.

Similarly to what we have done for the interior equilibrium, we can estimate the speed at which the economy is likely to reach an explosive equilibrium. Consider a collapse where the ratio of the household debt to output increases to infinity (i.e., $d_h \to +\infty$). Then, $\varepsilon_\lambda = \frac{1-c_-}{\nu} - (\alpha + \beta + \delta)$. If, say, $c_- \equiv 0.03$, then λ moves each year approximately 25% of the remaining distance towards 0, and it takes roughly 6 years for the zero employment rate to be reached, provided $\lambda(0)$ is already in the neighborhood of the explosive steady state, where the Taylor-series approximation is reliable. Of course, meanwhile, a number of political complications are likely to occur, such as social protests, political turmoil, etc. But at least, this provides an intuition of the forces that accelerate or decelerate the process towards a collapse. For instance, an increase in ν or c_- would slow down the fall of the economy.

The local stability properties of these different classes of equilibria are analyzed in Appendix A. We notice here that, according to (77) and (78), a necessary condition for the stability of any equilibrium with $\lambda = 0$ and $d_h \to +\infty$ is

$$1 - \nu(\alpha + \beta + \delta) < c_-. \tag{38}$$

But this is not compatible with Assumption 2.1 (a)-(ii), which in turn is a necessary condition in order to establish the existence of an interior equilibrium. Thus we can only have one of the two following situations: either there exists an interior equilibrium and the equilibria with $\lambda = 0$ and $d_h \to +\infty$ are locally unstable, or these undesirable equilibria are stable and there exists no interior equilibrium. Notice that, for typical parameter values, namely a capital-output ratio ν close to 3, population plus productivity growth $\alpha + \beta$ close to 3% and depreciation close to 4% one obtains $1 - \nu(\alpha + \beta + \delta) \approx 0.79$, which is comfortably above c_-, which in most countries should be of the order of a few percentage points of the output.[3] In other words, for parameter values within the range currently observed in most advanced economies, it is much more likely for the interior equilibrium to exist than for the explosive equilibria with $\lambda = 0$ and $d_h \to +\infty$ to be locally stable. This contrasts with the primal Keen model, where both types of equilibria are stable for typical parameters. On the other hand, notice that the condition

$$1 - \nu(\alpha + \beta + \delta) < c_+, \tag{39}$$

which is necessary for the stability of an explosive equilibrium with $\lambda = 0$ and $d_h \to -\infty$, is always compatible with assumption (a)-(ii). Observe further that if the capital to output ratio grows significantly, then both (38) and (39) are satisfied, whereas assumption (a)-(iii) is violated. In other words, the interior equilibrium ceases to exist and the explosive equilibria with $\lambda = 0$ and $d_h \to \pm\infty$ become more likely to be stable. These different scenarios are illustrated in Figure 1, where we observe first convergence to the interior equilibrium in the top panel, the emergence of business cycles with slowly increasing oscillations in the middle panel, and convergence to the explosive equilibrium in the top panel, with the parameter values and specific functional forms for the Philips curve and consumption as described in Appendix C.

Finally, observe that, since $g_\pm := (1 - c_\pm)/\nu - \delta$ is the growth rate for any equilibrium with $d_h \to \pm\infty$, we also see from (77) and (78) that another necessary condition for the stability of any equilibrium with $\lambda = 0$ and $d_h \to \pm\infty$ is

$$r > g_\pm + i(\overline{\omega}), \tag{40}$$

where either $i(\overline{\omega}) = i(\overline{\omega}_2) = -\eta_p$ or $i(\overline{\omega}) = i(\overline{\omega}_3)$. In either case, (40) is reminiscent of the controversial condition $r > g$ emphasized in Piketty (2014). Some differences, however, are worth mentioning:

[3] In France, for example, the cost of spending the minimal wage to the entire population would amount to $c_- \approx 0.03$

(1) Here, we deal with a monetary non-linear dynamical system *à la* Keen with endogenous saving rate and private debt, while Piketty (2014) deals with a non-monetary Solow model with exogenous saving rate and no debt, where in fact the condition $r > g$ does *not* imply a divergence between the incomes from work and capital, as observed in Acemoglu and Robinson (2015).[4]

(2) In Piketty (2014), r denotes the average return of capital, which includes the interest paid on government bonds, but also the return on many other types of financial assets. Here, r denotes the average rate of interest paid on private debt in the form of bank loans.

(3) Capital, here, is understood as productive capital K, and not in the all-embracing sense Piketty gives to it, which leads him to identify capital with wealth.

(4) Piketty (2014) argues that money should be neutral in the long run, even though no convincing argument is provided that would sustain this statement. Here, by contrast, money is neither neutral in the short-run, nor in the long-run. Indeed, as already emphasized, the nominal rate, r, and inflation, i, deeply shape the configuration space of system (28). Hence, following the present analysis, the entire dynamical landscape of an actual economy is likely to be affected by monetary policy.

Nevertheless, the connection between (40) and the stability of explosive equilibria prompts the question as to whether these type of equilibria would induce some divergence among income and wealth for different groups of households, a question to which we now turn.

3 A model with two classes of households

3.1 Workers and investors

We now consider a household sector divided into two classes: workers and investors. Workers are employed by the firm sector and hold deposits M_w and loans L_w in the banking sector. Their income therefore consists solely of wages and the difference between the interest received on deposits and paid on loans. Investors, on the other hand, have private ownership of both the firm and baking sector. Therefore, in addition to deposits M_i and loans L_i, their balance sheet consist of firm equity, E_f, and bank equity, E_b. Accordingly, the income of investors consists of dividends earned from their ownership of firms and banks, in addition to the interest rate differential between deposits and loans. The accounting structure of the economy with two classes of households is given by Table 2. We therefore see that the net worth (i.e., wealth) of workers is $X_w = -D_w$ and that of investors is $X_i = E_f + E_b - D_i$, where, as before, $E_f = pK + M_f - L_f = pK - D_f$ is the equity of firms and $E_b = (L_f + L_w + L_i) - (M_f + M_w + M_i)$ is the equity of banks. Once more we obtain

$$\begin{aligned} X = X_i + X_w &= E_f + E_b - D_i - D_w \\ &= pK - D_f + (D_f + D_i + D_w) - D_i - D_w \\ &= pK, \end{aligned}$$

but now observe that

$$X_i = pK - D_f + (D_f + D_i + D_w) - D_i = pK + D_w.$$

In other words, whereas the total wealth of the aggregate household sector (and therefore of the entire economy) still consists of the non-financial assets of the firm sector, the wealth of the investor class consists of these non-financial assets *plus* the net debt of workers.

The budget constraint for workers implies that their net debt evolves as

$$\dot{D}_w = -S_w = pC_w - \mathrm{w}\ell + rD_w, \tag{41}$$

[4]For a detailed analysis of Piketty's setting, see Giraud (2017).

