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ASSET PRICE BUBBLES IN INCOMPLETE MARKETS*

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This paper studies asset price bubbles in a continuous time model using the local martingale framework. Providing careful definitions of the asset's market and fundamental price, we characterize all possible price bubbles in an incomplete market satisfying the "no free lunch with vanishing risk (NFLVR)" and "no dominance" assumptions. We show that the two leading models for bubbles as either charges or as strict local martingales, respectively, are equivalent. We propose a new theory for bubble birth that involves a nontrivial modification of the classical martingale pricing framework. This modification involves the market exhibiting different local martingale measures across time—a possibility not previously explored within the classical theory. Finally, we investigate the pricing of derivative securities in the presence of asset price bubbles, and we show that: (i) European put options can have no bubbles; (ii) European call options and discounted forward prices have bubbles whose magnitudes are related to the asset's price bubble; (iii) with no dividends, American call options are not exercised early; (iv) European put-call parity in market prices must always hold, regardless of bubbles; and (v) futures price bubbles can exist and they are independent of the underlying asset's price bubble. Many of these results stand in contrast to those of the classical theory. We propose, but do not implement, some new tests for the existence of asset price bubbles using derivative securities.

Key Words: price bubbles, local martingales, NFLVR, no dominance.

1. INTRODUCTION

Asset price bubbles have fascinated economists for centuries, one of the earliest recorded price bubbles being the Dutch tulipmania in 1634–1637 (Garber 1989, 1990), followed by the Mississippi bubble in 1719–1720 (Garber 1990), the related South Sea bubble of 1720 (Garber 1990; Temin and Voth 2004), up to the 1929 U.S. stock price crash (White 1990; De Long and Shleifer 1991; Rappoport and White 1993; Donaldson and Kamstra 1996) and the more recent NASDAQ price bubble of 1998–2000 (Ofek and Richardson 2003; Brunnermeier and Nagel 2004; Cunando, Gil-Alana, and Perez de Gracia 2005; Battalio and Schultz 2006; Pastor and Veronesi 2006). Motivated by these episodes of sharp price increases followed by price collapses, economists have studied questions related to the existence of price bubbles, both theoretically and empirically.

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Sufficient conditions for the existence and nonexistence of price bubbles in economic equilibrium has been extensively investigated. Bubbles cannot exist in finite horizon rational expectation models (Tirole 1982; Santos and Woodford 1997). They can arise, however, in markets where traders behave myopically (Tirole 1982), where there are irrational traders (De Long et al. 1990), in infinite horizon growing economies with rational traders (see Tirole 1985; O'Connell and Zeldes 1988; Weil 1990), economies where rational traders have differential beliefs and when arbitrageurs cannot synchronize trades (Abreu and Brunnermeier 2003) or when there are short sale/borrowing constraints (Santos and Woodford 1997; Scheinkman and Xiong 2003a). For good reviews, see Camerer (1989) and Scheinkman and Xiong (2003b). In these models, albeit for different reasons, arbitrageurs cannot profit from and thereby eliminate price bubbles (via their trades). Equilibria with bubbles share many of the characteristics of sunspot equilibrium where extrinsic uncertainty can affect the allocation of resources solely because of traders' self-confirming beliefs (see Cass and Shell 1983; Balasko, Cass, and Shell 1995). Indeed, in bubble economies, the self-confirming beliefs often correspond to the expectation that one can resell the asset to another trader at a higher price (see Harrison and Kreps 1978; Scheinkman and Xiong 2003b).[2]

Equilibrium models impose substantial structure on the economy, in particular, investor optimality and a market-clearing mechanism equating aggregate supply to aggregate demand. Price bubbles have also been studied in less restrictive settings, using the insights and tools of mathematical finance. These papers are mainly concerned with the characterization of bubbles and the pricing of derivative securities in finite horizon economies satisfying the "no free lunch with vanishing risk (NFLVR)" hypothesis (see Loewenstein and Willard 2000a,b; Cox and Hobson 2005; Heston, Loewenstein, and Willard 2007). Herein, bubbles violate many of the classical option pricing theorems, and in particular, put-call parity. In contrast to equilibrium pricing, these violations occur due to the absence of sufficient structure on the economy within the NFLVR framework. Nevertheless, put-call parity is almost never empirically violated (e.g., see Klemkosky and Resnick 1908; Kamara and Miller 1995; Ofek and Richardson 2003),[3] suggesting that more structure than only NFLVR is needed to understand price bubbles in realistic economies.

