A mean-field approximation to stock-flow consistent agent-based models with state-dependent transition rates

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Abstract

We generalize the stock-flow consistent agent-based macroeconomic model proposed in [7] to the case of state-dependent transition probabilities between types of agents. We then propose a mean-field approximation, obtain the master equation associated with it, and the corresponding first and second order terms in a series expansion with respect to an appropriate scaling of the total number of agents. The first order term corresponds to the ordinary differential equation governing the deterministic mean of the fraction of agents of one type, whereas the second order term is the partial differential equation satisfied by the density of random perturbations around the mean. We perform numerical experiments to test the accuracy of the approximation and give examples of sensitivity analyses with respect to some of the parameters. We then use the model to investigate the relationship between stock markets with low returns and high volatility and the proportion of firms with fragile financial positions.

Keywords: stock-flow consistency, macroeconomics, mean-field approximation, master equation.

1 Introduction

Stock-flow consistent agent-based models (SF-ABM) are rapidly gaining popularity in the economic literature as a way to obtain more realistic macroeconomic models than the dominant Dynamic Stochastic General Equilibrium (DGSE) approach [2]. One practical limitation of this approach is its reliance on numerical simulations that can become prohibitively time-consuming in the interesting cases where the number of agents is very large. An appealing alternative to agent-based models for a large number of agents consists in the use of methods inspired by statistical physics as pioneered by [6] and further developed in [1]. In the specific context of SF-ABM, these techniques were used in [3] and [4] based on earlier work in [5].

Recent work in [7] uses a mean-field approximation to obtain fast and accurate simulations of a stock-flow consistent agent-based model that is inspired by [3] and [4], yet it is

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significantly different, both with respect to the assumptions for agent behaviour and the solution method. The model in [7] assumes that firms can be classified as either aggressive (Type 1) or conservative (Type 2) depending on their investment elasticity to past profits, whereas households can be classified as either non-investors (Type 1) or investors (Type 2) depending on whether or not they invest a portion of their wealth in the stock market. By contrast, the households in [3] are classified as borrowing and non-borrowing, depending on their financial position at the end of each time period, and firms in [4] are classified as speculative or hedge, again depending on their financial position at the end of each time period. Crucially, transitions between types occur independently and according to exogenously given constant transition rates in [7], whereas they depend on endogenous thresholds in [3] and [4].

The purpose of this paper is to generalize the results of [7] for the case where transition rates depend on the fraction of agents (i.e firms or households) of a given type (say Type 1 for concreteness), thereby introducing dependence between the agents, since each firm or household is affected by the state of all other firms or households, as well as endogeneity in the determination of the transition rates. Section 2 briefly recalls the stock-flow consistent agent-based model of [7] and introduces the state-dependent transition rates in equations (5) and (6).

Section 3 generalizes the mean-field approximation for this class of rates along the lines developed in [1]. In particular, we show that, if we write the fraction of firms of type 1 as a deterministic trend $\phi(t)$ plus stochastic fluctuations of the form $\xi(t)/\sqrt{N}$ and perform a series expansion of corresponding master equation, then we can obtain the ordinary different equation (69) for ϕ and the partial differential equation (72) for the density of $\xi(t)$, both in terms of the functions entering the specification of the transition probabilities. The general forms of these differential equation obtained here were not presented in [1], where only a few special cases are discussed, which we reproduce in Examples 1 to 3.

In Section 4 we elaborate on a class of transition probabilities discussed in [1], namely the state-dependent transition probabilities that arise when agents try to evaluate the relative merits of being of one type instead of another. If agents estimate that, when the fraction of agents of type 1 is x, the gains for being of type 1 are normally distributed with mean $g(x)$ and variance $2/(\pi\beta^2)$, then the probability of positive gains can be approximated by the function $\eta_1(x)$ given in (89). Adopting this function in the specification of the transition probabilities in (5) means that agents will be more likely to switch to type 1 when their estimate of the probability of positive gains for being in this class is higher. We investigate this class of transition probabilities for four different specifications of the function $g(x)$ and discuss the qualitative aspects of the dynamics that they generate.

Section 5 explores numerical examples for some of the transition rates discussed throughout the paper. We first show that the mean-field (MF) approximation gives rise to accurate proportions of the number of agents of each time when compared with the proportions obtained in the full agent-based model (ABM). Moreover, the MF approximation is also accurate for the calculation of aggregate variables such as equity prices and total economic output. We then use the MF approximation to explore the parameter space in a way that would be prohibitively slow with the ABM simulations. Namely we test the sensitivity of the growth rate of equity prices and output with respect to two parameters that are difficult to estimate directly and therefore are likely to be obtained in practice through calibration, which is exactly where fast approximations are most needed.

We conclude the paper with a practical illustration of the use of the model in an indepth exploration of a specific research question, namely the relationship between macroeconomic financial stability and the financial health of individual firms. More concretely, we recast Minskys classification of firms as hedge, speculative, and Ponzi in terms of our model and ask whether it is the case that periods of financial instability coincide with an increase in the proportion of Ponzi firms. The answer provided by the model is unequivocally positive, as can be seen in Figures 11 and 12. We conclude the paper in Section 5 with suggestions for further generalizations of the model and the approach, in particular by introducing a higher degree of interaction between agents through transition rates that also depend on mean-field variables.

2 The agent-based model

We follow the setup of [7] closely and refer to it for any unexplained notation. The economy consists of a bank, N firms and M households. Letting a denote a constant productivity per unit of labour and c denote a constant labour cost per unit of output, the output Q_t produced by firms determines the labour demand $L_t = Q_t/a$ and the wage bill $W_t = cQ_t$. With a constant markup $\chi \geq 1$ over unit cost, the price of each unit of output is given by

$$p_t = \chi c, \tag{1}$$

so that the wage share of output is constant and given by $\omega = \frac{W_t}{pQ_t} = \frac{1}{\chi}$, and consequently the profit share of output is also constant and given by $\pi = 1 - \omega = \frac{\chi - 1}{\chi}$. Finally, each household supplies L_t/M units of labour at time t, thereby receiving a wage rate $\mathrm{w}_t = \frac{W_t}{M} = ca\frac{L_t}{M}$, which we assume to be the same for all households.

Firm $n = 1, \ldots, N$ has capital with nominal value pk_t^n, net debt with nominal value b_t^n and ϵ_t^n shares at average price p_t^{en}. Treating the shares as liabilities leads to a net worth equal to $v_t^n = pk_t^n - b_t^n - p_t^{en}\epsilon_t^n$, whereas considering only debt as a liability leads to a shareholder equity (i.e "book value") equal to $\mathcal{E}^n = pk_t^n - b_t^n$.

Similarly, household $m = 1, \ldots, M$ has assets consisting of e_t^m shares at average price p_t^{em} and cash balances d_t^m deposited at the bank, leading to a net worth $v_t^m = p_t^{em}e_t^m + d_t^m$. Finally, the assets of the bank consist of the aggregate net borrowing by firms

$$B_t = \sum_{n=1}^{N} b_t^n, \tag{2}$$

plus cash reserves R_t, whereas its liabilities consist of aggregate net deposits of households

$$D_t = \sum_{m=1}^{M} d_t^m, \tag{3}$$

leading to a net worth of the form $V_t^b = B_t + R_t - D_t$.

Regarding equities, assume a homogenous behaviour for firms with respect to dividend payments and share issuance and buyback. Assume further that, instead of trading in shares for individual companies, investors buy and sell shares of an aggregated fund at a common price p_t^e, which in turn buys and sells shares from firms. The price p_t^e is then determined by an equilibrium condition for the supply and demand for equities under the constraint that

$$\sum_{n=1}^{N} e_t^n = E_t = \sum_{m=1}^{M} e_t^m. \tag{4}$$

We use $z_t^n \in \{1,2\}$ and $\zeta_t^m \in \{1,2\}$ to denote the type for firm n and household m at time t, respectively. Accordingly, we denote by N_t^1 and M_t^1 the number of firms and households of type 1 at time t and by $n_t^1 = N_t^1/N$ and $m_t^1 = M_t^1/M$ the corresponding proportions.

We further introduce the vectors $\boldsymbol{k}_t, \boldsymbol{b}_t, \boldsymbol{\epsilon}_t, \boldsymbol{z}_t \in \mathbb{R}^N$ to denote the capital, debt, number of issued shares, and type of the N firms at time t and $\boldsymbol{d}_t, \boldsymbol{e}_t, \boldsymbol{\zeta}_t \in \mathbb{R}^M$ to denote the deposits, number of owned shares, and type of the M households at time t. The agent-based model in [7] then consists of rules for updating the state variables $(Q_t, p_t^e, \boldsymbol{k}_t, \boldsymbol{b}_t, \boldsymbol{\epsilon}_t, \boldsymbol{z}_t, \boldsymbol{d}_t, \boldsymbol{e}_t, \boldsymbol{\zeta}_t)$, assuming that firms and households change their types according to constant, exogenously given probabilities.

In this paper, we drop the constraint of constant transition probabilities and assume instead that firm n changes type at time t according to the following probabilities:

$$
\begin{aligned}
&P(z_{t+1}^n = 1 | z_t^n = 2) = \lambda^f := \Psi^f(N)\bar{\lambda}^f \eta_1^f\left(n_t^1\right)\\
&P(z_{t+1}^n = 2 | z_t^n = 2) = 1 - \lambda^f\\
&P(z_{t+1}^n = 2 | z_t^n = 1) = \mu^f := \Psi^f(N)\bar{\mu}^f \eta_2^f\left(n_t^1\right)\\
&P(z_{t+1}^n = 1 | z_t^n = 1) = 1 - \mu^f
\end{aligned}
\tag{5}
$$

where $\bar{\lambda}^f$ and $\bar{\mu}^f$ are constants, $\eta_1^f(\cdot), \eta_2^f(\cdot)$ are functions of the current fraction of firms of type 1, and $\Psi^f(N)$ is a scaling factor. In other words, each firm decides to change type according to probabilities that are the same across all firms, but depend on the overall size of the system through the function $\Psi^f(\cdot)$ and current state of the system through the proportion of aggressive firms n_t^1.

Similarly, household m changes type at time t according to the following probabilities:

$$
\begin{aligned}
&P(\zeta_{t+1}^n = 1 | \zeta_t^m = 2) = \lambda^h := \Psi^h(M)\bar{\lambda}^h \eta_1^h\left(m_t^1\right)\\
&P(\zeta_{t+1}^m = 2 | \zeta_t^m = 2) = 1 - \lambda^h\\
&P(\zeta_{t+1}^m = 2 | \zeta_t^m = 1) = \mu^h := \Psi^h(M)\bar{\mu}^h \eta_2^h\left(m_t^1\right)\\
&P(\zeta_{t+1}^m = 1 | \zeta_t^m = 1) = 1 - \mu^h
\end{aligned}
\tag{6}
$$

where $\bar{\lambda}^h$ and $\bar{\mu}^h$ are constants, $\eta_1^h(\cdot), \eta_2^h(\cdot)$ are functions of the current fraction of households of type 1, and $\Psi^h(M)$ is a scaling factor. In other words, each household decides to change type according to probabilities that are the same across all households, but depend on the overall size of the system through the function $\Psi^h(\cdot)$ and current state of the system through the proportion of non-investor households m_t^1. Observe that setting $\eta_1^f = \eta_2^f = \Psi^f \equiv 1$ and $\Psi^h = \eta_1^h = \eta_2^h \equiv 1$ leads to the model with constant transition probabilities analyzed in [7].

For the rest of the model, namely the stock-flow dynamics of the balance-sheet variables of firms and households, we adopt the same specifications as in [7]. Namely, let α_z, β and γ denote, respectively, the sensitivity of investment by firms to gross profits, capacity utilization, and current level of debt, so that the investment demand of firm n at $t+1$ is given by

$$
i_{t+1}^n = (\alpha_{z_t^n}\pi + \beta)pq_t^n - \gamma b_t^n,
\tag{7}
$$

where it is assumed that the portion of total production Q_t allocated to firm n is

$$
q_t^n = \frac{k_t^n}{\sum_{n=1}^N k_t^n} Q_t,
\tag{8}
$$

that is, firms with higher capital attract a larger portion of total demand. It is further assumed that $\alpha_1 > \alpha_2$, that is to say, investment by aggressive firms is more sensitive to gross profits than for conservative ones. We then have that the capital for firm n at time $t+1$ is given by

$$pk^n_{t+1} = i^n_{t+1} + (1-\delta)pk^n_t, \tag{9}$$

where δ is a constant depreciation rate.

Denote further by $s^y_\zeta, s^w_\zeta \in [0,1]$ the saving rates from income and wealth for a household of type ζ, so the amount saved by household m in the period $[t, t+1]$ is

$$s^m_{t+1} = y^m_{t+1} - c^m_{t+1} = s^y_{\zeta^m_t} y^m_{t+1} - (1 - s^v_{\zeta^m_t})v^m_t, \tag{10}$$

where

$$y^m_{t+1} = (1-\pi)p\frac{Q_{t+1}}{M} + rd^m_t + \delta^e p^e_t e^m_t, \tag{11}$$

is the income received by household m when total production is equal to Q_{t+1}. As can be seen in the expression above, this income consists of wages (calculated by dividing the total wage bill $(1-\pi)Q_{t+1}$ among M households), plus interest income on deposits (calculated by multiplying a constant interest rate r by the value of deposits held at time t), plus dividend income on shares (calculated by multiplying a constant dividend yield δ^e by the value of shares held at time t). We assume that $s^y_1 \leq s^y_2$ and $s^v_1 \leq s^v_2$, so that non-investor households spend a higher proportion of both income and wealth than investor households.