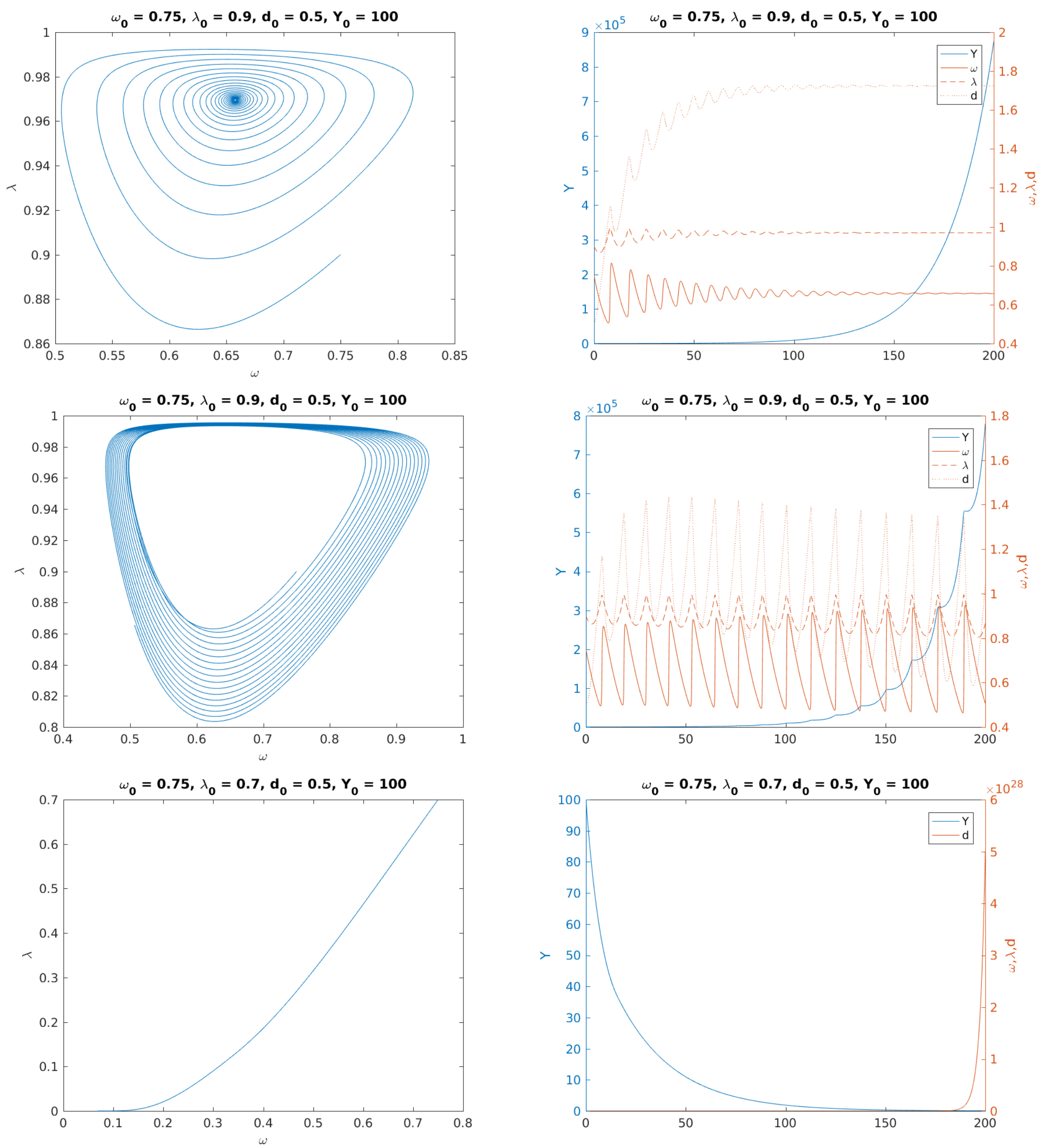

Figure 1: Simulations of the dual Keen model with parameter values as in Table 3. Top row shows convergence to an interior equilibrium using all baseline parameters. Middle row shows business cycles with slowly increased oscillations when $\eta_p = 0.45$ and $\gamma = 0.96$. Bottom row shows convergence to an explosive equilibrium when $\nu = 15$.

where C_w is the consumption of workers. On the other hand, the net debt of investors D_i evolves as

$$\dot{D}_i = -S_i = pC_i - r_k pK + rD_i - \Delta_b, \tag{42}$$

where C_i is the consumption of investors, $r_k pK$ corresponds to dividends paid by the firms with a rate of return on capital r_k (in general distinct from the loan interest rate r) and Δ_b are the dividends paid by the banking sector. Let us assume, as in Section 2, that the net equity E_b of the banking sector remains constant because its consumption is zero and all of its profits are distributed to investors, that is $C_b = 0$ and $\Delta_b = r(D_f + D_w + D_i)$. As we have seen in the previous section, the constant E_b plays no role in the dynamics of the system (or its equilibria), and from now on we assume that $E_b = 0$ without loss of generality, so that $D_w + D_f = -D_i$. It then follows that (42) reduces to

$$\dot{D}_i = pC_i - r_k pK + rD_i. \tag{43}$$

Extending the notation of the previous section, we assume that consumption of workers and investors is determined by functions $c_w(\cdot)$ and $c_i(\cdot)$ as follows

$$C_w := c_w(\omega - rd_w)Y \tag{44}$$

$$C_i := c_i(r_k\nu - rd_i)Y, \tag{45}$$

where $y_\omega = \omega - rd_w$ and $y_i = r_k\nu - rd_i$ are the shares of nominal output corresponding to the disposable income of workers and investors. Here $d_w = D_w/(pY)$, $d_i = D_i/(pY)$, and we used the fact that $r_k pK = r_k \nu pY$. In addition, we assume that investors have a lower consumption propensity with respect to income, that is to say, $c'(y_w) > c'(y_i)$.

3.2 Return on capital and corporate debt

The production side of the economy remains unchanged, given by (12), (25) and (1), with total consumption being $C = C_i + C_w$, except that firms now pay dividends to shareholders, which we assume to be a constant fraction $\Theta \in [0, 1]$ of profits (9). Accordingly, the rate of return on capital, r_k, can be found endogenously as:

$$\begin{aligned} r_k := r_k(\omega, d_w, d_i) &= \frac{\Theta(pY - \mathrm{w}\ell - rD_f - p\delta K)}{pK} \\ &= \frac{\Theta}{\nu}\left(1 - \omega - rd_f - \delta\nu\right) \\ &= \frac{\Theta}{\nu}\left(1 - \omega + r(d_w + d_i) - \delta\nu\right), \end{aligned} \tag{46}$$

where we have used again the fact that $D_f = -(D_i + D_w)$, since we assumed that $E_b = 0$. Savings for the firm sector are therefore given by retained profits, that is,

$$S_f = (1 - \Theta)(pY - \mathrm{w}\ell - rD_f - p\delta K) = pY - \mathrm{w}\ell - rD_f - p\delta K - r_k pK. \tag{47}$$

It then follows that the amount that needs to be raised externally to finance investment, namely the difference $[p(I - \delta K) - S_f]$, is given by

$$p(I - \delta K) - S_f = pI - pY + \mathrm{w}\ell + rD_f + r_k pK.$$

As in the Keen (1995) model, we assume that external financing is obtained solely through loans from the banking sector, rather than, for example, a combination of new loans and share issuance. It then follows from (47) that corporate debt evolves according to:

$$\dot{D}_f = pI - pY + \mathrm{w}\ell + rD_f + r_k pK = -(\dot{D}_w + \dot{D}_i), \tag{48}$$

where we have used (41) and (43).

	Workers	Investors	Firms		Banks	Sum
Balance sheet						
Capital stock			$+pK$			pK
Deposits	$+M_w$	$+M_i$	$+M_f$		$-M$	0
Loans	$-L_w$	$-L_i$	$-L_f$		$+L$	0
Equity		E	$-E_f$		$-E_b$	0
Column sum (Net worth)	X_w	X_i	0		0	pK
Transactions			Current	Capital		
Consumption	$-pC_w$	$-pC_i$	$+pC$		$-pC_b$	0
Investment			$+pI$	$-pI$		0
Accounting memo [GDP]			[pY]			
Wages	$+\mathrm{w}\ell$		$-\mathrm{w}\ell$			0
Depreciation			$-p\delta K$	$+p\delta K$		0
Interest on loans	$-rL_w$	$-rL_i$	$-rL_f$		$+rL$	0
Interest on deposits	$+rM_w$	$+rM_i$	$+rM_f$		$-rM$	0
Dividends		$+r_k pK + \Delta_b$	$-r_k pK$		$-\Delta_b$	0
Column sum (balances)	S_w	S_i	S_f	$-pI + p\delta K$	S_b	0
Flows of funds						
Change in capital stock			$+p\dot{K}$			$p\dot{K}$
Change in deposits	$+\dot{M}_w$	$+\dot{M}_i$	$+\dot{M}_f$		$-\dot{M}$	0
Change in loans	$-\dot{L}_w$	$-\dot{L}_i$	$-\dot{L}_i$		$+L$	0
Column sum (savings)	S_w	S_i	S_f		S_b	$p\dot{K}$
Change in firm equity		$+\dot{E}_f$	$-(S_f + \dot{p}K)$			0
Change in bank equity		$+\dot{E}_b$			$-S_b$	0
Change in net worth	S_w	$\dot{E} + S_i$	0		0	$\dot{p}K + p\dot{K}$

Table 2: Balance sheet, transactions and flow of funds for a three-sector economy with two types of households.

Remark 3.1. Observe that, whenever workers spend exactly their wages for consumption, that is, $pC_w \equiv w\ell$, then the accounting equations (9) and (12) yield

$$\Pi = pI + pC_i - rD_f - p\delta K. \tag{49}$$

If, moreover, $rD_f + p\delta K = 0$, this is but Kalecki's celebrated equation, of which (49) can be read as an extension. Kalecki's reasoning about the sense in which causality runs between the two sides of the equality still holds in its extended version. Indeed, whereas investors cannot choose the level of profit Π, they do decide their own consumption and therefore influence the level of investment and leverage of the firms they own. In other words, while "workers consume what they earn, investors earn what they consume and invest *less* the debt burden of their firms". Now, in actual economies, workers also borrow money in order to finance consumption. Thus, equation (41) leads us to modify (49) so that

$$\Pi = pI + pC_i + \dot{D}_w + rD_w - rD_f - p\delta K.$$

This sheds some light about the incentives of banks to provide loans to workers (e.g., during the decade prior to the subprime crisis of 2007-2009): investors also earn the debt burden of workers. Notice that a trade off shows up in the choice of the sort-run nominal rate, r. In countries where D_w is large, investors are likely to be in favour of increasing r, while in countries where D_f is large (relative to D_w), they are likely to put pressure on the Central Bank to decrease r.