The missing structure is the classical notion of *no dominance* (see Merton 1973), which has largely been forgotten in the mathematical finance literature. No dominance is stronger than NFLVR, but substantially weaker than imposing a market equilibrium. Adding this hypothesis to NFLVR, Jarrow, Protter, and Shimbo (2006) recently studied bubbles in *complete* market economies with infinite trading horizons. They show that the addition of no dominance excludes all asset price bubbles. Consequently, if bubbles are to exist, markets must be incomplete. This insight motivates this paper, which extends Jarrow, Protter, and Shimbo's analysis to incomplete markets.

Given in this paper is a nonnegative stochastic price process $S = (S_t)_{t\geq 0}$ and a risk-free money market account $r = (r_t)_{t\geq 0}$, both defined on a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. The First Fundamental Theorem of asset pricing (see Delbaen and Schachermayer 1994) states that there is no arbitrage in the sense of NFLVR, if and only if there exists an equivalent probability measure Q rendering S into a

[2]These models study bubbles in competitive markets where all traders act as price takers. Of course, bubbles can arise due to market manipulation behavior as well (see Jarrow 1992 and Bank and Baum 2004, for this class of models). These models are not discussed in this paper.

[3]In private discussions with professional option traders, put-call parity is uniformly viewed as holding across all assets classes, regardless of price bubbles.

σ-martingale. However, because S is nonnegative, it is bounded below (by zero). Because any σ-martingale bounded below is a *local martingale*, only local martingales need to be considered. Hence, under Q the price process S could be a uniformly integrable martingale, just a martingale, or even a strict local martingale. Which form the price process takes relates to whether or not price bubbles exist and their characterization.

To define the concept of a bubble, we first need to define the asset's *fundamental price*. Traditionally, in a complete market, the fundamental price was defined to be the arbitrage free price as in Harrison and Kreps (1978), i.e., the asset's discounted expected cash flows using the martingale measure. As the subject has evolved, however, fundamental prices and market prices have often been confused. We define each of these prices carefully, rigorously clarifying the distinction. Furthermore, there is another complication in an incomplete market. By the Second Fundamental Theorem of asset pricing, there are a multiplicity of local martingale measures that could be used to define the fundamental price. Using the insights of Jacod and Protter (2009) and Schweizer and Wissel (2008), we select the unique local martingale measure Q consistent with the market pricing of the traded derivative securities.

As shown by Jarrow et al. (2006) in the continuous time setting, but otherwise well-known in the discrete time economics literature (see Diba and Grossman 1987; Weil 1990), a problem with the current theory of bubbles is that bubbles can end, or "burst," but that they cannot be "born" after the model begins. That is, they must exist at the start of the model or not at all. Of course, this property contradicts economic intuition and historical experience. We introduce a new theory for bubble birth which involves a nontrivial modification of the classical martingale pricing framework. This modification involves the market exhibiting different local martingale measures across time—a possibility not previously explored within the classical theory. Shifting local martingale measures corresponds to regime shifts in the underlying economic fundamentals (endowments, beliefs, risk aversion, institutional structures, technology).