Consider next the following aggregate variables:

$$K_t = \sum_{n=1}^{N} k^n_t, \qquad Q^1_t = \sum_{\{z^n_t = 1\}} q^n_t, \qquad Q^2_t = \sum_{\{z^n_t = 2\}} q^n_t, \tag{12}$$

$$D^1_t = \sum_{\{\zeta^m_t = 1\}} d^m_t, \qquad D^2_t = \sum_{\{\zeta^m_t = 2\}} d^m_t, \qquad S^2_{t+1} = \sum_{\{\zeta^m_{t+1} = 2\}} s^m_{t+1} \tag{13}$$

$$D^{2,t+1}_t = \sum_{\{\zeta^m_{t+1} = 2\}} d^m_t, \quad E^{2,t+1}_t = \sum_{\{\zeta^m_{t+1} = 2\}} e^m_t \tag{14}$$

In these expressions, notice that the lower time index refers to the time in which the summands are evaluated, whereas the upper time index refers to the time in which the type z^m_{t+1} is evaluated. When these two times are equal we suppress the upper index. For example, Q^1_t denotes the total production of firms of type 1 at time t, whereas $D^{2,t+1}_t$ is the sum of deposits that were held at time t by households that are of type 2 at time $t+1$. Finally, define the auxiliary variables

$$I_{t+1} = \pi p(\alpha_1 Q^1_t + \alpha_2 Q^2_t) + \beta p Q_t - \gamma B_t \tag{15}$$

$$A_{t+1} = \pi p Q_{t+1} - rB_t - \delta p K_t - \delta^e p^e_t E_t \tag{16}$$

$$\begin{aligned} F_t = &\pi p(\alpha_1 Q^1_t + \alpha_2 Q^2_t) + \beta p Q_t - \gamma B_t + (1-s^y_1)rD^1_t + (1-s^v_1)D^1_t \\ &+ (1-s^y_2)(rD^2_t + \delta^e p^e_t E_t) + (1-s^v_2)(D^2_t + p^e_t E_t). \end{aligned} \tag{17}$$

Using the same arguments as in [7], it follows that if we know the time-t values of the types $\boldsymbol{z}_t, \boldsymbol{\zeta}_t$, total production Q_t, equity price p^e_t and balance sheet variables $\boldsymbol{k}_t, \boldsymbol{b}_t, \boldsymbol{\epsilon}_t, \boldsymbol{d}_t, \boldsymbol{e}_t$, then the types at time $(t+1)$ are obtained according to the transition probabilities in (5) and (6), that is:

$$z^n_{t+1}|_{z^n_t=2} = \begin{cases} 1 & \text{with probability } \lambda^f \\ 2 & \text{with probability } 1-\lambda^f \end{cases} \tag{18}$$

$$z^n_{t+1}|_{z^n_t=1} = \begin{cases} 1 & \text{with probability } 1-\mu^f \\ 2 & \text{with probability } \mu^f \end{cases} \tag{19}$$

$$\zeta^m_{t+1}|_{\zeta^m_t=2} = \begin{cases} 1 & \text{with probability } \lambda^h \\ 2 & \text{with probability } 1-\lambda^h \end{cases} \tag{20}$$

$$\zeta^m_{t+1}|_{\zeta^m_t=1} = \begin{cases} 1 & \text{with probability } 1-\mu^h \\ 2 & \text{with probability } \mu^h \end{cases} \tag{21}$$

$$\tag{22}$$

wheres the time-$(t+1)$ values of the other variables can be obtained sequentially as follows:

$$Q_{t+1} = \frac{F_t/p}{1-(1-\pi)[(1-s^y_1)m^1_t + (1-s^y_2)m^2_t]} \tag{23}$$

$$p^e_{t+1} = \frac{\varphi\left(D^{2,t+1}_t + S^2_{t+1}\right) - (1-\varpi)\left[I_{t+1} - \delta p K_t - A_{t+1}\right]}{E_t - \varphi E^{2,t+1}_t} \tag{24}$$

$$k^n_{t+1} = (\alpha_{z^n_t}\pi + \beta)\frac{k^n_t}{K_t}Q_t - \gamma\frac{b^n_t}{p} + (1-\delta)k^n_t \tag{25}$$

$$b^n_{t+1} = [1+\varpi(r-\gamma)]b^n_t + \varpi[(\alpha_{z^n_t}\pi+\beta)p\frac{k^n_t}{K_t}Q_t - \pi p\frac{k^n_{t+1}}{K_{t+1}}Q_{t+1} + \delta^e p^e_t \epsilon^n_t] \tag{26}$$

$$\epsilon^n_{t+1} = \epsilon^n_t + \frac{(1-\varpi)[(\alpha_{z^n_t}\pi+\beta)p\frac{k^n_t}{K_t}Q_t - \pi p\frac{k^n_{t+1}}{K_{t+1}}Q_{t+1} + (r-\gamma)b^n_t + \delta^e p^e_t \epsilon^n_t]}{p^e_{t+1}} \tag{27}$$

$$d^m_{t+1} = \begin{cases} v^m_t + s^m_{t+1} + (p^e_{t+1} - p^e_t)e^m_t & \text{if } z^m_{t+1} = 1 \\ (1-\varphi)(v^m_t + s^m_{t+1} + (p^e_{t+1} - p^e_t)e^m_t) & \text{if } z^m_{t+1} = 2 \end{cases} \tag{28}$$

$$e^m_{t+1} = \begin{cases} 0 & \text{if } z^m_{t+1} = 1 \\ \frac{\varphi[v^m_t + s^m_{t+1} + (p^e_{t+1} - p^e_t)e^m_t]}{p^e_{t+1}} & \text{if } z^m_{t+1} = 2 \end{cases} \tag{29}$$

3 Mean-Field Approximation

As can be seen from (18)-(22) and (23)-(29), the agent-based model (ABM) of the previous section requires the computation of four variables (the type z, capital k, debt b, and equity ϵ) for each firm and three variables for each household (the type ζ, deposits d, and share holdings e) for each household, in addition to the total production Q and equilibrium stock price p^e. Naturally, the model becomes very computationally intensive for a realistic number of firms and households. The purpose of this section is to extend the mean-field (MF) approximation introduced in [7] to the case of the state-dependent transition probabilities in (5) and (6). This then allows for much faster simulations of the model, which are particularly useful for the kind of exploration of the parameter space that is necessary in the absence of close-form analytic expressions to calibrate the model.

The key idea of the MF approximation consists of dividing the entire population of firms into two classes according to their type, namely aggressive and conservative, and only keep track of the average value of the variables of interest across each of the classes, with a similar procedure used for households.

Denote the mean-field value of a variable x for agents of type z at time t by $\overline{x}^z_t$, obtained by dividing the aggregate value of x for all agents of type z by the number of agents of type z. Its time evolution requires two steps: we first compute the deterministic value $\widetilde{x}^z_{t+1}$ *before* agents change type at time $t+1$ and then calculate the new mean-field value $\overline{x}^z_{t+1}$ taking into account the changes in type. This is necessary because agents carry their balance sheet items with them when they change type.

For example, the deterministic evolution of capital for firms of type z is assumed to be given by the mean-field analogue of equation (25), namely:

$$\widetilde{k}^z_{t+1} = (\alpha_z \pi + \beta)\frac{\overline{k}^z_t}{K_t} Q_t - \gamma \frac{\overline{b}^z_t}{p} + (1-\delta)\overline{k}^z_t, \tag{30}$$

where $\overline{k}^z_t$ and $\overline{b}^z_t$ are the mean-field capital and debt for firms of type z at time t. Now, since the expected value of the number of firms changing from type 2 to type 1 is $\lambda^f(N - N^1_t)$ (namely the transition probability for each firm multiplied by the number of firms of type 2 at time t) and the expected value of the number of firms changing from type 1 to type 2 at time $t+1$ is $\mu^f N^1_t$, we see that the expected aggregate capital of firms of type 1 after the change of type is

$$\lambda^f(N - N^1_t)\widetilde{k}^2_{t+1} + (1-\mu^f)N^1_t \widetilde{k}^1_{t+1}, \tag{31}$$

whereas the expected aggregate capital of firms of type 2 after the change of type is

$$\mu^f N^1_t \widetilde{k}^1_{t+1} + (1-\lambda^f)(N - N^1_t)\widetilde{k}^2_{t+1}. \tag{32}$$

We then declare that the mean-field values of capital at time $t+1$ to be these aggregate values divided by the realized number of firms of each type at time $t+1$.

To summarize, we set the mean-field values of x for firms of type z *after* a change of type at time $t+1$ as

$$\overline{x}^1_{t+1} = \frac{\lambda^f(N - N^1_t)\widetilde{x}^2_{t+1} + (1-\mu^f)N^1_t \widetilde{x}^1_{t+1}}{N^1_{t+1}} \tag{33}$$

and

$$\overline{x}^2_{t+1} = \frac{\mu^f N^1_t \widetilde{x}^1_{t+1} + (1-\lambda^f)(N - N^1_t)\widetilde{x}^2_{t+1}}{N - N^1_{t+1}}. \tag{34}$$

Similar expressions hold for mean-field variables for households, with M^ζ_t and M^ζ_{t+1} replacing N^z_t and N^z_{t+1}.

Analogously to what we did in the previous section, we want to show that given the time t values for the number of firms and households of each type, the total production Q_t, equity price p^e_t and mean-field balance sheet variables $\overline{k}^1_t, \overline{k}^2_t, \overline{b}^1_t, \overline{b}^2_t, \overline{\epsilon}^1_t, \overline{\epsilon}^2_t$ for firms and $\overline{d}^1_t, \overline{d}^2_t, \overline{e}^2_t$ for households (notice that $\overline{e}^1_t = 0$ for all t from the definition of a household of type 1), we can obtain their time $(t+1)$ values through appropriately modified versions of (18) to (29). As a first step, we let the change in type still be given by (18)–(22) and focus on the evolution of the other variables.

Start with the analogue of (10) for the mean-field savings for each type of households before a change of type at time $t+1$, namely:

$$\widetilde{s}^\zeta_{t+1} = s^y_\zeta \widetilde{y}^\zeta_{t+1} - (1 - s^v_\zeta)\overline{v}^\zeta_t, \tag{35}$$

where the mean-field income for the two types of households before a change of type is given by

$$\widetilde{y}^1_{t+1} = (1-\pi)p\widetilde{Q}_{t+1}/M + r\overline{d}^1_t \tag{36}$$

$$\widetilde{y}^2_{t+1} = (1-\pi)p\widetilde{Q}_{t+1}/M + r\overline{d}^2_t + \delta^e p^e_t \frac{E_t}{M - M^1_t}. \tag{37}$$

Observe that households of type 1 (non-investors) only receive income from wages and interest on deposit, whereas households of type 2 (investors) receive dividend income calculated as the total amount of dividends paid by firms $\delta^e p^e_t E_t$ divided by the number of investor households $(M - M^1_t)$. Accordingly, the mean-field wealth for the two types of households before changing type at $t+1$ are given by wealth at time t, plus savings, plus capital gains, that is

$$\widetilde{v}^1_{t+1} = \overline{v}^1_t + \widetilde{s}^1_{t+1} \tag{38}$$

$$\widetilde{v}^2_{t+1} = \overline{v}^2_t + \widetilde{s}^2_{t+1} + (p^e_{t+1} - p^e_t)\frac{E_t}{M - M^1_t}. \tag{39}$$

Next, compute the values $\overline{v}^1_{t+1}$ and $\overline{v}^2_{t+1}$ according to the rebalancing rules (33) and (34) (suitably modified for households instead of firms). Finally, define the mean-field analogues of (12) and (14) as

$$K_t = N^1_t \overline{k}^1_t + (N - N^1_t)\overline{k}^2_t, \qquad Q^1_t = \frac{\overline{k}^1_t}{K_t} Q_t N^1_t, \qquad Q^2_t = \frac{\overline{k}^2_t}{K_t} Q_t (N - N^1_t) \tag{40}$$

$$D^1_t = \overline{d}^1_t M^1_t, \qquad D^2_t = \overline{d}^2_t (M - M^1_t), \qquad S^2_{t+1} = \overline{s}^2_{t+1}(M - M^1_{t+1}) \tag{41}$$

$$D^{2,t+1}_t = \overline{d}^1_t \mu^h M^1_t + \overline{d}^1_t (1-\lambda^h)(M - M^1_t), \tag{42}$$

$$E^{2,t+1}_t = (1-\lambda^h)E_t, \tag{43}$$

which can be used to define the mean-field analogue of expression (15) to (17).