3.3 The main dynamical system

The evolution of wages is still provided by a short-run Phillips curve (5), so that the dynamic of wage share remains given by (7). The real growth rate of output is now

$$g(\omega, d_w, d_i) = \frac{\dot{Y}}{Y} = \frac{1 - c(\omega, d_w, d_i)}{\nu} - \delta, \tag{50}$$

where

$$c(\omega, d_w, d_i) := c_w(\omega - rd_w) + c_i(r_k\nu - rd_i). \tag{51}$$

The dynamics of employment is given by (4) with $g(\omega, d_w, d_i)$ as in (50). Finally, the net debt of workers and investors change according to (41) and (43) (with $E_b = 0$). We therefore end up with a 4-dimensional dynamical system for the state variables $(\omega, \lambda, d_w, d_i)$:

$$\begin{array}{rcl} \dot{\omega} & = & \omega[\Phi(\lambda) - \alpha - (1-\gamma)i(\omega)] \\ \dot{\lambda} & = & \lambda\left[\frac{1-c(\omega,d_w,d_i)}{\nu} - (\alpha+\beta+\delta)\right] \\ \dot{d}_w & = & d_w\left[r + \delta - \frac{1-c(\omega,d_w,d_i)}{\nu} - i(\omega)\right] + c_w(\omega - rd_w) - \omega \\ \dot{d}_i & = & d_i\left[r + \delta - \frac{1-c(\omega,d_w,d_i)}{\nu} - i(\omega)\right] + c_i(r_k\nu - rd_i) - r_k\nu \end{array} \tag{52}$$

where $r_k\nu = \Theta(1 - \omega + r(d_w + d_i) - \delta\nu)$ was defined in (46).

In order to study long-run steady states in this economy with two classes of households, we replace the Assumption 2.1 with the following.

Assumption 3.1. 1. The consumption function $c_i, c_w : \mathbb{R}^2 \to \mathbb{R}_+$ are locally Lipschitz, increasing on both arguments, and satisfy the following properties, for all $x \in \mathbb{R}$:

   (i) $y \leq 0 \Rightarrow c_i(y) = c_{i^-} > 0$ and $c_w(y) = c_{w^-} > 0$, with $c_{i^-} + c_{w^-} = c_-$.

   (ii) $\lim_{y\to+\infty} c_w(y) = c_{w^+}$ and $\lim_{y\to+\infty} c_i(y) = c_{i^+}$, with $c_{w^+} + c_{i^+} = c_+ \leq 1$.

   (iii) $c_- < 1 - \nu(\alpha + \beta + \delta) < c_{w^+}$

2. $\Phi : [0, 1) \to \mathbb{R}$ is $\mathcal{C}^1$, strictly increasing and admits a vertical asymptote at $\lambda = 1$. Moreover, $\Phi'(0) = 0$ and

$$\Phi(0) < \min\{\alpha, \gamma(\alpha + \beta) - \beta\} \tag{53}$$

3. $r + \eta_p \geq \alpha + \beta$

3.3.1 The balanced-growth path

As in Section 2.4.1, we start with an equilibrium with nonzero wage share and employment rate. As before, the equation for employment implies that the equilibrium growth rate is $\alpha+\beta$. With this in mind, define

$$\omega_0 = \frac{1}{m}\left[\frac{r-(\alpha+\beta)}{\eta_p}+1\right] \tag{54}$$

and observe that Assumption 3.1 implies that $\omega_0 \geq 0$. Moreover, we can see that $\alpha+\beta+i(\omega_0)=r$ and $\alpha+\beta+i(\omega)>r$ for all $\omega>\omega_0$. Define next d_{w0} as the solution to

$$c_w(\omega_0 - rd) = \omega_0, \tag{55}$$

which exists provided $c_{w-} < \omega_0 < c_{w+}$. Finally, define d_{i0} as the solution to

$$c_i(\Theta(1-\omega_0+rd_{w0}-\delta\nu)-(1-\Theta)rd) = \Theta(1-\omega_0+rd_{w0}+rd-\delta\nu), \tag{56}$$

which always exists, since the right-hand side is a linear function of d, whose image always contains the interval $[c_{i^-}, c_{i^+}]$.

Lemma 3.1. *Assume that ω_0, d_{w0} and d_{i0} defined as in* (54)-(56) *satisfy*

$$c_{w-} < \omega_0 < c_{w+} \tag{57}$$

and

$$(1-\Theta)\omega_0 + \Theta r(d_{w0}+rd_{i0}) < 1-\nu(\alpha+\beta+\delta)-\Theta(1-\delta\nu). \tag{58}$$

Then there exist an equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_{w1}, \overline{d}_{i1})$ for (52) *with $0 \leq \omega_0 < \overline{\omega}_1 < \infty$ and $\overline{\lambda}_1 > 0$.*

Proof. Consider the nonlinear equation satisfied by a candidate for equilibrium net debt of workers:

$$d_w = \frac{c_w(\omega - rd_w)-\omega}{(\alpha+\beta)+i(\omega)-r}. \tag{59}$$

For a fixed $\omega > \omega_0$, define

$$\Gamma_w(d) := c_w(\omega - rd) - \omega - d[(\alpha+\beta)+i(\omega)-r]$$

as a function of d. It then follows from the definition of ω_0 that $\Gamma_w(d)$ is strictly decreasing, so that (59) admits a solution $d_w(\omega)$ with $d_w(\omega)\to d_{w0}$ as $\omega\to\omega_0$ and $d_w(\omega)\to -1/\eta_p$ as $\omega\to+\infty$.

Consider next the nonlinear equation satisfied by a candidate for the net debt of investors:

$$d_i = \frac{c_i(r_k\nu - rd_i)-\Theta(1-\omega+rd_w+rd_i-\delta\nu)}{(\alpha+\beta)+i(\omega)-r}. \tag{60}$$

For fixed $\omega > \omega_0$ and $d_w(\omega)$ as determined above, define

$$\Gamma_i(d) := H(\omega, d_w(\omega), d) - G(\omega, d_w(\omega), d) - dF(\omega)$$

as a function of d, where

$$\begin{aligned} H(\omega, d_w(\omega), d) =& c_i(\Theta(1-\omega+rd_w(\omega)-\delta\nu)-(1-\Theta)rd) \\ G(\omega, d_w(\omega), d) =& \Theta(1-\omega+rd_w(\omega)+rd-\delta\nu) \\ F(\omega) =& (\alpha+\beta)+i(\omega)-r>0 \end{aligned}$$

It then follows again that $\Gamma_i(d)$ is strictly decreasing and (60) admits a solution $d_i(\omega)$ with $d_i(\omega)\to d_{i0}$ as $\omega\to\omega_0$ and $d_i(\omega)\to\Theta/\eta_p$ as $\omega\to+\infty$.

Finally, for $d_w(\omega)$ and $d_i(\omega)$ determined as above, we see that total consumption as a function of ω is given by

$$c(\omega) = c_w(\omega - rd_w(\omega)) + c_i(\Theta(1-\omega+rd_w(\omega)-\delta\nu)-(1-\Theta)rd_i(\omega)). \tag{61}$$

Observe that, if $\omega \to +\infty$, then Assumption 3.1 (a)-(i) implies that $c_i \to 0$, so that

$$c \to \lim_{\omega \to +\infty} c_w(\omega - r d_w(\omega)) = c_{w^+} > 1 - \nu(\alpha + \beta + \delta),$$

according to Assumption 3.1 (a)-(iii). Therefore, provided we have that

$$c(\omega_0) = c_w(\omega_0 - r d_{w0}) + c_i(\Theta(1 - \omega + r d_{w0} - \delta\nu) - (1 - \Theta) r d_i(\omega)) < 1 - \nu(\alpha + \beta + \delta), \quad (62)$$

then we can conclude that there exists some $\omega_0 < \overline{\omega} < \infty$ such that

$$c(\overline{\omega}) = 1 - \nu(\alpha + \beta + \delta).$$

But it is easy to see that (62) is equivalent to (58) and the conclusion holds.

We then define an interior equilibrium as the point $\overline{\omega}_1 = \overline{\omega}$, $\overline{d}_{w1} = d_w(\overline{\omega})$ and $\overline{d}_{i1} = d_i(\overline{\omega})$ just obtained and

$$\overline{\lambda}_1 = \Phi^{-1}\big(\alpha + (1 - \gamma) i(\overline{\omega}_1)\big), \quad (63)$$

which exists because of Assumption 3.1 (b). □

Before investigating how inequality evolves along this growth path, let us turn to undesirable equilibria.