The basic idea can be explained as follows. In an incomplete market, there are an infinite number of local martingale measures. When pricing derivatives, the market "chooses" a unique measure if enough derivatives trade (see Jacod and Protter 2009; Schweizer and Wissel 2008). This unique measure defines the fundamental price. A change in the measure selected can create bubbles. For example, at the start of the model, suppose that the market "chooses" a local martingale measure Q_0 which admits no bubbles. Then, at some future random time, the market exhibits a regime shift and it "chooses" a different measure Q_1 which creates a bubble. This change of measure leaves the price process unchanged, but will change the price of some derivative because the market is incomplete. This regime shift could be due to intrinsic uncertainty (Froot and Obstfeld 1991) or extrinsic uncertainty (Cass and Shell 1983)—"a sunspot." This change in measures can be thought of as roughly analogous to a phase change in an Ising model. This modification requires a nontrivial extension to standard NFLVR theory, which assumes a fixed local martingale measure for all times. Our paper contains this extension.

In the popular press, bubbles are conjectured to exist sector wide. Recent examples might include the NASDAQ price bubble of 1998–2000, or the "housing bubble" either here (Case and Shiller 2003) or earlier in Japan (Stone and Ziemba 1993). We show how the theory of bubbles for individual assets is easily extended to bubbles in market indexes and/or market portfolios.

Given the existence of bubbles in asset prices, an interesting set of questions arises as to how this existence impacts the pricing of derivative securities—calls, puts, forwards, futures; whether bubbles can independently exist in the derivative securities themselves;

and whether bubbles can, in fact, invalidate the well-known put-call parity relation. Partial answers to these questions were obtained in models using only the NFLVR assumption (see Cox and Hobson 2005). We revisit these questions herein using both the NFLVR and no dominance assumptions.

First, we extend the definition of an asset's fundamental price to the fundamental price for a derivative security. This involves one subtlety. The derivative security's payoffs are written on the market price, and not the fundamental price, of the underlying asset. Hence, the derivative's fundamental price must reflect this distinction. Given the proper definition, we show that European put options can have no bubbles, but that European call options can. In fact, the magnitude of the bubble in a European call option's price must be related to the magnitude of the bubble in the underlying asset's price. Alternatively stated, bubbles in the underlying stock price imply that there exists no local martingale measure such that the expected discounted value of the call option's payoff equals the market price. Thus, risk neutral valuation cannot be used to price call options in the presence of asset price bubbles.

Second, using Merton's (1973) original argument, but in our context, we show that European put-call parity always holds for market prices. In addition, put-call parity also holds for the fundamental prices of the relevant securities.

Third, we study American call option pricing under the standard no dividend assumption, and we show that the market price of a European call option must equal the market price of the American call option, even in the presence of asset price bubbles. This is due to the fact that American calls are not exercised early. This result extends a previous theorem of Merton's (1973) in this regard. Relative to its fundamental price, American call options themselves can have no bubbles, unlike their European counterparts. This follows because the fundamental value's stopping time (as distinct from the market price's exercise time) explicitly incorporates the impact of the price bubble. The fundamental value is stopped early because a bubble generates an effect on the asset's price process that is equivalent to a continuous dividend payment.

Finally, we study forward and futures prices. We show that the discounted forward price of a risky asset can have a bubble, and if it exists, it must equal the magnitude of the bubble in the asset's price. With respect to futures, in the existing finance literature, the characterization of a futures price implicitly (and sometimes explicitly) uses the existence of a given local martingale measure which makes the futures price a martingale (e.g., see Duffie 2001, p. 173 or Shreve 2004, p. 244). Because futures prices have bounded maturities, this excludes (by fiat), the existence of futures price bubbles. Thus, to study bubbles in futures prices, we first need to generalize the characterization of a futures price to remove this implicit (or explicit) restriction. Accomplishing this extension, we then show that futures prices can have bubbles, both positive and *negative,* and unlike discounted forward prices, the magnitude of a futures price bubble need not equal the magnitude of the underlying asset price's bubble. In a world of deterministic interest rates, however, no dominance implies that forward prices equal futures prices and therefore, their bubbles must be identical. This insight extends the classical work of Jarrow and Oldfield (1981) and Cox, Ingersoll, and Ross (1981) in this regard.