Using the same arguments as in [7], it follows that total production, equity price and mean-field balance sheet variables for households at time $t+1$ can be obtained as

$$Q_{t+1} = \frac{F_t/p}{1 - (1-\pi)[(1-s^y_1)m^1_t + (1-s^y_2)(1-m^1_t)]} \tag{44}$$

$$p^e_{t+1} = \frac{\varphi\left(D^{2,t+1}_t + S^2_{t+1}\right) - (1-\varpi)\left[I_{t+1} - \delta p K_t - A_{t+1}\right]}{[1 - \varphi(1-\lambda^h)]E_t} \tag{45}$$

$$\overline{d}^1_{t+1} = \overline{v}^1_{t+1}, \quad \overline{d}^2_{t+1} = (1-\varphi)\overline{v}^2_{t+1}, \quad \overline{e}^2_{t+1} = \frac{\varphi \overline{v}^2_{t+1}}{p^e_{t+1}} \tag{46}$$

Similarly, the capital, debt, and equity for firms before a change of type can be obtained as

$$\widetilde{k}^z_{t+1} = (\alpha_z \pi + \beta)\frac{\overline{k}^z_t}{K_t} Q_t - \gamma \frac{\overline{b}^z_t}{p} + (1-\delta)\overline{k}^z_t \tag{47}$$

$$\widetilde{b}^z_{t+1} = [1 + \varpi(r-\gamma)]\overline{b}^z_t + \varpi[(\alpha_z\pi + \beta)p\frac{\overline{k}^z_t}{K_t}Q_t - \pi p \frac{\widetilde{k}^z_{t+1}}{K_{t+1}} Q_{t+1} + \delta^e p^e_t \overline{\epsilon}^z_t] \tag{48}$$

$$\widetilde{\epsilon}^z_{t+1} = \overline{\epsilon}^z_t + \frac{(1-\varpi)[(\alpha_z\pi+\beta)p\frac{\overline{k}^z_t}{K_t}Q_t - \pi p \frac{\widetilde{k}^z_{t+1}}{K_{t+1}}Q_{t+1} + (r-\gamma)\overline{b}^z_t + \delta^e p^e_t \overline{\epsilon}^z_t]}{p^e_{t+1}} \tag{49}$$

from which we can obtain $\overline{k}^z_{t+1}, \overline{b}^z_{t+1}, \overline{\epsilon}^z_{t+1}$ by using the rebalancing rules (33) and (34).

The mean-field dynamics (44)-(49) is already considerably simpler than the agent-based dynamics (23)-(29). Further simplification can be achieved by treating the number of firms and households of each type as a continuous-time Markov chain, instead of keeping track of individual transitions for each agent as in (18)-(22). For this, consider the two-dimensional continuous-time Markov chain with state (N^1_t, M^1_t), that is, the numbers of aggressive firms and non-investor households at time t, and state space $\{0, 1, \dots, N\} \times \{0, 1, \dots, M\}$. Accordingly, we assume that the Markov chain at state (n, m) can jump to one of four neighbouring states $(n \pm 1, m \pm 1)$ with transition *rates* given by

$$\begin{aligned} b^f(n) &= \Psi^f(N)\bar{\lambda}^f \eta^f_1\left(\tfrac{n}{N}\right)(N-n), & d^f(n) &= \Psi^f(N)\bar{\mu}^f \eta^f_2\left(\tfrac{n}{N}\right) n \\ b^h(m) &= \Psi^h(M)\bar{\lambda}^h \eta^h_1\left(\tfrac{m}{M}\right)(M-m), & d^h(m) &= \Psi^h(M)\bar{\mu}^h \eta^h_2\left(\tfrac{m}{M}\right) m \end{aligned} \tag{50}$$

In other words, a jump from n to $n+1$, corresponding to the "birth" of a type 1 firm, occurs in an small time interval dt with probability $b^f(n)dt$ obtained as the probability λ^f of an individual firm to transition from type 2 to type 1 multiplied by the number $(N-n)$ of firms of type 2. Similarly, a jump from n to $n-1$, corresponding to the "death" of a type 1 firm, occurs in an small time interval dt with probability $d^f(n)dt$ obtained as the probability μ^f of an individual firm to transition from type 1 to type 2 multiplied by the number n of firms of type 1. The death and birth transition rates for households are obtained analogously. Observe that these calculations for transition rates assume that the change in type for different firms and households are independent random events, thus the multiplication of each individual transition probability by the number of agents undergoing that transition. Each firm, however, takes into account the size of the system through the function $\Psi^f(\cdot)$ and the choices of the other firms through the functions $\eta^f_1(\cdot)$ and $\eta^f_2(\cdot)$. A similar effect of the population of households on the decision of each household is modelled through the functions $\Psi^h(\cdot)$, $\eta^h_1(\cdot)$ and $\eta^h_2(\cdot)$.

The state of the Markov chain above is characterized by the joint probability

$$P(n, m; t) = \text{Prob}\left(N^1_t = n, M^1_t = m\right). \tag{51}$$

It follows from the Markov property that $P(n, m; t)$ satisfies the following so-called master equation (ME)

$$\begin{aligned} \frac{\partial P(n,m;t)}{\partial t} = \; & d^f(n+1)P(n+1,m;t) + b^f(n-1)P(n-1,m;t) \\ & + d^h(m+1)P(n,m+1;t) + b^h(m-1)P(n,m-1;t) \\ & - [d^f(n) + b^f(n) + d^h(m) + b^h(m)]P(n,m;t), \end{aligned} \tag{52}$$

with the obvious modifications at the boundaries $n = m = 0$, $n = N$, and $m = M$. Exact solutions of (52) are rarely available, and asymptotic approximations relying on the large number of agents are often used in practice. An approximation of this type was proposed in [5], based on a method adapted from [1] and earlier references, for the case of two types of firms and homogenous households. This was extended in [7] for the case of two types of firms and two types of households, but restricted to constant transition probabilities. In what follows, we adjust the method to the case of the more general transition probabilities in (5) and (6).

Assuming that firms and households choose their type independently from each other, we have that

$$P(n, m; t) = P(n, t)P(m, t), \tag{53}$$

where $P(n,t) = \text{Prob}\left(N_t^1 = n\right)$ and $P(m,t) = \text{Prob}\left(M_t^1 = m\right)$. Substituting (53) on both sides of (52) and assuming further that $P(n,t) \neq 0$ and $P(m,t) \neq 0$ for all n, m, we find that (52) decouples into the following equations:

$$\begin{aligned}\frac{\partial P(n,t)}{\partial t} &= d^f(n+1)P(n+1,t) + b^f(n-1)P(n-1,t) \\ &\quad - [d^f(n) + b^f(n)]P(n,t), \end{aligned} \tag{54}$$

$$\begin{aligned}\frac{\partial P(m,t)}{\partial t} &= d^h(m+1)P(m+1,t) + b^h(m-1)P(m-1,t) \\ &\quad - [d^h(m) + b^h(m)]P(m,t), \end{aligned} \tag{55}$$

which are identical to the master equation analyzed in [5]. We proceed the analysis in terms of firms, with the results for households following from obvious modifications. To simplify the notation, we set $d := d^f$, $b := b^f$, $\bar{\lambda} := \bar{\lambda}^f$, $\bar{\mu} := \bar{\mu}^f$, $\eta_1 := \eta_1^f$, $\eta_2 := \eta_2^f$ and $g := \Psi^f$.

As in [5], for a generic function $a(n)$ define the lead and lag operators as

$$L[a(n)] = a(n+1), \qquad L^{-1}[a(n)] = a(n-1), \tag{56}$$

so that we can rewrite (54) as

$$\frac{\partial P(n,t)}{\partial t} = (L-1)[d(n)P(n,t)] + (L^{-1}-1)[b(n)P(n,t)]. \tag{57}$$

Applying Taylor expansions to $a(n+1)$ and $a(n-1)$ at n we find that the operators $(L-1)$ and $(L^{-1}-1)$ can be written as:

$$(L-1)[a(n)] = a(n+1) - a(n) = \sum_{k=1}^{\infty} \frac{1}{k!}\frac{d^k a(n)}{dn^k} \tag{58}$$

and

$$(L^{-1}-1)[a(n)] = a(n-1) - a(n) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k!}\frac{d^k a(n)}{dn^k}. \tag{59}$$

We now make the ansatz that the numbers of firms of type 1 at time t can be written as

$$N_t^1 = N\phi(t) + \sqrt{N}\xi(t), \tag{60}$$

for a determinist function $\phi^f(t)$ corresponding to its trends and a stochastic processes $\xi^f(t)$ describing random fluctuations around the trend, we will now rewrite (57) in terms of $\phi(t)$ and $\xi(t)$. Observe first that, since $\phi(t)$ is assumed to be deterministic, we can write

$$P(n,t) = Q(\xi,t) = Q(\xi(t),t), \tag{61}$$

where $Q(\xi,t)$ is the distribution of the stochastic process $\xi(t)$. This leads to

$$\frac{\partial P(n,t)}{\partial t} = \frac{\partial Q(\xi,t)}{\partial t} + \frac{\partial Q(\xi,t)}{\partial \xi}\frac{d\xi}{dt} = \frac{\partial Q(\xi,t)}{\partial t} - \sqrt{N}\frac{\partial Q(\xi,t)}{\partial \xi}\frac{d\phi}{dt}, \tag{62}$$

where we differentiated the relation (60) at constant $N_t^1 = n$ to obtain $\frac{d\xi}{dt} = -\sqrt{N}\frac{d\phi}{dt}$. Rescaling time by $\tau = \Psi(N)t$, we obtain

$$\frac{\partial P}{\partial t} = \frac{\partial Q}{\partial \tau}\frac{d\tau}{dt} - \sqrt{N}\frac{\partial Q}{\partial \xi}\frac{d\phi}{d\tau}\frac{d\tau}{dt} = \Psi^f(N)\frac{\partial Q}{\partial \tau} - \Psi^f(N)\sqrt{N}\frac{\partial Q}{\partial \xi}\frac{d\phi}{d\tau}. \tag{63}$$

Next observe that the transition probabilities can be expressed as

$$b(n) = b(\xi,\tau) = \bar{\lambda}\cdot\eta_1\left(\phi(\tau)+\frac{\xi}{\sqrt{N}}\right)\cdot(N - N\phi(\tau)-\sqrt{N}\xi)\cdot\Psi(N) \tag{64}$$

$$d(n) = d(\xi,\tau) = \bar{\mu}\cdot\eta_2\left(\phi(\tau)+\frac{\xi}{\sqrt{N}}\right)\cdot(N\phi(\tau)+\sqrt{N}\xi)\cdot\Psi(N). \tag{65}$$

Finally, using the fact that $\frac{da(n)}{dn} = \frac{1}{\sqrt{N}}\frac{da(\xi)}{d\xi}$, we can re-write (58) and (59) as

$$(L-1)[a(\xi)] = \sum_{k=1}^{\infty}\frac{1}{k!N^{\frac{k}{2}}}\frac{d^k a(\xi)}{d\xi^k} \tag{66}$$

$$(L^{-1}-1)[a(\xi)] = \sum_{k=1}^{\infty}\frac{(-1)^k}{k!N^{\frac{k}{2}}}\frac{d^k a(\xi)}{d\xi^k} \tag{67}$$

Inserting (62) in the left-hand side of (57) and (65)-(67) in the right-hand side we obtain

$$\begin{aligned}\Psi(N)\frac{\partial Q}{\partial\tau} - \Psi(N)\sqrt{N}\frac{\partial Q}{\partial\xi}\frac{d\phi}{d\tau} = &\ (L-1)[d(\xi,t)Q(\xi,\tau)] + (L^{-1}-1)[b(\xi,\tau)Q(\xi,\tau)] \quad (68)\\ = \left(\sum_{k=1}^{\infty}\frac{1}{k!N^{\frac{k}{2}}}\frac{d^k}{d\xi^k}\right)&\left[\bar{\mu}\cdot\eta_2\left(\phi(\tau)+\frac{\xi}{\sqrt{N}}\right)\cdot(N\phi(t)+\sqrt{N}\xi)\cdot\Psi(N)\cdot Q(\xi,\tau)\right]\\ + \left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!N^{\frac{k}{2}}}\frac{d^k}{d\xi^k}\right)&\left[\bar{\lambda}\cdot\eta_1\left(\phi(\tau)+\frac{\xi}{\sqrt{N}}\right)\cdot(N-N\phi(t)-\sqrt{N}\xi)\cdot\Psi(N)\cdot Q(\xi,\tau)\right]\end{aligned}$$

Collecting terms of order $O(\Psi(N)\sqrt{N})$ in the equation above leads to

$$\frac{d\phi}{d\tau} = F(\phi), \tag{69}$$

where

$$F(x) = \bar{\lambda}\eta_1(x)(1-x) - \bar{\mu}\eta_2(x)x. \tag{70}$$

We see that (69) has an equilibrium at ϕ^* satisfying $F(\phi^*)=0$, which is equivalent to

$$\frac{\eta_1(\phi^*)}{\eta_2(\phi^*)} = \frac{\bar{\mu}\phi^*}{\bar{\lambda}(1-\phi^*)}. \tag{71}$$

Similarly, collecting terms of order $O(\Psi(N))$ in (68) leads to

$$\begin{aligned}\frac{\partial Q}{\partial\tau} =& \left\{\bar{\lambda}[\eta_1(\phi)-\eta_1'(\phi)(1-\phi)] + \bar{\mu}[\eta_2(\phi)+\eta_2'(\phi)\phi]\right\}\frac{\partial(\xi Q)}{\partial\xi}\\ &+ \frac{\bar{\lambda}\eta_1(\phi)(1-\phi)+\bar{\mu}\eta_2(\phi)\phi}{2}\frac{\partial^2 Q}{\partial\xi^2}.\end{aligned} \tag{72}$$