3.3.2 Equilibria with collapsing employment

In order to investigate the properties of the system for large values of net debt for workers and investors, we consider first the changes of variables $u_w = 1/d_w$ and $u_i = 1/d_i$, leading to the modified system

$$\begin{array}{rcl} \dot{\omega} & = & \omega[\Phi(\lambda) - \alpha - (1 - \gamma) i] \\ \dot{\lambda} & = & \lambda\left[\frac{1-c}{\nu} - (\alpha + \beta + \delta)\right] \\ \dot{u_w} & = & u_w\left[\frac{1-c}{\nu} + i - r - \delta\right] - u_w^2\left[c_w - \omega\right] \\ \dot{u_i} & = & u_i\left[\frac{1-c}{\nu} + i - r - \delta\right] - u_i^2\left[c_i - \Theta\left(1 - \omega + \frac{r}{u_w} + \frac{r}{u_i} - \delta\nu\right)\right]. \end{array} \quad (64)$$

The appearance of the term u_i^2/u_w in the differential equation for u_i above makes it unclear whether $u_i = u_w = 0$ corresponds to an equilibrium of the modified system. We then follow Grasselli and Costa Lima (2012) and consider the $u_w = 1/d_w$ together with the ratio $v = u_i/u_w = d_w/d_i$. This leads to the further modified system

$$\begin{array}{rcl} \dot{\omega} & = & \omega[\Phi(\lambda) - \alpha - (1 - \gamma) i] \\ \dot{\lambda} & = & \lambda\left[\frac{1-c}{\nu} - (\alpha + \beta + \delta)\right] \\ \dot{u}_w & = & u_w\left[\frac{1-c}{\nu} + i - r - \delta\right] - u_w^2\left[c_w - \omega\right] \\ \dot{v} & = & v(1 + v) r\Theta + v u_w[c_w - \omega] - v^2 u_w\left[c_i - \Theta(1 - \omega - \delta\nu)\right] \end{array} \quad (65)$$

We can then see that $(\omega, \lambda, u_w, v) = (0, 0, 0, 0)$ and $(\omega, \lambda, u_w, v) = (0, 0, 0, -1)$ are equilibria of the modified system (65) corresponding to $(\omega, \lambda, u_w, u_i) = (0, 0, 0, 0)$ in (64) along trajectories with $u_i^2/u_w = u_i v \to 0$, which in turn correspond to $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_{w2}, \overline{d}_{i2}) = (0, 0, \pm\infty, \pm\infty)$ for the original system (52) along a trajectory with $d_w/d_i \to 0$, that is to say, with the debt of investors growing faster than the debt of workers.

Finally, yet another type of long-term steady state can be reached of the form $(\overline{\omega}_3, \overline{\lambda}_3, \overline{d}_{w3}, \overline{d}_{i3}) = (\overline{\omega}_3, 0, \overline{d}_{w3}, \overline{d}_{i3})$ where

$$\overline{\omega}_3 = \frac{\Phi(0) - \alpha}{m(1 - \gamma)\eta_p} + \frac{1}{m}, \quad (66)$$

and $\overline{d}_{w3}$ is either a finite solution of

$$d\Big[r + \delta - \frac{1 - c)}{\nu} - i\Big] = \overline{\omega}_3 - c_w, \quad (67)$$

or else $\overline{d}_{w3} = \pm\infty$, whereas $\overline{d}_{i3}$ is either a finite solution of

$$d\Big[r(1-\Theta)+\delta-\frac{1-c}{\nu}-i\Big] = \Theta(1-\overline{\omega}_3+r\overline{d}_{w3}-\delta\nu)-c_i \tag{68}$$

for finite $\overline{d}_{w3}$, or else $\overline{d}_{i3} = \pm\infty$.

The stability of these classes of equilibrium is discussed in Appendix B and numerical simulations are showing in Figure 2.

3.4 Asymptotic inequality

Using the definitions in Table 2, we see that the total nominal income for the two types of households and retained profits of the firm sector are given by

$$\begin{aligned} Y_w^n &= \mathrm{w}\ell - rD_w \\ Y_i^n &= r_k pK - rD_i = \Theta(pY-\mathrm{w}\ell-rD_f-p\delta K)-rD_i \\ \Pi_r &= (1-\Theta)(pY-\mathrm{w}\ell-rD_f-p\delta K), \end{aligned}$$

so that, using again $D_f = -(D_w + D_i)$, we find that their sum equals total income, that is

$$Y_w^n + Y_i^n + \Pi_r = (pY - \delta pK). \tag{69}$$

Moreover, since investors are the private owners of the firms, we follow the terminology adopted in Piketty (2014), according to which retained profits should be added to dividends to obtain the total income arising from capital. In other words, the total income from capital corresponds to the sum $(Y_i^n + \Pi_r)$.

Accordingly, the shares of output corresponding to the income of workers, income of investors, and retained profits are given by

$$y_w = \frac{Y_w^n}{pY} = \omega - rd_w \tag{70}$$

$$y_i = \frac{Y_i^n}{pY} = r_k\nu - rd_i = \Theta(1-\omega+rd_w-\delta\nu)-(1-\Theta)rd_i \tag{71}$$

$$\pi_r = \frac{\Pi_r}{pY} = (1-\Theta)(1-\omega-rd_f-\delta\nu), \tag{72}$$

whereas the share of output corresponding to total capital income is

$$y_c = y_i + \pi_r = 1-\omega+rd_\omega-\delta\nu = 1-y_\omega-\delta\nu, \tag{73}$$

in accordance with (69).

3.4.1 When inequality remains constant in the long-run

Let us now examine whether a divergence between investors' and workers' real incomes can occur in the long-run. The growth rate of total real income for workers $Y_w^n/p = (\omega - rd_w)Y$ is

$$g_w(\omega,\lambda,d_w,d_i) = \frac{(\dot{\omega}-r\dot{d}_w)}{\omega-rd_w}+\frac{\dot{Y}}{Y}.$$

At equilibrium, we have that $\dot{d}_w = \dot{\omega} = 0$, so that

$$g_w(\overline{\omega}_1,\overline{\lambda}_1,\overline{d}_{w1},\overline{d}_{i1}) = \frac{\dot{Y}}{Y} = \alpha+\beta, \tag{74}$$

provided $\overline{\omega}_1 \neq r\overline{d}_{w1}$.

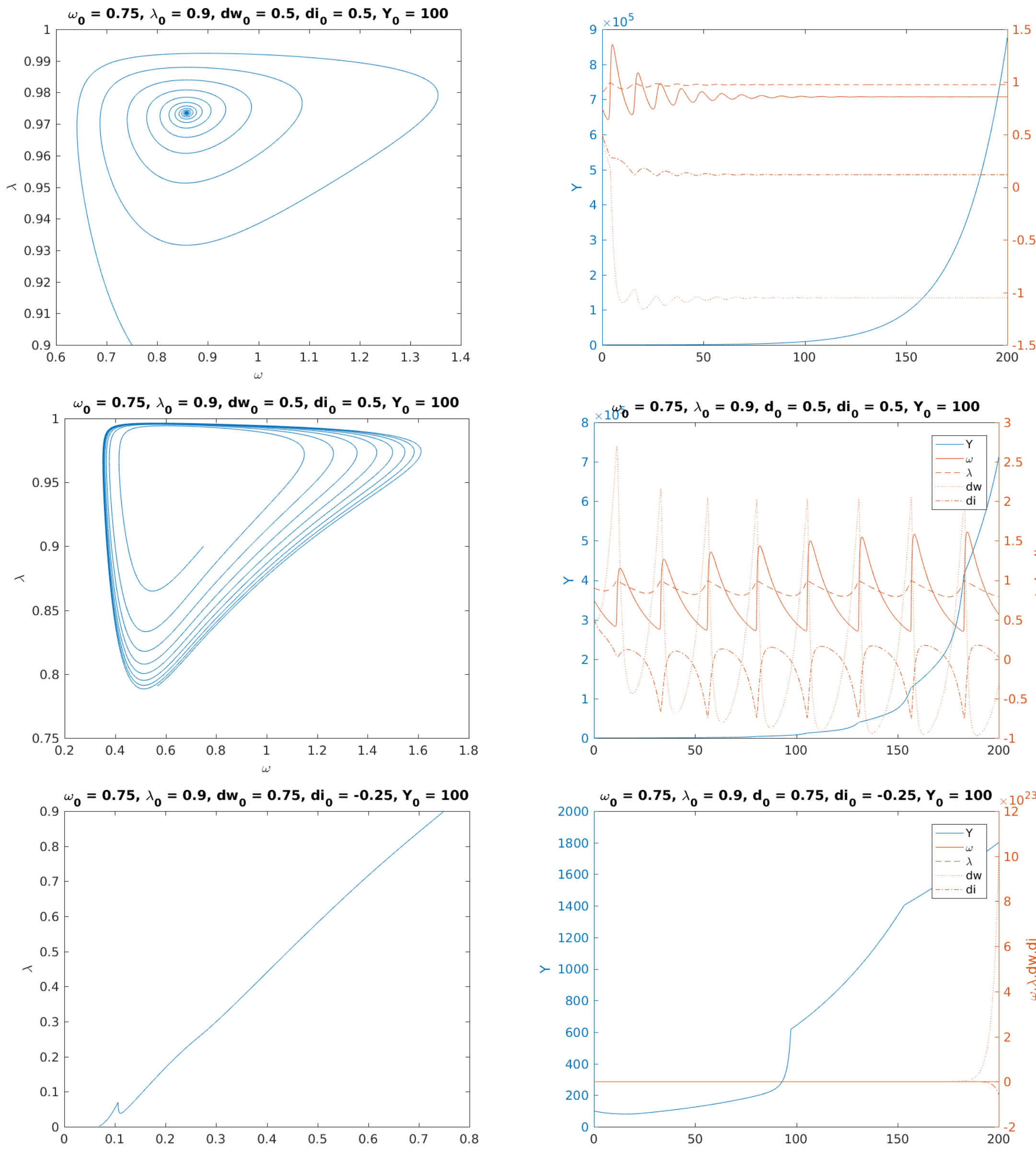

Figure 2: Simulations of the dual Keen model with two classes of households using parameter values as in Table 3. Top row shows convergence to an interior equilibrium with $\overline{d}_w > 0$ and $\overline{d}_i < 0$ using all baseline parameters. Middle row shows business cycles with slowly increased oscillations when $\eta_p = 0.45$ and $\gamma = 0.96$. Bottom row shows convergence to the explosive equilibrium with $d_w \to +\infty$ and $d_i \to -\infty$ when $\nu = 15$