Our extension also generates an unexpected insight. Traditionally, the study of bubbles has been viewed from two apparently different perspectives, one we call the *local martingale approach*, which we discussed earlier, and the other based on finitely additive linear operators (or "charges"), as typified in Gilles (1988), Gilles and Leroy (1992), and Jarrow and Madan (2000). We show these two approaches are, in fact, the same in Theorem 8.3.

A related issue, in a portfolio context, has been studied by Cvitanic, Schachermayer, and Wang (2001).

Before concluding, we comment on the existing literature testing for asset price bubbles in various markets (e.g., Flood and Garber 1980; Evans 1986; West 1987, 1988; Diba and Grossman 1988; Donaldson and Kamstra 1996). As is well known, testing for price bubbles in the asset prices themselves involves the specification of the local martingale measure Q, and hence represents a joint hypothesis. We add no new insights in this regard. However, given our increased understanding of the pricing of derivative securities with asset price bubbles, some new tests using call and put prices are proposed. Empirical implementation of these proposed tests await subsequent research.

An outline for this paper is as follows. Section 2 provides the model setup, whereas Section 3 defines the fundamental price and price bubbles. Section 4 characterizes all possible asset price bubbles. Examples are provided in Section 5. Section 6 studies derivatives securities and Section 7 clarifies forward and futures price bubbles. Section 8 connects the local martingale approach with the charge approach to price bubbles. Finally, Section 9 concludes with a brief discussion of the empirical literature with respect to price bubbles.

2. THE MODEL

Important in studying bubbles is the precise mathematical definition of a bubble. Historically, there are two approaches: one we term the *local martingale approach* (Loewenstein and Willard 2000a,b; Cox and Hobson 2005; and Heston et al. 2007) and the other we call the *charges approach* (Gilles 1988; Gilles and Leroy 1992; Jarrow and Madan 2000). In Section 8, we show that these two approaches are the same. Therefore, without loss of generality, we first present the local martingale approach. This section presents the necessary model structure.

2.1. The Traded Assets

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered complete probability space. We assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the "usual hypotheses." (See Protter 2005, for the definition of the usual hypotheses and any other undefined terms in this paper.) We assume that our economy contains a traded risky asset and a money market account. We take the money market account as a numeraire. In particular, the price of one unit of the money market account is the constant value 1. Changing the numeraire is standard in this literature, and after the change of numeraire, the spot interest rate is zero. Consequently, all prices and *cash flows* defined later are relative to the price of the money market account.

Let τ be a stopping time which represents the maturity (or life) of the risky asset. Let $D = (D_t)_{0\leq t<\tau}$ be a càdlàg semimartingale process adapted to $\mathbb{F}$ and representing the cumulative dividend process of the risky asset. Let $X_\tau \in \mathcal{F}_\tau$ be the time τ terminal payoff or liquidation value of the asset. We assume that both $X_\tau, D \geq 0$. Throughout this paper, we use either $(X_t)_{t\geq 0}$ or X to denote a stochastic process and X_t to denote the value of the process sampled at time t. We also adopt a convention that if we give a value of a process at each t in the definition of a process, we define the process by choosing its càdlàg version unless otherwise stated (see Protter 2001, for a related discussion).

The *market price* of the risky asset is given by the nonnegative càdlàg semimartingale $S = (S_t)_{0\leq t\leq\tau}$. Note that for t such that $\triangle D_t > 0$, S_t denotes a price *ex-dividend*, because S is càdlàg.

Let W be the wealth process associated with the market price of the risky asset, i.e.,

$$W_t = \mathbf{1}_{\{t<\tau\}} S_t + \int_0^{t\wedge\tau} dD_u + X_\tau \mathbf{1}_{\{\tau\le t\}}. \tag{2.1}$$

The market value of the wealth process is the position in the stock plus all accumulated dividends, and the terminal payoff if $t \geq \tau$. Because the risky asset does not exist after τ, we focus on $[0, \tau]$ by stopping every process at τ, and then $\mathcal{F} = \mathcal{F}_\tau$.