We see that (69) is the ordinary differential equation providing the dynamics of the deterministic trend for the fraction of firms of type 1, whereas (72) is the partial differential equation providing the dynamics of the distribution of fluctuations around the trend, known as the Fokker-Planck equation. At the equilibrium ϕ^*, we find that the stationary solution to (72) is a Gaussian distribution with mean zero and variance

$$\sigma^2 = \frac{1}{2}\frac{\bar{\lambda}\eta_1(\phi^*)(1-\phi^*) + \bar{\mu}\eta_2(\phi^*)\phi^*}{\bar{\lambda}[\eta_1(\phi^*) - \eta_1'(\phi^*)(1-\phi^*)] + \bar{\mu}[\eta_2(\phi^*) + \eta_2'(\phi^*)\phi^*]} \tag{73}$$

It follows that the fraction $n^1_\tau = N^1_\tau/N$ of firms of type 1 can be approximated by a OrnsteinUhlenbeck process $\tilde{n}^1_\tau$ of the form

$$d\tilde{n}^1_\tau = \kappa_{ou}\left(\phi^* - \tilde{n}^1_\tau\right)d\tau + \sigma_{ou}dW_\tau, \tag{74}$$

where the parameters κ_{ou} and σ_{ou} are chosen to match the variance σ^2/N, that is to say, satisfying

$$\frac{\sigma^2_{ou}}{2\kappa} = \frac{\sigma^2}{N}. \tag{75}$$

Example 1. As we mentioned in Section 2, choosing $\eta_1(x) = \eta_2(x) = 1$ and $\Psi(N) = 1$ leads to the constant transition rates considered in [7], namely

$$b(n) = \bar{\lambda}(N-n), \qquad d(n) = \bar{\mu}n. \tag{76}$$

In other words, with constant transition probabilities for individual agents in (5) and (6), the aggregate birth rate is a linear decreasing function of the current proportion of agents of type 1, wheres the aggregate death rate is a linear increasing function of n/N. In this case, the equation for the deterministic trend reduces to

$$\frac{d\phi}{dt} = \bar{\lambda}(1-\phi) - \bar{\mu}\phi, \tag{77}$$

whose solution is

$$\phi(t) = \frac{\bar{\lambda}}{\bar{\lambda}+\bar{\mu}} + e^{-(\bar{\lambda}+\bar{\mu})t}\left(\phi(0) - \frac{\bar{\lambda}}{\bar{\lambda}+\bar{\mu}}\right) \tag{78}$$

and converges asymptotically to the equilibrium

$$\phi^* = \frac{\bar{\lambda}}{\bar{\lambda}+\bar{\mu}}. \tag{79}$$

We then find that the stationary solution to (72) is a Gaussian distribution with mean zero and variance

$$\sigma^2 = \frac{\bar{\mu}\bar{\lambda}}{(\bar{\mu}+\bar{\lambda})^2}. \tag{80}$$

Example 2. If we take $\eta_1(x) = \eta_2(x) = x$ and $\Psi(N) = 1$, the transition rates take the more interesting form [1]

$$b(n) = \bar{\lambda}\frac{n(N-n)}{N}, \qquad d(n) = \bar{\mu}\frac{n^2}{N}. \tag{81}$$

[1] Replacing the factor n^2 in $d(n)$ by $n(n-1)$ and the scaling function does $\Psi(N) = 1$ by $\Psi(N) = 1/N$ doe not change the asymptotic approximation (apart from a re-scaling of time) and leads to the same model considered on page 35 of [1], where a justification for these transition probabilities is given in terms of population dynamics.

We find that the equation for the deterministic trend becomes

$$\frac{d\phi}{dt} = \bar{\lambda}\phi(1-\phi) - \bar{\mu}\phi^2. \tag{82}$$

whose solution is the logistic curve

$$\phi(t) = \frac{\bar{\lambda}}{\bar{\lambda} + \bar{\mu} - ce^{-\beta t}}, \tag{83}$$

with $c = 1 - \frac{\bar{\lambda}}{\phi(0)(\bar{\lambda}+\bar{\mu})}$ and $\beta = \frac{(\bar{\mu}+\bar{\lambda})^2}{\bar{\lambda}}$, which also converges asymptotically to the equilibrium $\phi^* = \frac{\bar{\lambda}}{\bar{\lambda}+\bar{\mu}}$. We find that the stationary solution is also a Gaussian distribution with the same variance as in the previous example, namely $\sigma^2 = \frac{\bar{\mu}\bar{\lambda}}{(\bar{\mu}+\bar{\lambda})^2}$

Example 3. Next, consider $\eta_1(x) = x/(1-x)$ and $\eta_2(x) = x$ and $\Psi(N) = 1$. In this case[2], the transition rates take the form

$$b(n) = \bar{\lambda}n, \qquad d(n) = \bar{\mu}\frac{n^2}{N}. \tag{84}$$

We then find that the equation for the trend becomes

$$\frac{d\phi}{dt} = \bar{\lambda}\phi - \bar{\mu}\phi^2, \tag{85}$$

whose equilibrium is $\phi^* = \frac{\bar{\lambda}}{\bar{\mu}}$. In this case the stationary solution to (72) is a Gaussian distribution with mean zero and variance $\sigma^2 = \bar{\lambda}/\bar{\mu}$.

4 Evaluating Alternatives

The simple examples at the end of the previous section lead to aggregate dynamics with essentially the same asymptotic properties of the case with constant transition probabilities. In this section, we explore a more elaborate example described in detail in [1, Section 5.3], where the transition probabilities reflect the perceived advantage between the two classes of agents. As before, we describe the results in terms of firms, with the results for households being obtained through obvious modifications.

Start by observing that the (54) admits a stationary solution of the form

$$\overline{p}(n) = p_0 \left(\frac{\bar{\lambda}}{\bar{\mu}}\right)^n \frac{N!}{n!(N-n)!} \prod_{k=0}^{n} \frac{\eta_1\left(\frac{k}{N}\right)}{\eta_2\left(\frac{k}{N}\right)}, \tag{86}$$

where p_0 is a normalization constant obtained from the condition $\sum_{n=0}^{N} \overline{p}(n) = 1$. This is a complicated expression, and the purposes of the ansatz (60) and the corresponding series expansion techniques introduced in the previous section is to provide an accurate yet tractable approximation to it. We now introduce a different approximation based on the binomial coefficient appearing in (86).

[2] Taking $\Psi(N) = 1/N$ instead of $\Psi(N) = 1$ does not change the asymptotic approximation (apart from re-scaling time) and leads to the the model presented on page 37 of [1] as an example with a nonlinear death rate.

If we rewrite (86) as

$$\overline{p}(n) = \frac{1}{Z} e^{-\beta N U\left(\frac{n}{N}\right)}, \tag{87}$$

for a potential functions U, parameters $\beta > 0$ and normalization constant Z, we find that

$$-\beta N U\left(\frac{n}{N}\right) - \log Z = \log p_0 + n \log\left(\frac{\bar{\lambda}}{\bar{\mu}}\right) + \log\left(\frac{N!}{n!(N-n)!}\right) + \sum_{k=0}^{n} \log\left(\frac{\eta_1\left(\frac{k}{N}\right)}{\eta_2\left(\frac{k}{N}\right)}\right). \tag{88}$$

This is still a complicated expression, so we make the additional assumption that

$$\eta_1(x) = \frac{e^{\beta g(x)}}{e^{\beta g(x)} + e^{-\beta g(x)}} \tag{89}$$

$$\eta_2(x) = \frac{e^{-\beta g(x)}}{e^{\beta g(x)} + e^{-\beta g(x)}} = 1 - \eta_1(x) \tag{90}$$

for some function g. Using Stirling's approximation for the binomial coefficient, we obtain from (88) that the potential function can be written as

$$U\left(\frac{n}{N}\right) = -\frac{2}{N} \sum_{k=0}^{n} g\left(\frac{k}{N}\right) - \frac{1}{\beta}\left[H\left(\frac{n}{N}\right) - \frac{n}{N} \log\left(\frac{\bar{\lambda}}{\bar{\mu}}\right)\right] + O\left(\frac{\log N}{N}\right), \tag{91}$$

where $H(x) = -x \log x - (1-x)\log(1-x)$ is an entropy function. For N sufficiently large, we see that (91) can be approximated by the following expression

$$U(x) = -2 \int_0^x g(s) ds - \frac{1}{\beta}\left[H(x) - x \log\left(\frac{\bar{\lambda}}{\bar{\mu}}\right)\right]. \tag{92}$$

Differentiating this expression and setting the derivative equal to zero we see that the potential has a critical point at x^* satisfying

$$e^{2\beta g(x^*)} = \frac{\bar{\mu} x^*}{\bar{\lambda}(1-x^*)}. \tag{93}$$

Recalling that (69) has a stationary solution for ϕ^* satisfying (71), we see that, when η_1 and η_2 are given by (89) and (90), the stationary solution of the dynamics for the trend ϕ and the critical points of the potential $U(x)$ coincide. We see that this corresponds to the fraction of agents of type 1 for which the stationary probability $\overline{p}(k)$ has a maximum, corresponding to the minimum of the potential $U(x)$. This in turn is compatible with the ansatz (60), which is expected to approximate well the fraction of agents of type 1 when the probability $P(n,t)$ has a well-defined peak of order N and fluctuations of order $\sqrt{N}$, as is the case near a stationary distribution of the form (86).

Apart from this a posteriori justification of the ansatz (60), the choice of functions η_1 and η_2 in (89) and (90) allow us to interpret the transition probabilities in terms of how the agents perceive the comparative advantages of belonging to each class. For this, suppose that the gain $G(x)$ from being an agent of type 1 instead of type 2 conditional on a fraction x of agents of type 1 is given by a normal random variable with mean $g(x)$ and variance $2/(\pi\beta^2)$. If we then compute the probability that this gain is positive for a given value of x we find

$$\text{Prob}[G(x) \geq 0] = \frac{1}{2}[1 + \text{erf}(u)], \tag{94}$$

where we used the error function

$$\operatorname{erf}(G) = \frac{2}{\sqrt{\pi}} \int_0^u e^{-y^2} dy, \tag{95}$$

for $u = \beta g(x)\sqrt{\pi}/2$. Using the approximation $\operatorname{erf}(x) \approx \tanh(2x/\sqrt{\pi})$ we find that

$$\operatorname{Prob}[G(x) \geq 0] \approx \frac{e^{\beta g(x)}}{e^{\beta g(x)} + e^{-\beta g(x)}} = \eta_1(x). \tag{96}$$

In other words, for a given fraction x of agents of type 1, the higher the probability of realizing a positive gain by being of type 1, the higher the transition rate from type 2 to type 1. Moreover, the parameter β measures the degree of confidence that agents have in assessing the advantage of being of one type over another: a small variance in the distribution of gains from being of type 1 means a large value of β and consequently a larger value of $\eta_1(x)$ whenever $g(x) > 0$. Conversely, near total ignorance about the distribution of gains means $\beta \approx 0$ and $\eta_1(x) \approx \eta_2(x) \approx 1/2$.

Pursuing this interpretation further, for large values of β we see from (92) that

$$U'(x) \approx -2g(x),$$

so that the critical points of the potential $U(x)$ are close to the zeros of the function $g(x)$. In other words, a maximum of the stationary distribution (86) corresponds to a fraction x^* of agents of type 1 for which the agents are indifferent between the two types.

Observe further that the function $g(x)$ can be used to investigate the stability of the equilibrium of (69). Namely, recalling (70) and using the expression

$$g(x) = \frac{1}{2\beta} \log\left(\frac{\eta_1(x)}{\eta_2(x)}\right) \tag{97}$$

together with the equilibrium condition (71) we find that

$$F'(x) = \frac{\bar{\lambda}\eta_1(x)}{x}\left[2\beta x(1-x)g'(x) - 1\right], \tag{98}$$

from which we conclude that the condition for local stability of an equilibrium for (69) is

$$2\beta x(1-x)g'(x) < 1. \tag{99}$$

In Figures 4 to 3 we explore four examples based on different specifications of the function g. We take $\bar{\lambda} = \bar{\mu} = 0.5$ in all cases, so that there is no exogenous bias towards either type of agent. The equilibrium condition (71) is then equivalent to

$$\phi^* = \eta_1(\phi^*) = \frac{e^{\beta g(\phi^*)}}{e^{\beta g(\phi^*)} + e^{-\beta g(\phi^*)}}, \tag{100}$$

which we recognize as a Gibbs distribution for a two-level system. The stationary solution to (72) in this case has variance

$$\sigma^2 = \frac{\phi^*(1-\phi^*)}{1-\eta_1'(\phi^*)}. \tag{101}$$

We take $\beta = 10$, so that the standard deviation in the agents' perception of gains from being in a given class is approximately 0.08.

Example 4. For $g(x) = x - 0.2$ we see that the function $\eta(x)$ takes the form of the logistic curve shown in Figure 2: when $x \to 0$ (most agents are of type 2), we have that $g(x) \to -0.2$ (there are significant perceived losses associated with being an agent of type 1) and $\eta_1(x) \to 0$ (the individual transition probability from type 2 to type 1 is negligible). Conversely, when $x \to 1$ (most agents are of type 1), we see that $g(x) \to 0.8$ (there are significant perceived gains associated with being an agent of type 1) and $\eta_1(x) \to 1$ (the individual transition probability from type 2 to type 1 is at its maximum). There is an equilibrium at $\phi_1^* = 0.0369$ and another at $\phi_2^* = 0.727$. Using (99), we can easily determine that the first equilibrium stable whereas the second is unstable, as we can confirm from the corresponding graph of the function $F(x)$ on Figure 3.