Similarly, the growth rate of total real income for investors $Y_i^n/p = \big(r_k\nu - rd_i\big)Y$ is

$$
\begin{aligned}
g_i(\omega,\lambda,d_w,d_i) &= \frac{\dot r_k\nu - r\dot d_i}{r_k\nu - rd_i} + \frac{\dot Y}{Y} \\
&= \frac{-\Theta\dot\omega + \Theta r\dot d_w - (1-\Theta)r\dot d_i}{\Theta(1-\omega+rd_w-\delta\nu)-(1-\Theta)rd_i} + \frac{\dot Y}{Y}.
\end{aligned}
$$

where we have used (46) in the last equation. At equilibrium, we again have that $\dot\omega = \dot d_w = \dot d_i = 0$, so that

$$
g_i(\overline\omega_1,\overline\lambda_1,\overline d_{w1},\overline d_{i1}) = \alpha+\beta, \tag{75}
$$

provided $r_k(\overline\omega_1,\overline d_{w1},\overline d_{i1})\nu \neq r\overline d_{i1}$.

Finally, the growth rate of real retained profits $\Pi_r/p = (1-\Theta)(1-\omega-rd_f-\delta\nu)Y$ is

$$
g_r(\omega,\lambda,d_w,d_i) = \frac{-\dot\omega + r(\dot d_w+\dot d_i)}{(1-\Theta)[1-\omega+r(d_w+d_i)-\delta\nu]} + \frac{\dot Y}{Y}.
$$

Once more, at equilibrium $\dot\omega = \dot d_w = \dot d_i = 0$, so that

$$
g_r(\overline\omega_1,\overline\lambda_1,\overline d_{w1},\overline d_{i1}) = \alpha+\beta, \tag{76}
$$

provided $1-\overline\omega_1 + r(\overline d_{w1}+\overline d_{i1}) - \delta\nu \neq 0$.

In other words, in this case both real income from labour and from capital grow at the same pace, namely the real growth rate of the economy. Furthermore, at the interior equilibrium, the rate of return on capital r_k and the income shares y_w, y_i and π_r all converge towards a constant level, fully justifying its characterization as a "balanced growth path".

3.4.2 Inequality as a hallmark of inefficiency

Next, what happens at one of the undesirable equilibria? As shown in Section 3.3.2, there are four possible asymptotic crisis states corresponding to infinite levels of net debt for workers and investors and vanishing wage share and employment, namely: $(\overline\omega_2,\overline\lambda_2,\overline d_{w2},\overline d_{i2}) = (0,0,\pm\infty,\pm\infty)$.

Let us start with the two cases where the debt ratio of workers increases without bound. In the case $d_w\to+\infty$ and $d_i\to-\infty$, observe first that we have that

$$
d_f = -(d_w+d_i) = -(v+1)d_i \to +\infty,
$$

since this corresponds to an equilibrium with $v\to 0$. We therefore see that (70), (71) and (72) with $r>0$ (as we assume throughout) give $y_w\to-\infty$, whereas

$$
y_i = \Theta(1-\delta\nu) + \Theta rd_w - (1-\Theta)rd_i \to +\infty,
$$

and $\pi_r \to -\infty$. In other words, the income share of investors explodes while profits and the income of workers plunges, showing that the distributional conflict here does not pit workers against investors, but rather banks (and their owners) against firms and workers. Nevertheless, (73) shows that $y_c\to+\infty$, that is, the share of total capital income still explodes, despite the fall in profits π_r and return on capital r_k, courtesy of interest on debt paid by workers.

On the other hand, in the case $d_w\to+\infty$ and $d_i\to+\infty$, observe that we now have

$$
d_f = -(d_w+d_i) = -(v+1)d_i \to -\infty.
$$

We then have that $y_w\to-\infty$ and $\pi_r\to+\infty$, whereas, provided $0\le\Theta<1$, we have that

$$
y_i = \Theta(1-\delta\nu) + \Theta(rd_w+rd_i) - rd_i = \Theta(1-\delta\nu) + rd_i(\Theta v+\Theta-1) \to -\infty,
$$

where we again used that $v \to 0$. In other words, the income shares of both investors and workers plunges, while the net profit share, which depends on $d_f = -(d_w + d_i)$, explodes. The corner case of $\Theta = 1$ restores the situation where $y_i \to +\infty$, since in this case net profits are distributed in full to investors, and the share of profits is zero. For all $0 \leq \Theta \leq 1$, however, we have that $y_c \to +\infty$, still courtesy of the interest on debt paid by workers.

Conversely, consider now the cases where workers have positive cash balances, that is to say negative debt, growing faster than income. In the case $d_w \to -\infty$ and $d_i \to +\infty$ we find that

$$d_f = -(d_w + d_i) = -(v + 1)d_i \to -\infty.$$

We then have $y_w \to +\infty$, whereas

$$y_i = \Theta(1 - \delta\nu) + \Theta r d_w - (1 - \Theta) r d_i \to -\infty,$$

and $\pi_r \to +\infty$. In other words, the income of workers soars together with profits, whereas the income of investors plunges. We also have that $y_c \to -\infty$, so total capital income also collapses, despite increased profits, this time courtesy of the interest paid by investors (who are the owners of the banks) to workers. Similarly, in the case $d_w \to -\infty$ and $d_i \to -\infty$ we find that

$$d_f = -(d_w + d_i) = -(v + 1)d_i \to +\infty,$$

so that we have $y_w \to +\infty$ and $\pi_r \to -\infty$, whereas, provided $0 \leq \Theta < 1$, we have that

$$y_i = \Theta(1 - \delta\nu) + \Theta(r d_w + r d_i) - r d_i = \Theta(1 - \delta\nu) + r d_i(\Theta v + \Theta - 1) \to +\infty.$$

In other words, the income share of both investors and workers soars, while this time the net profit share plunges. The corner case of $\Theta = 1$ restores the situation where $y_i \to -\infty$, since in this case the (negative) net profits of firms are distributed in full to investors, and the share of profits is zero. In either case, $y_c \to -\infty$, still courtesy of the interest paid by investors to workers.

We therefore see that in all four cases where $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_{w2}, \overline{d}_{i2}) = (0, 0, \pm\infty, \pm\infty)$, income inequality between workers and investors grows without bound. In the two cases where $d_w \to +\infty$, such extreme inequality favours capitalists, whereas in the two cases when $d_w \to -\infty$ the converse is true. This result sharply contrasts with those of Piketty (2014), according to which any efficient economy (where it is likely that the condition $r \geq g$ holds) should experience an ever growing inequality. We obtain exactly the opposite conclusion: an "inefficient" economy (i.e., one that converges to a disastrous equilibrium) experiences an ever growing inequality, while an efficient economy (one that ultimately follows the balanced growth path) converges to a constant state of inequality.

Finally, consider the deflationary equilibrium $(\overline{\omega}_3, 0, \overline{d}_{w3}, \overline{d}_{i3})$. The situation when $(\overline{d}_{w3}, \overline{d}_{i3}) \to (\pm\infty, \pm\infty)$ is entirely analogous to the four cases analyzed above. In the case of finite $\overline{d}_{w3}$ and $\overline{d}_{i3}$, however, the income share of both populations will be respectively $r_k\nu - r\overline{d}_{i3}$ and $\overline{\omega}^3 - r\overline{d}_{w3}$ and therefore both finite. This is, however, an artifact of the fact that prices are falling faster than real output. In other words, nominal income ratios remain artificially constant, but the real income of both populations collapse. In other words, this time, there is no divergence because *both* types of households end up ruined! Appendix A.3. provides the necessary and sufficient conditions for the local stability of such a liquidity trap at large time scale. As before, it turns out that increasing the capital-income ratio, ν, makes it more likely for the asymptotic liquidity trap to be locally stable. As a consequence, a good policy to drive an actual economy out of the basin of attraction of such a catastrophic steady-state consists in reducing ν.

The analysis of wealth inequality is straightforward, since we have $X_w = -D_w$ for the wealth of workers and $X_i = pK + D_w$ for the wealth of investors. Namely, at the interior equilibrium the wealth-to-income ratios $x_w = X_w/(pY)$ and $x_i = X_i/(pY)$ converge to $\overline{x}_w = -\overline{d}_{w1}$ and $\overline{x}_i = \nu + \overline{d}_{w1}$ respectively. As discussed in Section (3.3.1), we have that $\overline{d}_{w1} \geq 0$ provided $c_w(\overline{\omega}_1, 0) \geq \overline{\omega}_1$, that is to say, when consumption for workers with zero debt is larger than wage income, which is likely to hold in reality, leading to an equilibrium situation where $\overline{x}_i > \overline{x}_w$, but with constant ratios.