2.2. No Free Lunch with Vanishing Risk

Key to understanding an arbitrage opportunity is the notion of a trading strategy. A *trading strategy* is defined to be a pair of adapted processes (π, η) representing the number of units of the risky asset and money market account held at time t with $\pi \in L(W)$.[4] The corresponding wealth process V of the trading strategy (π, η) is given by

$$V_t^{\pi,\eta} = \pi_t S_t + \eta_t. \tag{2.2}$$

Assume temporarily that π is a semimartingale. Then, a *self-financing trading strategy* with $V_0^\pi = 0$ is a trading strategy (π, η) such that the associated wealth process $V^{\pi,\eta}$ is given by

$$\begin{aligned} V_t^{\pi,\eta} &= \int_0^t \pi_u dW_u \\ &= \int_0^t \pi_u dS_u + \int_0^{t\wedge\tau} \pi_u dD_u + \pi_\tau X_\tau \mathbf{1}_{\{\tau\le t\}} \\ &= \left(\pi_t S_t - \int_0^t S_{u-} d\pi_u - [\pi^c, S^c]_t\right) + \int_0^{t\wedge\tau} \pi_u dD_u + \pi_\tau X_\tau \mathbf{1}_{\{\tau\le t\}} \\ &= \pi_t S_t + \eta_t \end{aligned} \tag{2.3}$$

where we have used integration by parts, and where

$$\eta_t = \int_0^{t\wedge\tau} \pi_u dD_u + \pi_\tau X_\tau \mathbf{1}_{\{\tau\le t\}} - \int_0^t S_{u-} d\pi_u - [\pi^c, S^c]_t. \tag{2.4}$$

Discarding the temporary assumption that π is a semimartingale, we can define a *self-financing trading strategy* (π, η) to be a pair of processes, with π predictable and η optional such that:

$$V_t^{\pi,\eta} = \pi_t S_t + \eta_t = \int_0^t \pi_u dW_u = (\pi W)_t,$$

where $\pi \in L(W)$ for P. As noted, a self-financing trading strategy starts with zero dollars, $V_0^{\pi,\eta} = 0$, and all proceeds from purchases/sales of the risky asset are financed/invested in the money market account. Because equation (2.4) shows that η is uniquely determined by π if a trading strategy is self-financing, without loss of generality, we represent (π, η) by π.

To avoid doubling strategies (see Harrison and Pliska 1981), we need to restrict the class of self-financing trading strategies further.

[4]See Protter (2005) for the definition of $L(W)$. Here we are still working under the original (objective) measure P.

DEFINITION 2.1 (Admissibility). Let V^π be a wealth process given by equation (2.3). We say that the trading strategy π is *a−admissible* if it is self-financing and $V_t^\pi \geq -a$ for all $t \geq 0$ almost surely. We say a trading strategy is *admissible* if it is self-financing and there exists an $a \in \mathbb{R}_+$ such that $V_t^\pi \geq -a$ for all t almost surely. We denote the collection of admissible strategies by $\mathcal{A}$.

The notion of admissibility corresponds to a lower bound on the wealth process, an implicit inability to borrow if one's collateralized debt becomes too large (e.g., see Loewenstein and Willard 2000a, for a related discussion). The restriction to admissible trading strategies is the reason bubbles can exist in our economy (see Jarrow et al. 2006).

We can now introduce the meaning of an arbitrage-free market. As shown in the mathematical finance literature (see Delbaen and Schachermayer 1994, 1998a,b; Protter 2001), the appropriate notion is that of *No Free Lunch with Vanishing Risk* (NFLVR). Let[5]

$$\mathcal{K} = \left\{ W_\infty^\pi = (\pi \cdot W)_\infty : \pi \in \mathcal{A} \right\} \tag{2.5}$$

$$\mathcal{C} = \left(\mathcal{K} - L_0^+ \right) \cap L_\infty \tag{2.6}$$

DEFINITION 2.2 (NFLVR). We say that a market satisfies NFLVR if

$$\bar{\mathcal{C}} \cap L_\infty^+ = \{0\} \tag{2.7}$$

where $\bar{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ in the sup-norm topology on L_∞.