Example 5. The opposite situation arises when $g(x) = -x + 0.8$, with the function $\eta(x)$ taking the form of the reversed logistic curve shown in Figure 2, that is to say, decreasing from $\eta_1(x) \to 1$ when $x \to 0$ to $\eta_1(x) \to 0$ when $x \to 1$. The equilibrium fraction of agents of type 1 is now reached at $\phi^* = 0.7461$, which is seen to be stable according to (99), as can be confirmed from the graph of the function $F(x)$ on Figure 3.

Example 6. Consider next the function $g(x) = -x^2 + x - 0.16$. In this case, the agents are indifferent between the two types when the fraction of agents of type 1 is either $x = 0.2$ or $x = 0.8$. As we can see in Figure (4), the solutions to $x = \eta_1(x)$, which correspond to the equilibria of (69), converge to these values when $\beta \to \infty$. When $\beta < \bar{\beta} \approx 13$, the equilibrium for (69) is unique. For $\beta = 10$, we can see in Figure 2 that $\eta_1(x)$ peaks around $x = 0.5$, where the perceived advantage of being an agent of type 1 is the largest, and decreases as x approaches either 0 or 1, where it is advantageous to be an agent of type 2. The unique equilibrium in this case is $\phi^* = 0.7117$, which is easily seen to be stable according to (99), as we can confirm from the graph of the function $F(x)$ in Figure 3.

Example 7. Conversely, for $g(x) = x^2 - x + 0.16$, Figure 2 shows that the function $\eta_1(x)$ has a minimum at $x = 0.5$, where agents deem to be least advantageous to be of type 1. The equilibrium in this case is reached at $\phi^* = 0.2883$ and is also stable as can be seen from the graph of the function $F(x)$ in Figure 3.

5 Numerical Experiments

In this section we illustrate the properties of the model by simulating both the full agent-based model and the mean-field approximation for different specifications of the transition probabilities. We use the base parameters described in Table 1, which were chosen consistently with the assumption that the discrete-time equations in the model correspond to $\Delta t = 0.25$ years. In particular, the one-period depreciation rate δ, the dividend yield δ^e, and the interest rate r were chosen consistently with annualized rates of 4% for each variable. As in [7], we perform the agent-based simulations with $p \equiv 1.4$, $p_0^e = 1$ and initialize the aggregate balance sheet items for firms at $pK_0 = 1400$, $B_0 = 667$, $E_0 = 333$, leading to initial aggregate net worth of the firm sector equal to $V_0^F = 400$, and aggregate balance sheet items the household sector at $D_0 = 1067$ and $p_0^e E_0 = 333$, leading to initial aggregate net worth of the household sector equal to $V_0^H = 1400$. We also assume a constant level of cash reserves for the bank $R_0 = 400$, so that the initial net worth of the bank implied by the aggregate balance sheets of firms and households is $V_0^B = 0$. We then assume these aggregate amounts are uniformly distributed among individual firms and households respectively.

Symbol	Value	Description
N	1000	number of firms
M	4000	number of households
a	1	labour productivity
c	1	unit labour cost
χ	1.4	markup factor
α_1	0.575	profit elasticity of investment for aggressive firms
α_2	0.4	profit elasticity of investment for conservative firms
β	0.16	utilization elasticity of investment
γ	0.05	debt elasticity of investment
r	0.01	one-period interest rate on loans and deposits
δ	0.01	one-period depreciation rate
δ^e	0.01	one-period dividend yield
s_1^y	0.15	propensity to save from income for non-investors
s_2^y	0.4	propensity to save from of income for investors
s_1^v	0.85	propensity to save from wealth for non-investors
s_2^v	0.85	propensity to save out of wealth for investors
$\bar{\mu}^f$	0.6	transition probability from aggressive to conservative type for firms
$\bar{\lambda}^f$	0.4	transition probability from conservative to aggressive type for firms
$\bar{\mu}^h$	0.2	transition probability from non-investors to investors type for households
$\bar{\lambda}^h$	0.3	transition probability from investors to non-investor type for households
ϖ	0.6	proportion of external financing for firms obtained issuing new debt
φ	0.5	proportion of investor household wealth allocated to stocks

Table 1: Baseline parameter values

5.1 Accuracy of the approximation and parameter sensitivity

In [7], the accuracy of the approximation for constant transition rates is tested by comparing the number of firms and households of each type obtained from the agent-based simulation and the mean-field approximation. We perform the same comparison for the transition probabilities in Examples 2 and 3 in Section 3 and two different specifications of the function $g(x)$ in Section 4, namely Examples 5 and 6. The results are shown in Figure 5, where we focus on the number of firms only and omit the analogous results for the number of households for the sake of brevity. As we can see in the graphs, in both the ABM simulation and the MF approximation the proportions of aggressive and conservative firms oscillate around the asymptotic limit ϕ^* with variance σ^2/N as expected. In Figure 6, we compute the time evolution for equity prices for the same examples and observe a close match between the computationally intensive agent-based model and its mean-field approximation.

Next in Figures 7 to 8 we focus on two specifications of the transition probabilities, namely the cases $g(x) = x - 0.8$ and $g(x) = -x^2 + x - 0.16$ in Section 4, and use the mean-field approximation to perform sensitivity tests with respect to the discretionary parameters ϖ and φ. The results for the proportion ϖ of external financing that firms raised through new debt confirm the findings of [7] for constant transition probabilities, namely that the average growth rates of both equity prices and output are increasing functions of ϖ and tend to flatten out around the base value $\varpi = 0.6$ adopted in Table 1.

Regarding φ, we see in Figures 7 to 8 that the average growth rate of output increases with φ, similarly to what was reported in [7] for constant transition probabilities, albeit with an increasing standard deviation, whereas [7] found that the standard deviation of output growth decreases with φ. Similarly, we find that the average growth rate of equity prices increases with φ, although less steeply that in [7]. Finally, in accordance with [7], equity prices become more volatile as φ approaches either zero or one.

5.2 Exploration: the Financial Instability Hypothesis

In the previous section, we presented evidence that the baseline parameters in Table 1 lead to plausible simulation outcomes. Moreover, for some of the parameters that are least likely to be directly estimated from observed data, such as ϖ and φ, we showed how perturbations from the baseline values affect aggregate variables such as equity prices and nominal output. In this section, we use the model to explore the consequences of heterogeneity in firms and households on one particular macroeconomic aspect: the link between equilibrium equity prices and the financial fragility of firms. As we mentioned in the Introduction, this is motivated by Minsky's Financial Instability Hypothesis (FIH), according to which periods of financial turmoil are predicated on a higher proportion of financially fragile firms (see for example [8]).

Accordingly, we begin by defining the usual Minsky classes of firms - namely hedge, speculative, and Ponzi - in the context of our model. In Minsky's classification, hedge firms are those for which profits are enough to meet all financial obligations and still decrease the amount of net debt. In our model, this corresponds to a situation where retained profits, defined as

$$a^n_{t+1} = \pi p q^n t + 1 - r b^n_t - k^n_t - \delta^e p^e_t e^n_t \tag{102}$$

being larger than net investment, that is,

$$a^n_{t+1} > i^n_{t+1} - \delta p k^n_t, \tag{103}$$

as this would lead to a reduction in debt according to (26). Conversely, Ponzi firms are those that need to borrow even to meet their basic financial obligations, such as paying interest on debt or agreed dividends. We interpret this in our model as as situation in which

$$a^n_{t+1} < 0, \tag{104}$$

so that debt increases even at the level of zero net investment, that is $i^n_{t+1} = \delta p k^n_t$. The intermediate class consists of speculative firms, namely firms for which

$$0 \le a^n_{t+1} \le i^n_{t+1} - \delta p k^n_t, \tag{105}$$

so that their debt increases if they choose to invest more than their level of retained profits, presumably in the expectation that demand, and therefore profits, will increase in future.

In the context of our model, the equilibrium equity price is an indicator of overall financial stability in the market, with stable periods corresponding to moderate growth and low volatility and unstable ones characterized by boom and busts and higher volatility. As an implication of the FIH, one can then conjecture that the overall proportion of Ponzi firms tends to be higher around unstable periods. To test this conjecture, we perform ABM simulations of the model for three different specifications of transition probabilities.

Experiment 1: Consider transition probabilities as in Example 1, namely corresponding to the constant transition probabilities used in [7]. We begin by noticing that the baseline scenario obtained from the parameters in Table 1 corresponds to the quite stable equilibrium equity prices reproduced at the top-left panel of Figure 9. The overall proportion of hedge, speculative, and Ponzi firms in the entire population are shown in the top-right panel. As one can see, speculative firms are the dominant group, with Ponzi and hedge firms representing much smaller fractions. The bottom panels in Figure 9 show that these proportions are largely unchanged if one considers only aggressive or conservative firms.

To obtain a less stable scenario, we change the values for the fraction of external financed raised from debt and the fraction of household wealth invested in stock to $\varpi = 0.3$ and $\varphi = 0.3$ respectively. As discussed in connection with Figures 4 and 5 of [7], the predicted effect of each of these changes is to lower the returns and increase volatility of stock prices. This is confirmed in the top-left panel of Figure 10, where we see an equilibrium stock price initially increasing, followed by a prolonged downturn with higher volatility. In the top-right corner of the figure we can see that, in accordance with the FIH, this is accompanied by a much higher proportion of Ponzi firms in the population, namely around 40% versus less than 10% in Figure 9. Moreover, the onset of the decline in stock prices coincide with a precipitous drop in the proportion of hedge firms. As in the previous case, the bottom panels show that these proportions are not affected by considering only aggressive or conservative firms.

We extend this analysis to six additional scenarios described in Table 2, where we explore different degrees of heterogeneity in the populations of firms and households. In each scenario the average values for profit elasticity α and propensity to save from income s^y are kept constant and equal to the averages obtained from the baseline parameters in Table 1. For example, for Scenario 1 we have

$$\overline{\alpha} = \frac{\bar{\lambda}_f}{\bar{\mu}_f + \bar{\lambda}_f}\alpha_1 + \frac{\bar{\mu}_f}{\bar{\mu}_f + \bar{\lambda}_f}\alpha_2 = (0.4) \cdot (0.575) + (0.6) \cdot (0.4) = 0.47$$

and

$$\bar{s}^y_1 = \frac{\bar{\lambda}_h}{\bar{\mu}_h + \bar{\lambda}_h} s^y_1 + \frac{\bar{\mu}_h}{\bar{\mu}_h + \bar{\lambda}_h} s^y_2 = (0.2) \cdot (0.05) + (0.8) \cdot (0.3) = 0.25,$$

where we recall that $\frac{\bar{\lambda}_f}{\bar{\mu}_f+\bar{\lambda}_f}$ and $\frac{\bar{\lambda}_h}{\bar{\mu}_h+\bar{\lambda}_h}$ are the long-term proportions of firms and households of type 1 (respectively, aggressive firm and non-investor household) in the population. The differences between the scenarios are the corresponding spreads in α and s^y implied by the transition probabilities and profit elasticity and propensity to save for each type. For example, we see that the standard deviation in profit elasticity decreases from 0.0857 to 0.035 as we move from Scenario 1 to 6 in Table 2, corresponding to a decrease of heterogeneity in the population of firms. Similarly, we see that the standard deviation in propensity to save from income is relatively high for Scenarios 1 to 4, moderate for Scenario 5, and much smaller in Scenario 6 (namely decreasing from around 0.1 to 0.0131).

The resulting equilibrium stock prices and proportions of hedge, speculative, and Ponzi firms are shown in Figures 11 and 12. The remarkable pattern we observe is that in the scenarios where the stock price displays decent growth and low volatility, namely Scenarios 1 and 4, the proportion of Ponzi firms in the economy remains very low, namely around 10%, whereas in all the scenarios with lower growth and higher volatility the proportion of Ponzi firms is much higher, namely around 50%. In each case we have also calculated the corresponding proportions within the subgroups of aggressive and conservative firms and found that they remain largely unaffected, which indicates that the classification according to financial health (that is, hedge, speculative, and Ponzi) is independent from firm type with respect to investment demand as defined in this paper (namely aggressive and conservative).

Scenario	$\bar{\mu}_f$	$\bar{\lambda}_f$	α_1	α_2	σ_α	$\bar{\mu}_h$	$\bar{\lambda}_h$	s_1^y	s_2^y	σ_{s^y}
1	0.6	0.4	0.575	0.4	0.0857	0.8	0.2	0.05	0.3	0.1
2	0.6	0.4	0.575	0.4	0.0857	0.3	0.7	0.1857	0.4	0.0982
3	0.5	0.5	0.54	0.4	0.07	0.3	0.7	0.1857	0.4	0.0982
4	0.3	0.7	0.5	0.4	0.0458	0.4	0.6	0.15	0.4	0.1224
5	0.3	0.7	0.5	0.4	0.0458	0.7	0.3	0.1333	0.3	0.0764
6	0.2	0.8	0.4875	0.4	0.035	0.3	0.7	0.2414	0.27	0.0131

Table 2: Scenarios for Figures 11 and 12.

Experiment 2: We now consider state-dependent transition probabilities as Examples 5 and 6 and focus on Scenario 3 of Table 2. The corresponding equity prices and proportions of hedge, speculative, and Ponzi firms are shown in Figure 13 and confirm the pattern observe in Figures 11 and 12, namely with the proportion of Ponzi firms rising when the stock markets undergoes periods of low growth and high volatility. Observe that in both cases the equilibrium stock price collapses to zero much faster than in the corresponding scenario with constant transition probabilities, namely the bottom row of Figure 11, indicating that actively choosing a type based on relative performance can exacerbate the negative effects of a period of crisis.