Even when workers have positive cash balances at equilibrium (that is, $\overline{d}_{w1} < 0$), we could still have $\overline{x}_i > \overline{x}_w$ provided $\nu > -2\overline{d}_{w1}$.

At the explosive equilibria, it is clear that $x_w \to -\infty$ and $x_i \to +\infty$ whenever $d_w \to +\infty$, so that we observe extreme levels of wealth inequality in favour of investors, with the reverse situation when $d_w \to -\infty$. Finally, at the deflationary equilibrium $(\overline{\omega}_3, 0, \overline{d}_{w3}, \overline{d}_{i3})$ with finite $\overline{d}_{w3}$ and $\overline{d}_{i3}$ we have finite wealth-to-income ratios of the form $\overline{x}_w = -\overline{d}_{w3}$ and $\overline{x}_i = \nu + \overline{d}_{w3}$, but only because prices are falling faster than real income in this collapsing economy.

4 Discussion and future work

In the previous sections we introduced a monetary stock-flow consistent model to analyze the interplay between household debt, inequality and growth. The version of the model with an aggregate household sector presented in Section 2 displays the characteristic behaviour of Keen-type models analyzed in Grasselli and Costa Lima (2012), namely a multiplicity of equilibria, one of them exhibiting an ever growing debt-to-income ratio and accompanying decrease in employment rate and wage share. As observed in Section 2.4, the conditions for stability of this type of equilibrium are likely to be violated for currently observed parameter values, therefore favouring the existence and stability of an interior equilibrium with a finite debt ratio and non-zero employment instead. In addition, we have shown that this interior long-run steady state is but an extension of Solow's famous balanced-growth path, as it satisfies all the properties of the latter, including the golden rule. Finally, employment along this balanced growth path turns out to converge towards a positive NAIRU, thus providing an out-of-equilibrium foundation for Tobin's concept.

This can be reversed, however, if some of the model parameters change in the way that is predicted in Piketty (2014), notably by an increase in the capital-income ratio ν and a decrease in the long-run "natural" growth rate $\alpha + \beta$, both of which would improve the stability of the deflationary equilibrium with explosive debt levels. Since both trends are associated with increasing income and wealth inequality in Piketty (2014), we then considered a version of the mode with two classes of households, namely workers and investors, in Section 3 to investigate the connection between the two phenomena. As shown in Section 3.4, we indeed observe that the income shares of workers and investors diverge without bounds whenever the economy approaches any of the possible deflationary equilibria with explosive debt-to-income ratios for households. Among these catastrophic long-run steady states, some of them exhibit a debt-deflationary trend, typical of what has been described by Irving Fisher and that several papers recently tried to capture (see, e.g., Eggertsson and Krugman (2011) and Giraud and Pottier (2016)). Both papers, however, confined themselves to an equilibrium approach of the liquidity trap, neglecting the transitional dynamics that may lead to it. Moreover, Eggertsson and Krugman (2011) considers the effect of debt-overhang in a real set-up, where the interaction between money and deflation cannot be tackled. Here, by contrast, we provide a monetary characterization of the road that leads to such a trap, and its hallmark turns out to be the ultimate convergence of real income of both workers and investors because all of them end up ruined. In most cases, including the asymptotic liquidity trap just mentioned, it turns out that increasing the capital-to-income ratio, ν, reinforces the stability of these catastrophic long-run steady states. As a consequence, we reach the rather unexpected conclusion that a good policy to escape from the current savings glut that seems to condemn the world economy to stagnation would consist in boldly reducing the capital-to-income ratio. More generally, our findings shed light on the old dichotomy between efficiency and fairness: The pace at which the size of the pie increases cannot be considered in isolation from the manner it is divided within a given society. Distribution (of income and debt) does have an impact on growth, and more egalitarian economies are more likely to be efficient in the long-run. One reason why this conclusion has hardly been reached in earlier analyses lies in the neglect of the macroeconomic impact of debt. Indeed, it has long been believed that, one debt being just the counterpart of someone else's liability, private debt has little consequence, if any, on the macroeconomic trajectory. This wisdom turns out to be wrong, as we have shown in this paper. One first consequence is that long-term growth cannot be understood within a framework with a single "representative" consumer (who, by construction, cannot exhibit debt). Moreover, capital and income taxation become tools of paramount importance in order to foster the growth of national income, or at least to prevent it from declining.

We kept the model deliberately simple to highlight the interplay between debt and inequality in a straightforward way. We now discuss several extensions of our setting and their likely effect on our results. Let us begin with the assumption of a constant capital-to-output ratio in (1) and purely accommodating investment function of the form $(1 - c(y))$ in (25). It is clear that these simplifications are overoptimistic, in the sense that investment demand keeps aggregate demand artificially high even at times of very low consumption. A more realistic setting, for example along the Steindlian lines adopted in Dutt (2006), would assume that firms keep excess capacity and adjust output to meet aggregate demand, with investment demand being a function of utilization. This modification would result in reduced output when the debt ratio of workers increases (because of lower total consumption) therefore reducing employment and consequently wages, leading to even lower consumption. It is therefore clear that, absent other mechanism, a more realistic production side would strengthen the link between household debt and inequality in our model.

A second modification pertains the inclusion of capital markets in the model. Following Skott (2013), one can consider ownership of firms by investors through an active stock market where share prices are determined by equilibrium (as opposed to the prices of goods, which are assumed to follow the dynamics (6) in an imperfectly competitive market). As shown in Skott (2013), when the demand for shares depends on expected returns, which in turn follow an adaptive dynamics based on observed returns, the stock market is prone to inequity-induced bubbles. This mechanism is also likely to exacerbate the divergence of incomes in our set-up.

Two final modifications, on the other hand, do have the potential to alter the conclusions of the model: default and government intervention. As mentioned in Nguyen-Huu and Pottier (2016), the introduction of bankruptcy by firms in the original Keen (1995) model can prevent debt ratios from increasing indefinitely and lead to limit cycles instead. The same mechanism can prevent households from accumulating increasingly large debt ratios. As default also implies losses to asset holders, the corresponding redistribution of wealth and income can decrease inequality, much as the losses caused by the turmoil of wars and depression did in the first half of the last century, as documented in Piketty (2014). Finally, as shown in Costa Lima et al. (2014), government intervention, both in the form of taxes and government spending, can destabilize the explosive equilibrium in the original Keen (1995) model, as well as prevent employment from remaining arbitrarily low, provided the intervention is responsive enough at periods of crisis. We expect that a similar stabilization role can be played by income and wealth taxes levied in a sufficiently progressive way. Moreover, the level of the short-run nominal interest rate turns out to have a long-lasting impact on the economy's trajectory. It is therefore to be expected that a countercyclical monetary policy (the Taylor rule is but one of them, aiming at stabilizing inflation) might help avoid a catastrophic steady-state. And since we have observed that, at large time scale, a trade-off between employment and inflation emerges along the trajectories that lead to the balanced growth path, such a monetary policy might as well try to favour employment.

Last but not least, an important test for the relevance of our framework will consist in bringing it to empirical data. This has been done with some success for the primal Keen model (see Grasselli and Maheshwari (2016) and Bovari et al. (2017)). It remains to be checked whether this can be done for our dual version, and its extension with two classes of households.

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References

Acemoglu, D. and Robinson, J. A. (2015). The rise and decline of general laws of capitalism. *Journal of Economic Perspectives*, 29(1):3–28.

Akerlof, G. A. and Stiglitz, J. E. (1969). Capital, wages and structural unemployment. *The Economic Journal*, 79(314):269–281.

Bastidas, D., Giraud, G., and McIsaac, F. (2017a). Multisectoral and monetary dynamics of a stock-flow consistent open-economy. Working-paper, AFD.

Bastidas, D., Giraud, G., and McIsaac, F. (2017b). Growth, underemployment, capital and international trades in brazil: a multisectorial, monetary and stock-flow consistent analysis. Working-paper, AFD.

Bovari, E., Giraud, G., McIsaac, F., and Zatsepina, E. (2017). Coping with the collapse: A stock-flow consistent monetary macrodynamics of global warming. Working-paper 2017-29bis, AFD.

Costa Lima, B., Grasselli, M., Wang, X.-S., and Wu, J. (2014). Destabilizing a stable crisis: Employment persistence and government intervention in macroeconomics. *Structural Change and Economic Dynamics*, 30:30 – 51.

Debreu, G. (1974). Excess demand functions. *Journal of Mathematical Economics*, 1(1):15–21.

Dutt, A. K. (2006). Maturity, stagnation and consumer debt: A steindlian approach. *Metroeconomica*, 57(3):339–364.