Roughly, NFLVR effectively excludes all self-financing trading strategies that have zero initial investment, and that generate nonnegative cash flows for sure and strictly positive cash flows with positive probability (called, simple arbitrage opportunities), plus sequences of trading strategies that approach these simple arbitrage opportunities. We assume that our market satisfies NFLVR.

ASSUMPTION 2.3. *The market satisfies NFLVR.*

Key to characterizing a market satisfying NFLVR is an equivalent local martingale measure.

DEFINITION 2.4 (Equivalent Local Martingale Measure). Let Q be a probability measure equivalent to P such that the wealth process W is a Q-local martingale. We call Q an Equivalent Local Martingale Measure (ELMM), and we denote the set of ELMMs by $\mathcal{M}_{\text{loc}}(W)$.

By the First Fundamental Theorem of Asset Pricing (Delbaen and Schachermayer 1998a,b), this implies that the market admits an equivalent σ-martingale measure. By Proposition 3.3 and Corollary 3.5, Ansel and Stricker (1994, pp. 307, 309), a σ-martingale bounded from below is a local martingale. (For the definition and properties of σ-martingales, see Emery 1980; Delbaen and Schachermayer 1998a,b; Jacod and Shiryaev 2003, Section III.6e; Protter 2005). Thus, we have the following theorem:

[5] L_∞ is the set of a.s. bounded random variables and L_0^+ is the set of nonnegative finite-valued random variables.

THEOREM 2.5 (First Fundamental Theorem). *A market satisfies NFLVR if and only if there exists an ELMM.*

Theorem 2.5 holds even if the price process is not locally bounded due to the assumption that W_t is nonnegative.[6] In Jarrow et al. (2006), we studied the existence and characterization of bubbles under NFLVR in complete markets. In this paper, we discuss market prices and bubbles under NFLVR in incomplete markets. Hence, by the Second Fundamental Theorem of asset pricing (see, e.g., Protter 2005), this implies that the ELMM is not unique in general, that is $|\mathcal{M}_{\text{loc}}(S)| \geq 2$, where $|\cdot|$ denotes cardinality. The next section studies the properties of $\mathcal{M}_{\text{loc}}(S)$ in an incomplete market.

There is a literature that relates the difference between an equivalent strict local martingale measure and an equivalent martingale measure, in the context of the first fundamental theorem of asset pricing and NFLVR, to the choice of a numeraire that bounds the asset price process (see Sin 1998; Yan 1998, 2002; Xia and Yan 2002). This literature shows that a theory of bubbles depends crucially on the choice of the numeraire.[7] For our analysis, we choose the money market account as the numeraire, under which a risky asset price process is typically not bounded. From an economic perspective, this is the natural choice because a money market account represents the "riskless" investment alternative for any investor in the economy.

2.3. The Set of Equivalent Local Martingale Measures

Let $\mathcal{M}_{\text{UI}}(W)$ be the collection of equivalent measures Q such that W is a Q-uniformly integrable martingale. We call such a measure an Equivalent Uniformly Integrable Martingale Measure (EUIMM). Then, $\mathcal{M}_{\text{UI}}(W)$ is a subset of $\mathcal{M}_{\text{loc}}(W)$. Let

$$\mathcal{M}_{\text{NUI}}(W) = \mathcal{M}_{\text{loc}}(W) \backslash \mathcal{M}_{\text{UI}}(W) \tag{2.8}$$

be the subset of $\mathcal{M}_{\text{loc}}(W)$ such that W is not a uniformly integrable martingale. In general, both of the sets $\mathcal{M}_{\text{UI}}(W)$ and $\mathcal{M}_{\text{NUI}}(W)$ are nonempty. To see this in a particular case, consider the following lemma.