6 Concluding remarks and further work

The framework introduced in Sections 2 and 3 significantly expands the domain of applicability of the mean-field approximation to stock-flow consistent models with state-dependent transition probabilities. In particular, the explicit form of the differential equation (69) in terms of the functions $\eta_1(x)$ and $\eta_2(x)$ allow modellers to explore more complex specifications of transition probabilities that lead to multiple equilibria with many different stability properties. More specifically, the functional form presented in Section 4 gives rise to a rich set of dynamic behaviour depending on how agents value the relative gains from being of one type or another. The numerical experiments in Section 5 confirm the accuracy of the mean-field approximation and show how it can be used to explore the parameter space in a way that would be impossible in practice with agent-based models alone.

Once reasonable parameters are selected, one can then return to the more detailed description provided by agent-based models to investigate other questions, some of which require keeping track of features of the agents that are neglected in the mean-fields approximation. We provided an example of such detailed investigation in the context of Minsky's Financial Instability Hypotheses, where we simulated different scenarios for the agent-based model and found that whenever the stock market undergoes periods of low growth and high volatility we observe a corresponding increase in the proportion of Ponzi firms, that is to say, firms that need to increase their borrowing even to meet basic financial obligations.

Further dependence between agents can be achieved if we allow, for example, the transition probabilities for firms to depend on the current fractions of firms *and* households of each type (and similarly for the transition probabilities for households). Economically this means that firms will based their behaviour on the aggregate behaviour of both their competitors and their consumers, which is not at all unrealistic. Mathematically this means that the mean-field approximation will lead to a two-dimensional system of coupled differential equations for the trends $\phi^f(t)$ and $\phi^h(t)$ of the fraction of firms and households of type 1. Accordingly, the monotone convergence to a stable equilibrium or divergence from an unstable one might be replaced by more interesting dynamics, such as the appearance of limit cycles for the two coupled state variables.

References

[1] M. Aoki. *Modeling Aggregate Behavior and Fluctuations in Economics: Stochastic Views of Interacting Agents*. Cambridge University Press, 2002.

[2] A. Caiani, A. Godin, E. Caverzasi, M. Gallegati, S. Kinsella, and J. E. Stiglitz. Agent based-stock flow consistent macroeconomics: Towards a benchmark model. *Journal of Economic Dynamics and Control*, 69:375 – 408, 2016.

[3] L. Carvalho and C. Di Guilmi. Income inequality and macroeconomic instability: a stock-flow consistent approach with heterogeneous agents. CAMA Working Paper 60/2014, 2014.

[4] C. Di Guilmi and L. Carvalho. The dynamics of leverage in a Minskyan model with heterogeneous firms. Journal of Economic Behavior and Organization, forthcoming, 2017.

[5] C. Di Guilmi, M. Gallegati, and S. Landini. Financial fragility, mean-field interaction and macroeconomic dynamics: A stochastic model. In *Institutional and Social Dynamics of Growth and Distribution*, chapter 13. Edward Elgar Publishing, 2010.

[6] D. K. Foley. A statistical equilibrium theory of markets. *Journal of Economic Theory*, 62(2):321 – 345, 1994.

[7] M. R. Grasselli and P. X. Li. A stock-flow consistent macroeconomic model with heterogeneous agents: the master equation approach. Submitted to the Journal of Network Theory in Finance, 2017.

[8] H. P. Minsky. *Can 'it' happen again ?* M E Sharpe, Armonk, NY, 1982.

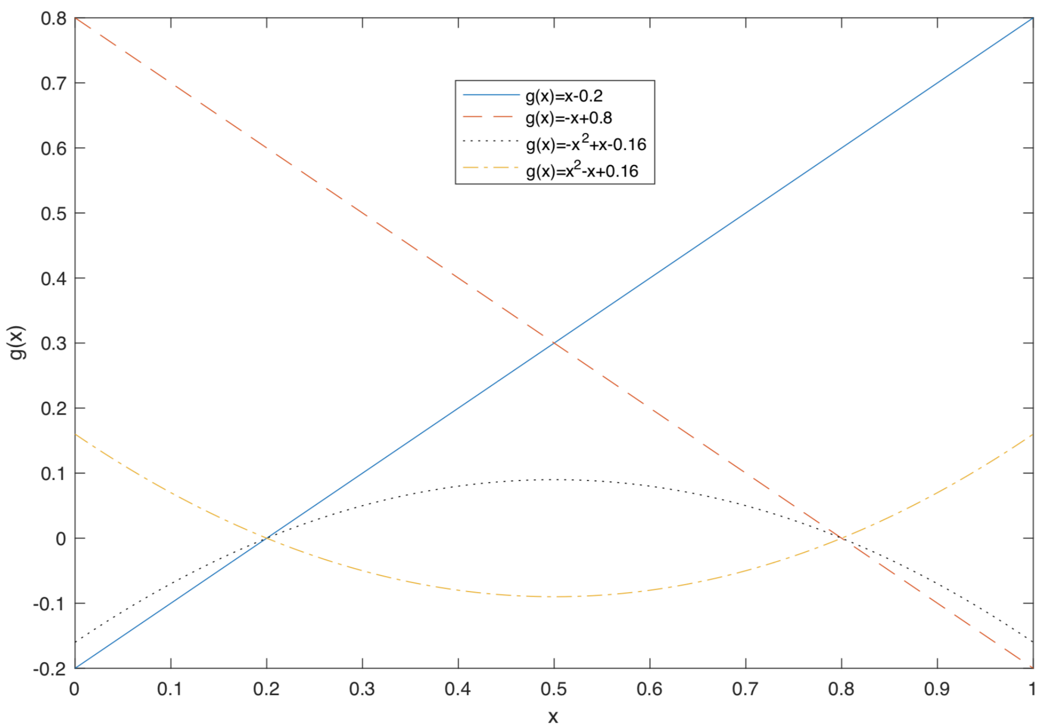


Figure 1: Mean $g(x)$ of the perceived gain for being an agent of type 1 when the fraction of agents of type 1 is x.

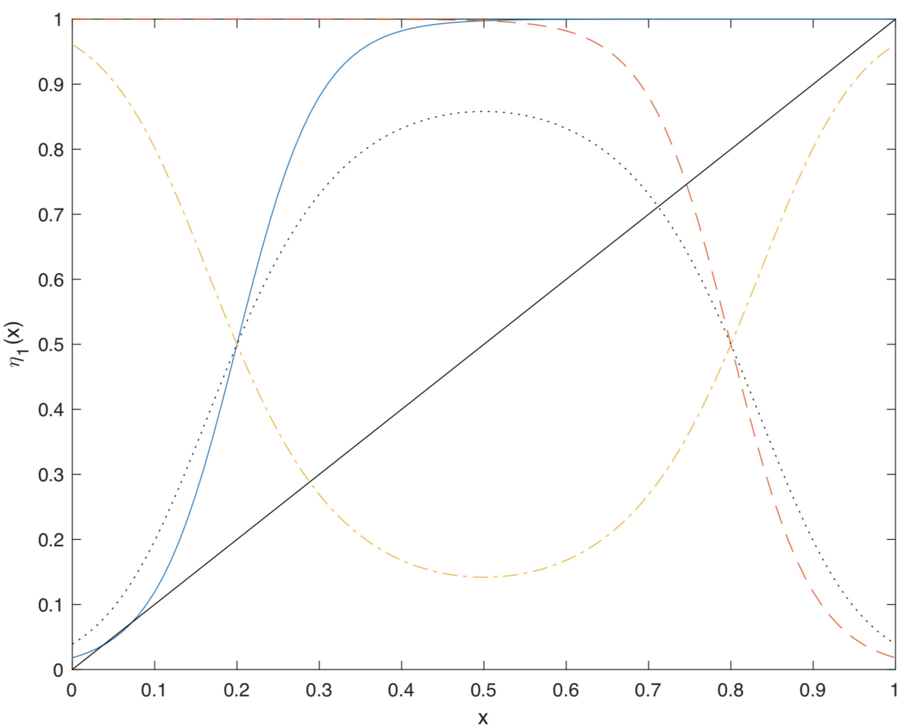


Figure 2: Function $\eta_1(x)$ appearing in the transition probability from type 2 to type 1 given in (5) when the fraction of agents of type 1 is x. For a function of the form (89), this is approximately equal to the probability that the perceived gain from being of type 1 is positive. Each graph corresponds to a different function $g(x)$ according to the legend in Figure 4. The intersection with the line $y = x$ denotes the equilibrium point $x^* = \eta_1(x^*)$ for (69).

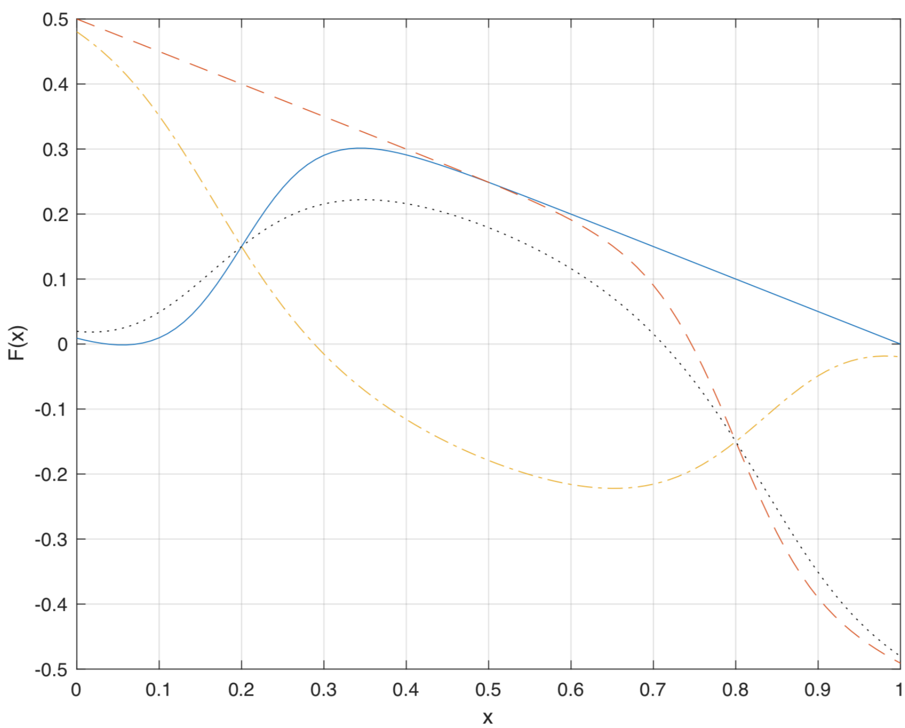


Figure 3: Function $F(x)$ appearing on the right-hand side of (69). Each graph corresponds to a different function $g(x)$ according to the legend in Figure 4. The equilibrium point x^* is characterized by $F(x^*) = 0$ and the condition for stability is $F'(x^*) < 0$.

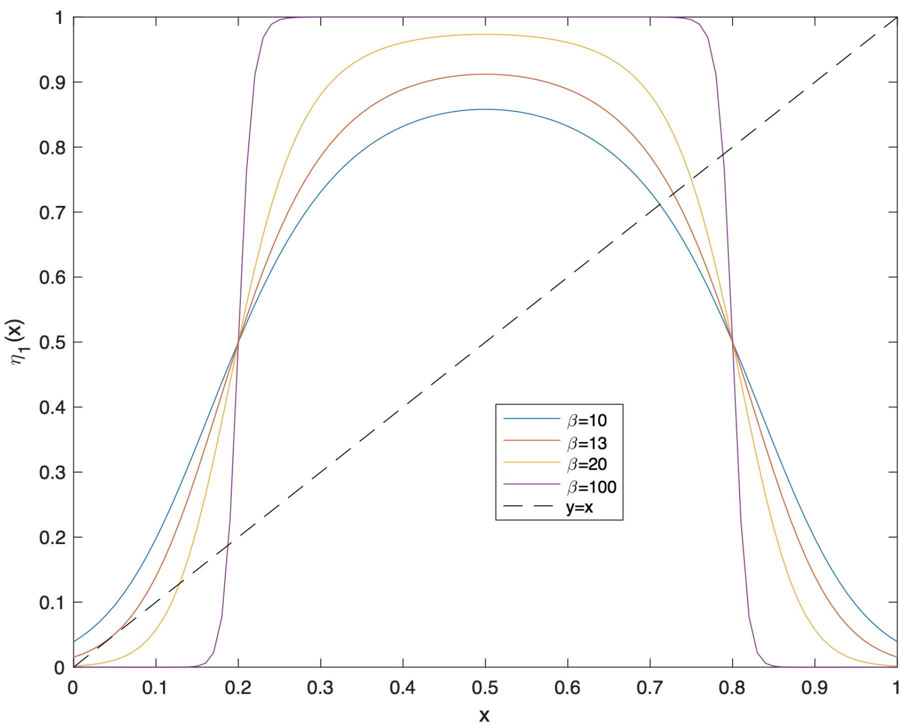
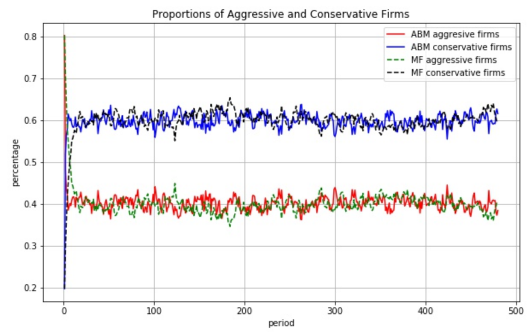
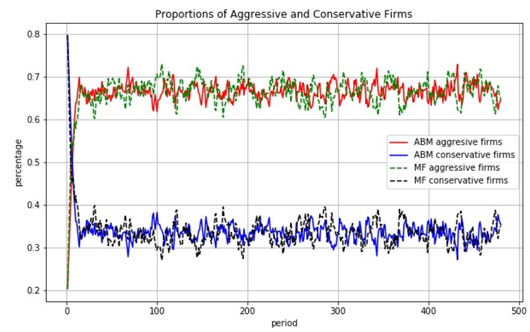


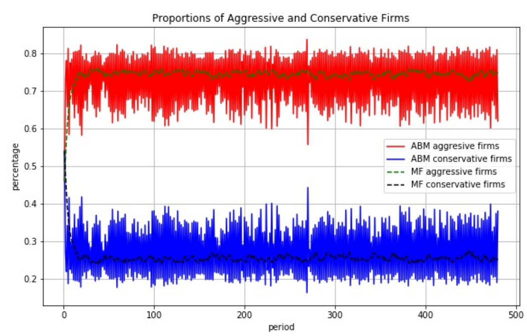
Figure 4: Function $\eta_1(x)$ in (89) for $g(x) = -x^2 + x - 0.16$ and different values of β. As β increases, the function $\eta_1(x)$ approaches a step function rapidly increasing from zero to one near $x = 0.2$ and then dropping from one to zero near $x = 0.8$. The intersection of the functions $\eta_1(x)$ with the line $y = x$ correspond to the equilibrium points of (69).