Eggertsson, G. and Krugman, P. (2011). Debt, deleveraging, and the liquidity trap: A fisher-minsky-koo approach. *Quarterly Journal of Economics*, 3(127):1469–1513.

Friedman, M. (1968). The role of monetary policy. *The American Economic Review*, 58(1):1–17.

Giraud, G. (2017). Turning piketty right side up: Inequality and growth in *capital in the twenty-first century*. Working-paper, AFD.

Giraud, G. and Pottier, A. (2016). Debt-deflation versus liquidity trap: the dilemma of non-conventional monetary policy. *Economic Theory*, 1(62):383–408.

Goodwin, R. M. (1967). A growth cycle. In Feinstein, C. H., editor, *Socialism, Capitalism and Economic Growth*, pages 54–58, London. Cambridge University Press.

Gordon, J. (2014). The Demise of U.S. Economic Growth: Restatement, Rebuttal, and Reflections. NBER Working Papers 19895, National Bureau of Economic Research, Inc.

Gordon, R. (2011). The history of the phillips curve: Consensus and bifurcation. *Economica*, 111(78):10–50.

Gordon, R. (2013). The phillips curve is alive and well: Inflation and the nairu during the slow recovery. NBER Working Papers 19390, National Bureau of Economic Research, Inc.

Grasselli, M. R. and Costa Lima, B. (2012). An analysis of the Keen model for credit expansion, asset price bubbles and financial fragility. *Mathematics and Financial Economics*, 6(3):191–210.

Grasselli, M. R. and Maheshwari, A. (2016). Econometric estimation of Goodwin growth models. Submitted to Metroeconomica.

Grasselli, M. R. and Nguyen Huu, A. (2015). Inflation and speculation in a dynamic macroeconomic model. *Journal of Risk and Financial Management*, 8(3):285.

Gregory, N. M. (2001). The inexorable and mysterious tradeoff between inflation and unemployment. *The Economic Journal*, 111(471):45–61.

Gregory, N. M. (2014). *Principles of macroeconomics*. Cengage Learning.

Hansen, A. H. (1939). Economic progress and declining population growth. *The American Economic Review*, 29(1):1–15.

Kalecki, M. (1971). *Selected Essays on the Dynamics of the Capitalist Economy*. Cambridge University Press.

Keen, S. (1995). Finance and economic breakdown: Modeling Minsky's "Financial Instability Hypothesis". *Journal of Post Keynesian Economics*, 17(4):607–635.

Modigliani, F. and Papademos, L. (1975). Targets for monetary policy in the coming year. *Brookings Papers on Economic Activity*, 1975(1):141–165.

Nguyen-Huu, A. and Pottier, A. (2016). Debt and investment in the keen model: a reappraisal of modeling minsky. Preprint.

Palley, T. I. (1994). Debt, aggregate demand, and the business cycle: An analysis in the spirit of kaldor and minsky. *Journal of Post Keynesian Economics*, 16(3):371–390.

Phelps, E. S. (1968). Money-wage dynamics and labor-market equilibrium. *Journal of Political Economy*, 76(4, Part 2):678–711.

Piketty, T. (2014). *Capital in the Twenty-First Century*. Harvard University Press.

Skott, P. (2013). Increasing inequality and financial instability. *Review of Radical Political Economics*, 45(4):478–488.

Solow, R. M. (1956). A contribution to the theory of economic growth. *The Quarterly Journal of Economics*, 70(1):65–94.

Steindl, J. (1952). *Maturity and stagnation in American capitalism*. Blackwell Oxford.

Stiglitz, J. E. (2015). New theoretical perspectives on the distribution of income and wealth among individuals: Part i. the wealth residual. *NBER Working Paper*, (21189).

Summers, L. (2016). The age of secular stagnation: What it is and what to do about it. Foreign Affairs.

Tobin, J. (1980). Stabilization Policy Ten Years After. *Brookings Papers on Economic Activity*, 11(1, Tenth):19–90.

A Local stability analysis for (28)

A.1 Interior equilibria

Due to the similarity with the monetary version of Keen's model, studied by Grasselli and Nguyen-Huu (2015), the local stability turns out to be quite similar. Assume the existence of an interior equilibrium, as defined by (31), (32), (33). The Jacobian matrix of the system (28) is

$$\begin{bmatrix} \Phi(\lambda) - \alpha + (1-\gamma)\eta_p(1-2\omega m) & \omega\Phi'(\lambda) & 0 \\ -\lambda\frac{c'(\omega - rd_h)}{\nu} & \frac{1-c(\omega-rd_h)}{\nu} - (\alpha+\beta+\delta) & \lambda r\frac{c'(\omega-rd_h)}{\nu} \\ c'(\omega - rd_h)(\frac{d_h}{\nu}+1) - 1 - d\eta_p m & 0 & K_4 \end{bmatrix}$$

with $K_4 := r + \delta - \frac{1-c(\omega-rd_h)}{\nu} - i(\omega) - rc'(\omega - rd_h)(1+\frac{d_h}{\nu})$. At the equilibrium point, $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$, this matrix becomes

$$J(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1) = \begin{bmatrix} K_0 & K_1 & 0 \\ -K_2 & 0 & rK_2 \\ K_3 - \eta_p m\overline{d}_1 & 0 & K_4 \end{bmatrix},$$

with $K_0 := (\gamma-1)\eta_p m\overline{\omega}_1 < 0$, $K_1 := \overline{\omega}_1\Phi'(\overline{\lambda}_1) > 0$ and $K_2 := \frac{\overline{\lambda}_1}{\nu}c'(\eta) > 0$ having obvious signs, and

$$K_3 := c'(\eta)\Big(\frac{\overline{d}_1}{\nu}+1\Big) - 1 \quad \text{ and } K_4 := r\Big[1 - c'(\eta)(1+\frac{\overline{d}_1}{\nu})\Big] - \big(\alpha+\beta+i(\overline{\omega}_1)\big).$$

With these notations at hand, the characteristic polynomial of $J(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ is quite similar to the one studied by Grasselli and Nguyen-Huu (2015), and the discussion of conditions for local stability are parallel. Details are left to the reader.

A.2 Explosive equilibria

Next, the Jacobian associated to modification of (28) after the change in variable $u := 1/d_h$ at the equilibrium $(0,0,0)$ is given by:

$$\begin{bmatrix} \Phi(0) - \alpha + (1-\gamma)\eta_p & 0 & 0 \\ 0 & \frac{1-c_\pm}{\nu} - (\alpha+\beta+\delta) & 0 \\ 0 & 0 & \frac{1-c_\pm}{\nu} - \delta - \eta_p - r \end{bmatrix},$$

where we used the fact that $c(-1/u) \to c_- > 0$ if $u \to 0^+$ (that is, $d_h \to +\infty$) and $c(-1/u) \to c_+ \leq 1$ if $u \to 0^-$ (that is, $d_h \to -\infty$). The necessary and sufficient condition for such equilibrium to be locally stable are therefore

$$\Phi(0) < \alpha - (1-\gamma)\eta_p, \quad 1 - \nu(\alpha+\beta+\delta) < c_\pm, \quad \frac{1-c_\pm}{\nu} - \delta - \eta_p < r. \tag{77}$$

A.3 Deflationary equilibria

Finally, when applied to the deflationary equilibrium $(\overline{\omega}_3, 0, \overline{d}_3)$ with $\overline{d}^3 < \infty$ described in subsection 2.4, the Jacobian becomes

$$\begin{bmatrix} (\gamma-1)\eta_p m\overline{\omega}_3 & \overline{\omega}_3\Phi'(0) & 0 \\ 0 & \frac{1-c(\overline{\omega}_3 - r\overline{d}_3)}{\nu} - (\alpha+\beta+\delta) & 0 \\ \widetilde{K}_3 - \eta_p m\overline{d}_3 & 0 & \widetilde{K}_4 \end{bmatrix},$$

where
$$\widetilde{K}_3 := c'(\overline{\omega}_3 - r\overline{d}_3)(\frac{\overline{d}_3}{\nu} + 1) - 1$$
and
$$\widetilde{K}_4 := r + \delta - \frac{1 - c(\overline{\omega}_3 - r\overline{d}_3)}{\nu} - i(\overline{\omega}_3) - rc'(\overline{\omega}_3 - r\overline{d}_3)(1 + \frac{\overline{d}_3}{\nu}).$$

The determinant of this matrix readily yields its eigenvalues, and therefore the necessary and sufficient conditions for local stability:
$$(\gamma - 1)\eta_p\overline{\omega}_3 < 0\ ,\ \frac{1 - c(\overline{\omega}_3 - r\overline{d}_3)}{\nu} < \alpha + \beta + \delta \ \text{ and } \widetilde{K}_4 < 0.$$
While the first condition is always met, the two others are imposing non-trivial restrictions. Since the real growth rate of output at this equilibrium is given by
$$g = \frac{1 - c(\overline{\omega}_3 - r\overline{d}_3)}{\nu} - \delta,$$
the second condition says that the real growth rate at this long-run new equilibrium must be lower than the growth rate at an interior equilibrium, namely $\alpha + \beta$.