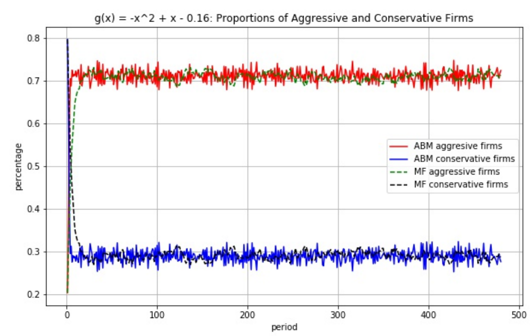
(a) Example 2 with $\phi^* = 0.4$ and $\frac{\sigma}{\sqrt{N}} = 0.0155$



(b) Example 3 with $\phi^* = 0.67$ and $\frac{\sigma}{\sqrt{N}} = 0.0155$

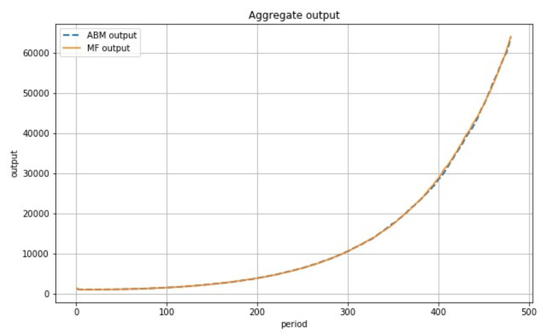


(c) Example 5 with $\phi^* = 0.75$ and $\frac{\sigma}{\sqrt{N}} = 0.0089$

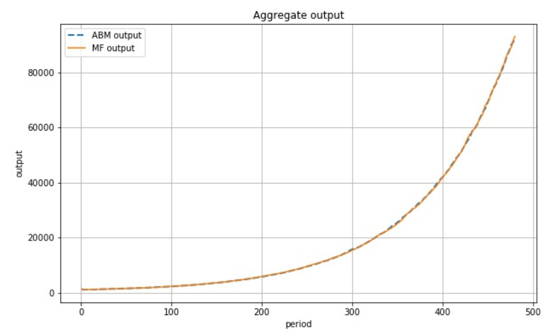


(d) Example 6 with $\phi^* = 0.71$ and $\frac{\sigma}{\sqrt{N}} = 0.0122$

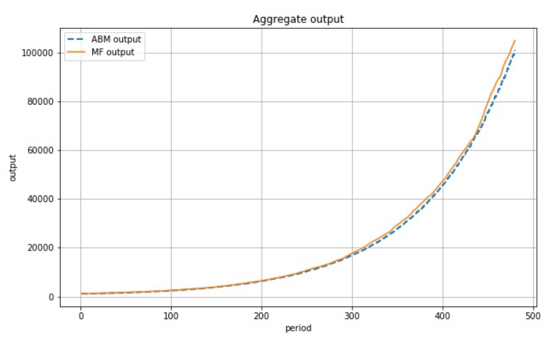
Figure 5: Proportions of firms of each type obtained from ABM simulations and the MF approximation for different specifications of state-dependent transition probabilities.



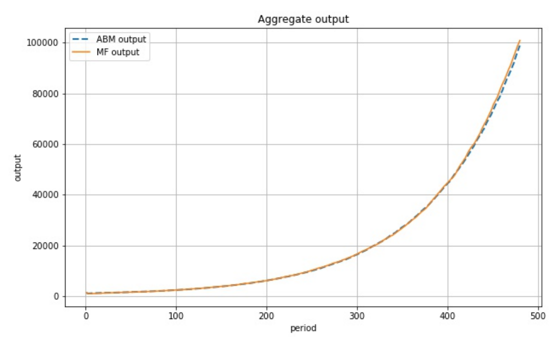
(a) Example 2



(b) Example 3



(c) Example 5



(d) Example 6

Figure 6: Aggregate output from ABM simulations and the MF approximation for different specifications of state-dependent transition probabilities.

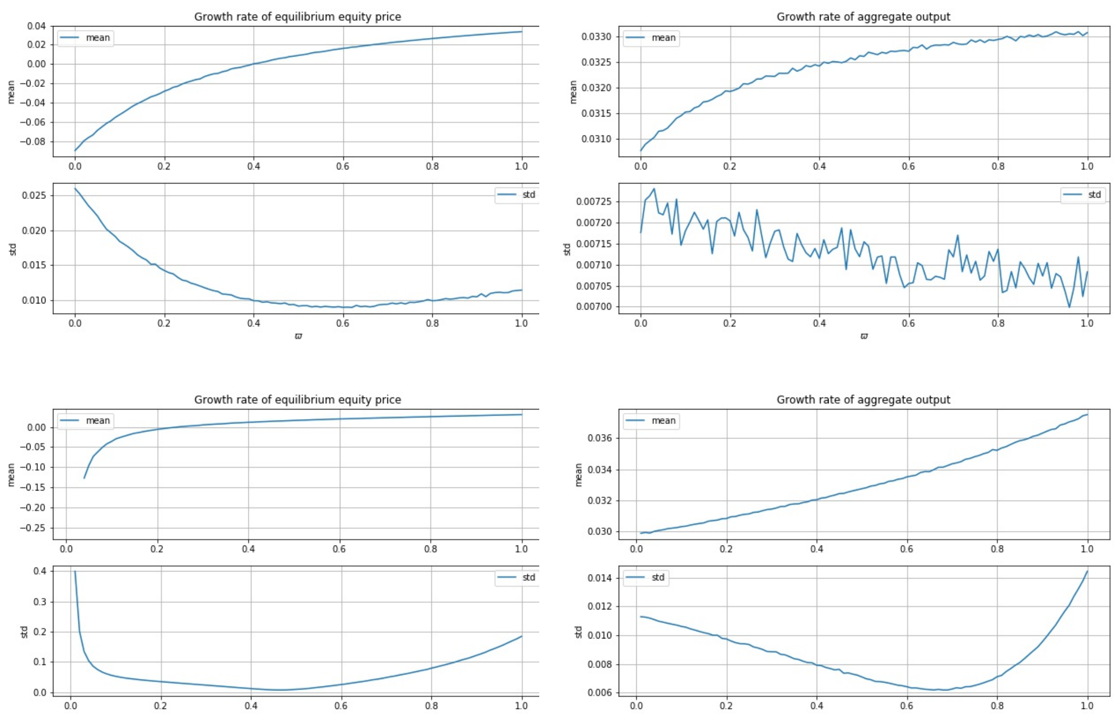


Figure 7: Sensitivity of equity price and aggregate output to the proportion ϖ of external financing raised by debt (top row) and the fraction φ of household wealth invested in the stock market (bottom row) for transition probabilities as in Example 5.

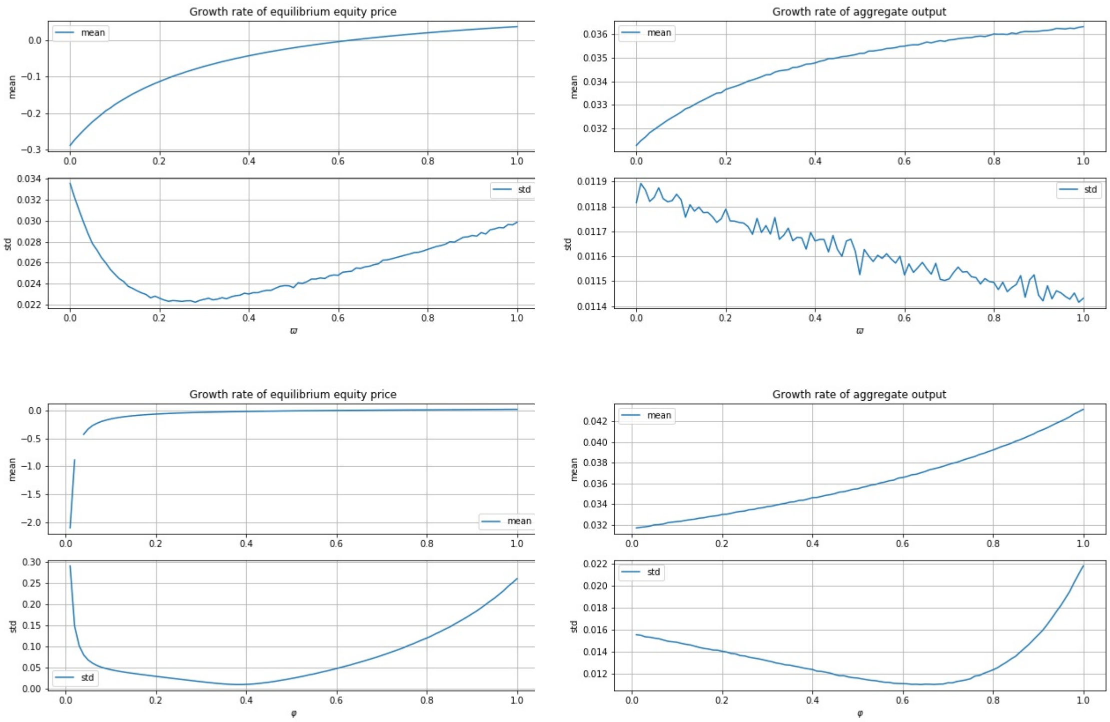


Figure 8: Sensitivity of equity price and aggregate output to the proportion ϖ of external financing raised by debt (top row) and the fraction φ of household wealth invested in the stock market (bottom row) for transition probabilities as in Example 6.

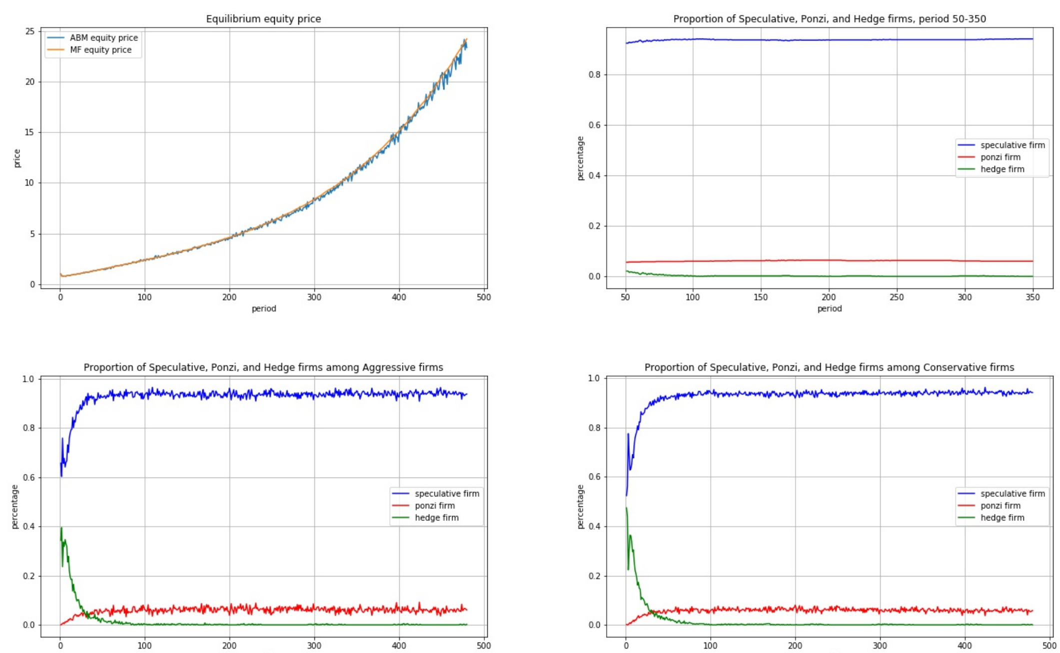


Figure 9: Equity price and proportions of hedge, speculative, and Ponzi firms in a scenario of high growth and low volatility. Parameter values are as described in Table 1.