Lastly, when $\overline{d}_3 = \pm\infty$ we have the following Jacobian matrix at equilibrium:
$$\begin{bmatrix} \Phi(0) - \alpha + (1-\gamma)\eta_p & \overline{\omega}_3\Phi'(0) & 0 \\ 0 & \frac{1-c_\pm}{\nu} - (\alpha+\beta+\delta) & 0 \\ 0 & 0 & \frac{1-c_\pm}{\nu} - \delta + i(\overline{\omega}_3) - r \end{bmatrix}.$$
We therefore see that this equilibrium is locally stable if an only if the following variant of (77) holds:
$$\Phi(0) < \alpha - (1-\gamma)\eta_p, \quad 1 - \nu(\alpha+\beta+\delta) < c_\pm, \quad \frac{1-c_\pm}{\nu} - \delta + i(\overline{\omega}_3) < r. \tag{78}$$

B Local stability analysis for (52)

B.1 Interior equilibrium

First note that our standing assumption on the marginal propensities of consumption by workers and investors, namely $c_w'(y_w) > c_i'(y_i)$, implies that
$$\begin{aligned} \frac{\partial c}{\partial \omega} &= c_w'(y_w) - \Theta c_i'(y_i) > 0 \\ \frac{\partial c}{\partial d_w} &= r\left[\Theta c_i'(y_i) - c_w'(y_w)\right] < 0 \\ \frac{\partial c}{\partial d_i} &= -r(1-\Theta)c_i'(y_i) < 0 \end{aligned}$$
Next, let $x \in \mathbb{R}^4$ denote the point $(\omega, \lambda, d_w, d_i)$, so that the interior equilibrium defined in Section (3.3.1) is denoted by $\overline{x}_1$. In addition, we write
$$\overline{y}_{w1} = \overline{\omega}_1 - r\overline{d}_{w1}, \quad \overline{y}_{i1} = \Theta(1 - \overline{\omega}_1 + r\overline{d}_{w1} - \delta\nu) - r(1-\Theta)r\overline{d}_{i1}$$
for the disposable income of workers and investors at this equilibrium. The Jacobian matrix of system (52) at this interior equilibrium is then
$$J_1 = J\left(\overline{\omega}_1, \overline{\lambda}_1, \overline{d_{w1}}, \overline{d_{i1}}\right) = \begin{bmatrix} K_0 & K_1 & 0 & 0 \\ K_2 & 0 & K_3 & K_4 \\ K_5 & 0 & K_6 & K_7 \\ K_8 & 0 & K_9 & K_{10} \end{bmatrix} \tag{79}$$

with the terms

$$K_0 = -(1-\gamma)\eta_p m\overline{\omega}_1 < 0, \quad K_1 = \overline{\omega}_1 \Phi'(\overline{\lambda}_1) > 0, \quad K_2 = -\frac{\overline{\lambda}_1}{\nu}\frac{\partial c}{\partial \omega}(\overline{x}_1) < 0$$
$$K_3 = -\frac{\overline{\lambda}_1}{\nu}\frac{\partial c}{\partial d_w}(\overline{x}_1) > 0, \quad K_4 = -\frac{\overline{\lambda}_1}{\nu}\frac{\partial c}{\partial d_i}(\overline{x}_1) > 0, \quad K_7 = -\frac{\overline{d}_{w1}}{\nu} r(1-\Theta)c_i'(y_i) < 0$$

having well-defined signs, and the following terms

$$\begin{aligned}
K_5 &= \overline{d}_{w1}\left[\frac{1}{\nu}(c_w'(\overline{y}_{w1}) - \Theta c_i'(\overline{y}_{i1})) - \eta_p m\right] + c_w'(\overline{y}_{w1}) - 1, \\
K_6 &= \delta - i(\overline{\omega}_1) - \frac{1-c(\overline{x}_1)}{\nu} + r\left[1 + \frac{\overline{d}_{w1}}{\nu}\Big(\Theta c_i'(\overline{y}_{i1}) - c_w'(\overline{y}_{w1})\Big) - c_w'(\overline{y}_{w1})\right] \\
K_8 &= \overline{d}_{i1}\left[\frac{1}{\nu}(c_w'(\overline{y}_{w1}) - \Theta c_i'(\overline{y}_{i1})) - \eta_p m\right] + \Theta[1 - c_i'(\overline{y}_{i1})] \\
K_9 &= r\left[\frac{\overline{d}_{i1}}{\nu}\Big(\Theta c_i'(\overline{y}_{i1}) - c_w'(\overline{y}_{w1})\Big) + c_i'(\overline{y}_{i1}) - \Theta\right] \\
K_{10} &= \delta - i(\overline{\omega}_1) - \frac{1-c(\overline{x}_1)}{\nu} + (1-\Theta)r\left[1 - \frac{\overline{d}_{i1}}{\nu}c_i'(y_i) - c_i'(\overline{y}_{i1})\right]
\end{aligned} \tag{80}$$

having signs that depend on the parameters and the form of the consumption function. One can then find the characteristic polynomial for J_1 in terms of the K_i above and obtain the corresponding Routh-Hurwitz conditions for stability, which then need to be checked numerically for given values of the parameters.

B.2 Local stability of income divergence

The Jacobian of (64) associated with the equilibrium $(\omega, \lambda, u_w, u_i) = (0,0,0,0)$ is

$$\begin{bmatrix}
\Phi(0) - \alpha + (1-\gamma)\eta_p & 0 & 0 & 0 \\
 & \frac{1-c_0}{\nu} - (\alpha+\beta+\delta) & 0 & 0 \\
(*) & & \frac{1-c_0}{\nu} + i - r - \delta & 0 \\
 & & & \frac{1-c_0}{\nu} + i - r(1-\Theta) - \delta
\end{bmatrix}$$

where we used the fact that $c(\omega, d_w, d_i) \to c_0$ depending on which combination of $u_w \to 0^{\pm}$ and $u_i \to 0^{\pm}$ is considered. Namely, recalling that we are considering trajectories where d_i grows faster than d_w, we have that

$$c_0 = \begin{cases} c_{w^-} + c_{i^+} & \text{if } u_w \to 0^+, u_i \to 0^- \\ c_{w^-} + c_{i^-} & \text{if } u_w \to 0^+, u_i \to 0^+ \\ c_{w^+} + c_{i^-} & \text{if } u_w \to 0^-, u_i \to 0^+ \\ c_{w^+} + c_{i^+} & \text{if } u_w \to 0^-, u_i \to 0^- \end{cases} \tag{81}$$

Consequently, the necessary and sufficient conditions for stability of the corresponding equilibrium are

$$\Phi(0) < \alpha - (1-\gamma)\eta_p, \quad 1 - \nu(\alpha+\beta+\delta) < c_0, \quad \text{and} \quad \frac{1-c_0}{\nu} - \delta - \eta_p < r(1-\Theta). \tag{82}$$

Finally, the Jacobian of (65) at the equilibria $(\omega, \lambda, u_w, v) = (0,0,0,v)$, with $v = 0$ and $v = -1$, is identical to (B.2) except for the last row, which now has $(1+2v)r\Theta$ as the diagonal entry. We therefore see that, provided the conditions in (82) hold, the equilibrium with $v = 0$ is unstable, whereas the equilibrium with $v = -1$ is stable.

The analysis of stability of the deflationary equilibria of the type $(\overline{\omega}_3, \overline{\lambda}_3, \overline{d}_{w3}, \overline{d}_{i3}) = (\overline{\omega}_3, 0, \overline{d}_{w3}, \overline{d}_{i3})$ follows similarly to that of Appendix A.3 above and is not included here for space reasons.

C Parameters and functions for numerical simulations

The baseline parameters for our simulations are provided in Table 3. Alternative values for some specific parameters are provided in the legend of each figure. We use a Philips curve of the form

$$\Phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0, \tag{83}$$

with parameters specified in Table 3. For Figure 1, which illustrates the dual Keen model of Section 2, we used a consumption function for the aggregate household sector of the form

$$c(y) = \max\left\{c_-, A_c + \frac{K_c - A_c}{(C_c + Q_c e^{-B_c y})^{1/\nu_c}}\right\} \tag{84}$$

that is to say, a generalized logistic function truncated at c_- on the negative half-line, with parameters given in Table 3.

Symbol	Value	Description
ν	3	capital-to-output ratio
α	0.025	productivity growth rate
β	0.02	population growth rate
δ	0.05	depreciation rate
r	0.03	real interest rate
η_p	0.35	adjustment speed for prices
m	1.6	markup factor
γ	0.8	inflation sensitivity in the bargaining equation
ϕ_0	0.0401	Philips curve parameter
ϕ_1	6.41×10^{-5}	Philips curve parameter
c_-	0.03	hard lower bound for consumption function
A_c	0	asymptotic lower bound for consumption function
B_c	0.5	growth rate for consumption function
K_c	0.9	upper bound for consumption function
C_c	1	consumption function parameter
ν_c	0.2	consumption function parameter
Q_c	1.0123	consumption function parameter

Table 3: Baseline parameter values