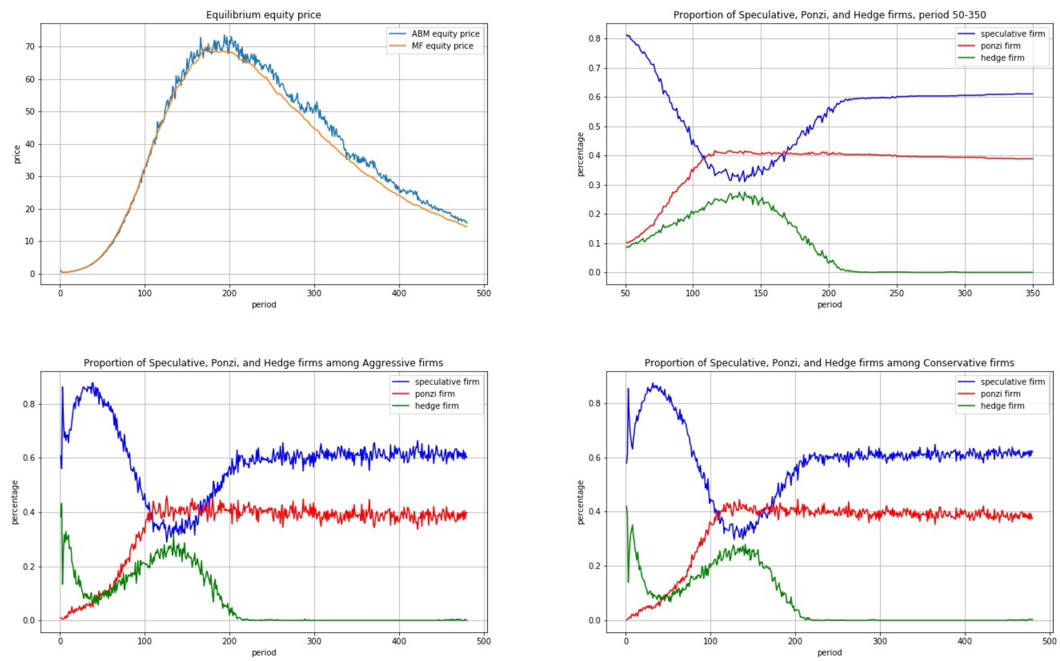


Figure 10: Equity price and proportions of hedge, speculative, and Ponzi firms in a scenario of low growth and high volatility. Parameter values are as described in Table 1, with the exception of $\varpi = 0.3$ and $\varphi = 0.3$.

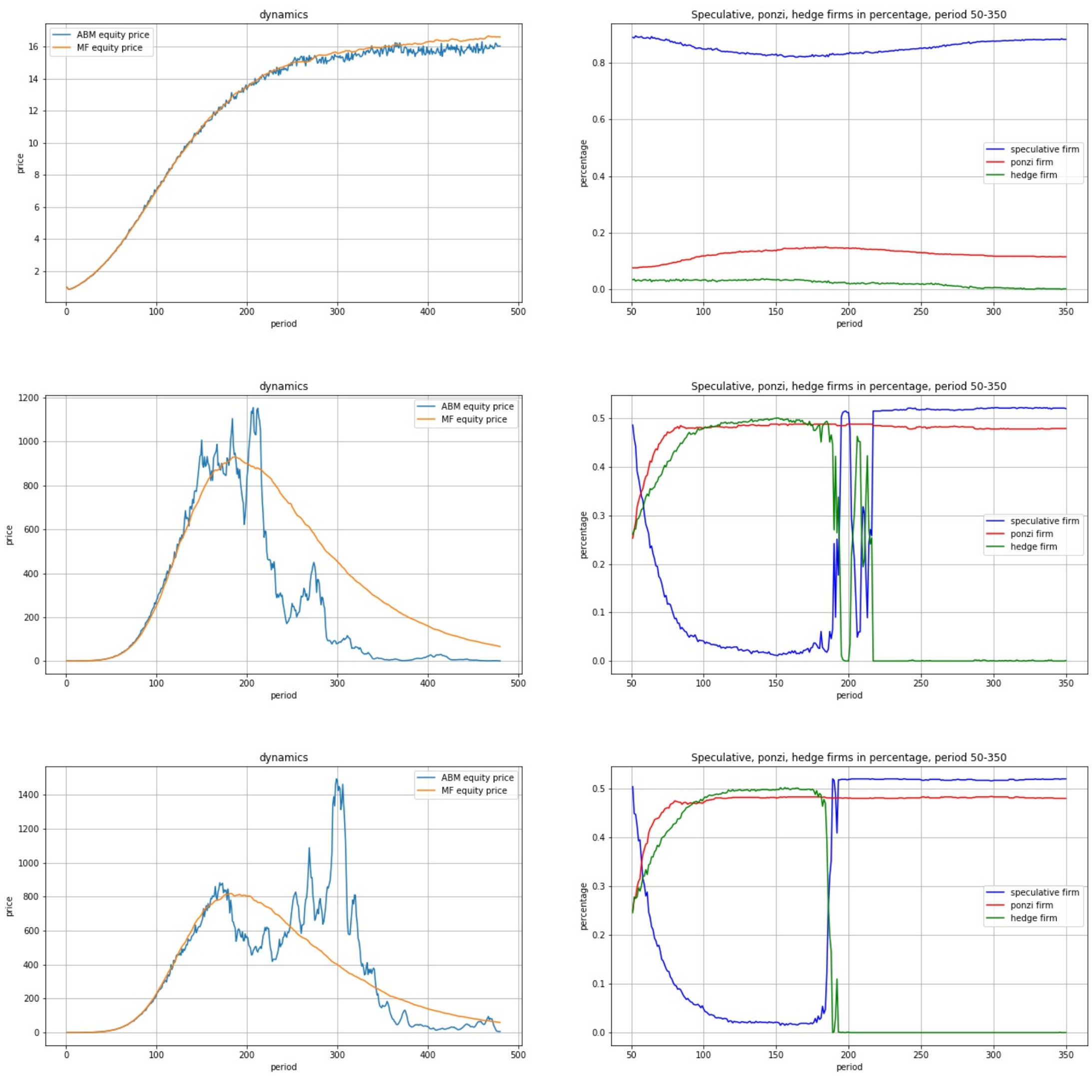


Figure 11: Equity price and proportions of hedge, speculative, and Ponzi firms for Scenarios 1 (top row) to 3 (bottom row) from Table 2. All other parameter values are as described in Table 1, with the exception of $\varpi = 0.3$ and $\varphi = 0.3$.

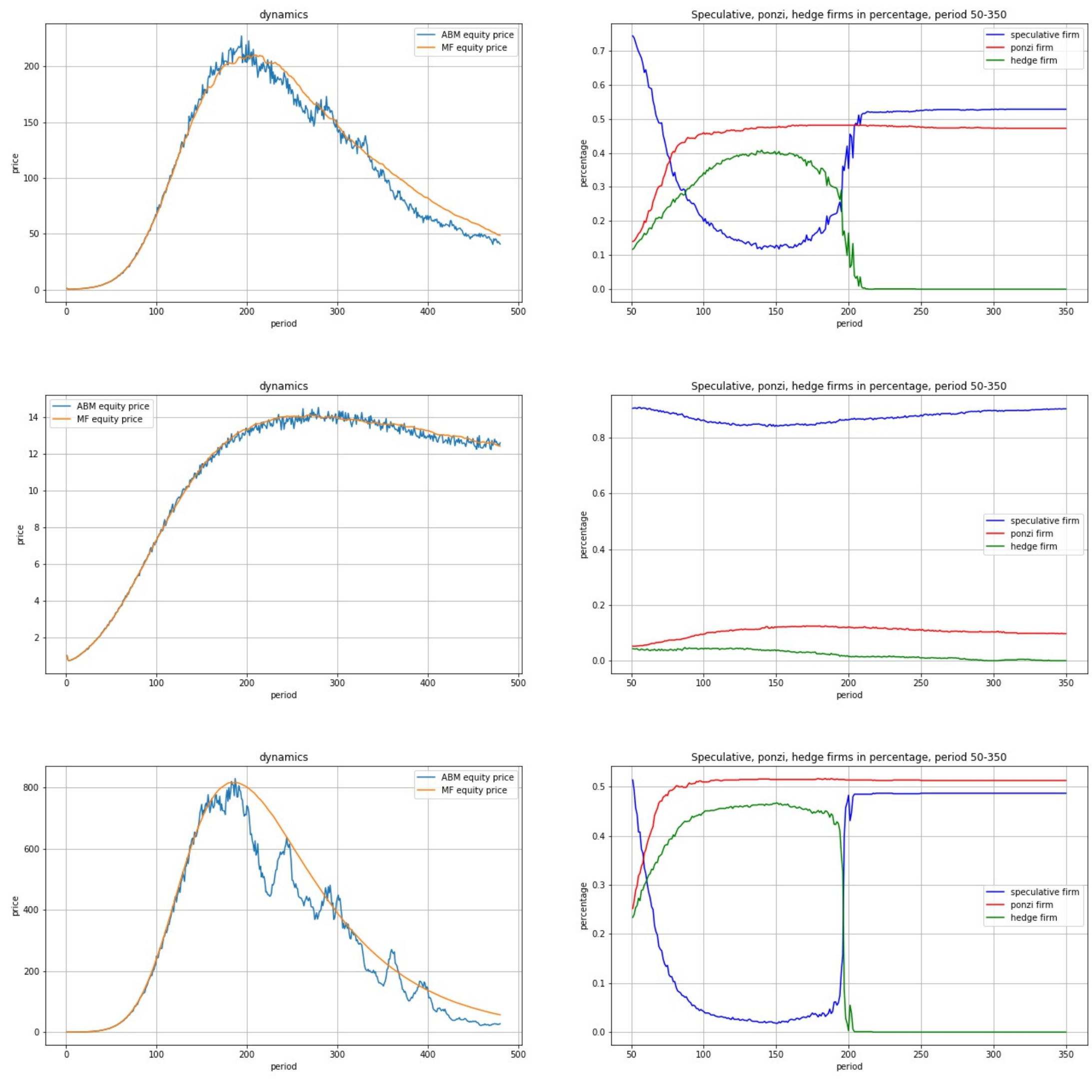


Figure 12: Equity price and proportions of hedge, speculative, and Ponzi firms for Scenarios 4 (top row) to 6 (bottom row) from Table 2. All other parameter values are as described in Table 1, with the exception of $\varpi = 0.3$ and $\varphi = 0.3$.

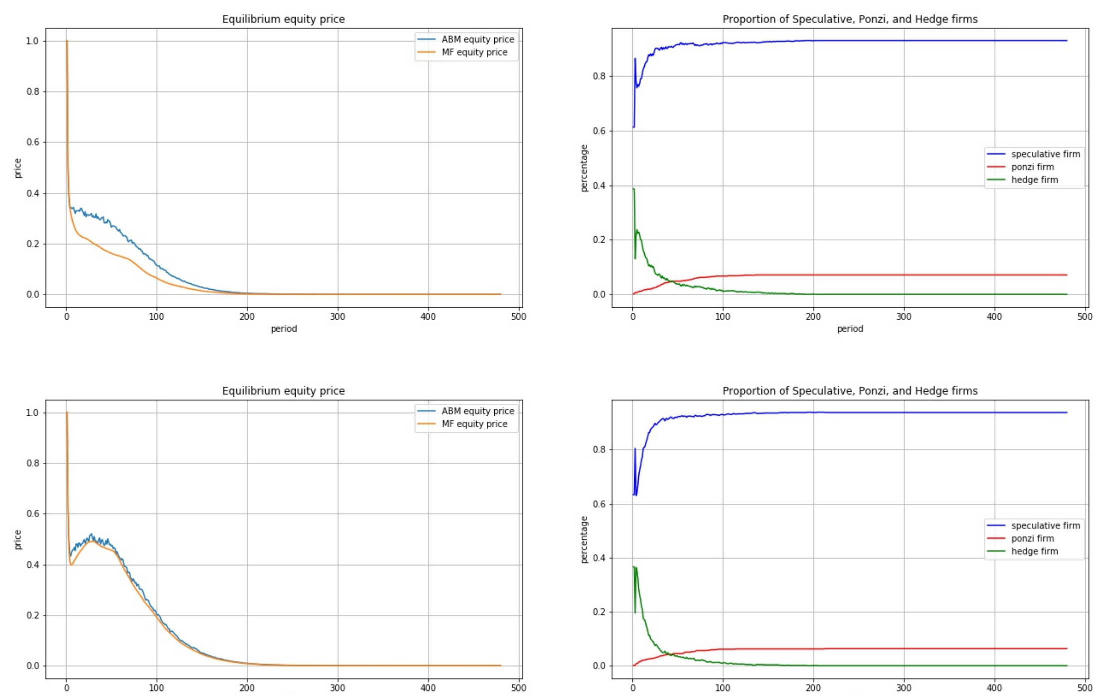


Figure 13: Equity price and proportions of hedge, speculative, and Ponzi firms for Scenarios 3 and transition probabilities as in Examples 5 (top row) and 6 (bottom row). All other parameter values are as described in Table 1, with the exception of $\varpi = 0.3$ and $\varphi = 0.3$.