LEMMA 2.6. *This example is a simplified version of the example in Delbaen and Schachermayer (1998a,b). Let B^1, B^2 be two independent Brownian motions. Fix $k > 1$ and let $\sigma = \inf\{\mathcal{E}(B^2)_t = k\}$, where $\mathcal{E}(X)$ is the stochastic exponential of X given as the solution of the stochastic differential equation $dY_t = Y_{t-} dX_t$, $Y_0 = 1$. (Here we need to assume that X has no jumps smaller than -1 to ensure that X is always positive.) Define the processes Z and M by*

$$Z_t = \mathcal{E}(B^2)_{t\wedge\sigma}, \qquad M_t = \mathcal{E}(B^1)_{t\wedge\sigma}. \tag{2.9}$$

Then, Z is a uniformly integrable martingale, M is a nonuniformly integrable martingale, and the product ZM is a uniformly integrable martingale.

[6] In Delbaen and Schachermayer (1998a,b), the driving semimartingale (price process) takes value in $\mathbb{R}^d$ and is not locally bounded from below.

[7] Indeed, if two assets have bubbles relative to the money market account, they need not have a bubble relative to each other (using one of them as the numeraire).

Proof. Observe that M and Z are nonnegative local martingales. Since B^1 and B^2 are independent, $[B^1, B^2] \equiv 0$ and

$$M_t Z_t = \mathcal{E}(B^1 + B^2 + [B^1, B^2])_{t\wedge\sigma} = \mathcal{E}(B^1 + B^2)_{t\wedge\sigma}. \tag{2.10}$$

In particular, MZ is a local martingale. Z is a uniformly integrable martingale because it is bounded.

$$\begin{aligned} E[M_\infty] &= E[M_\sigma \mathbf{1}_{\{\sigma<\infty\}}] + E[M_\infty \mathbf{1}_{\{\sigma=\infty\}}] = E[M_\sigma \mathbf{1}_{\{\sigma<\infty\}}] \\ &= E\left[\int M_u P(\sigma \in du)\right] \\ &= \int E[M_u] P(\sigma \in du) \\ &= P(\sigma < \infty), \end{aligned} \tag{2.11}$$

where the second line of the equations above follows because σ and M are independent. Moreover, because the stopping time σ is the hitting time of k, we have

$$0 \le \mathcal{E}(B^2)_{t\wedge\sigma} \le k,$$

which implies that $E[\mathcal{E}(B^2)_\sigma] = 1$ (since $\mathcal{E}(B^2)$ is bounded). However,

$$1 = E[\mathcal{E}(B^2)_\sigma] = 0\,P(\sigma = \infty) + k P(\sigma < \infty),$$

which implies that $P(\sigma < \infty) = \frac{1}{k}$, and finally $E[M_\infty] = P(\sigma < \infty) = \frac{1}{k}$. It follows that M is not a uniformly integrable martingale, since $M_0 = 1 \neq \frac{1}{k}$. Similarly, we can show that

$$E[M_\infty Z_\infty] = E[M_\infty Z_\infty \mathbf{1}_{\{\sigma<\infty\}}] = k E[M_\sigma \mathbf{1}_{\{\sigma<\infty\}}] = k \times \frac{1}{k} = 1 \tag{2.12}$$

and it follows that MZ is a uniformly integrable martingale. □

COROLLARY 2.7. *There exists an NFLVR economy such that both $\mathcal{M}_{UI}(W)$ and $\mathcal{M}_{NUI}(W)$ are nonempty.*

Proof. In lemma 2.6, let Q be a probability measure under which B^1, B^2 are independent Brownian motions. Since M is not uniformly integrable under Q, $Q \in \mathcal{M}_{\mathrm{NUI}}(M)$. Define a new measure R on $\mathcal{F}_\infty$ by $dR = Z_\infty dQ$. Then by construction $R \in \mathcal{M}_{\mathrm{UI}}(M)$. □

2.4. No Dominance

As shown in Jarrow et al. (2006), NFLVR is not sufficient in a complete market to exclude bubbles. Also needed is the additional hypothesis of *no dominance* (originally used by Merton 1973). This section introduces the necessary structure for the notion of no dominance.

For each admissible trading strategy $\pi \in \mathcal{A}$, its wealth process V is given by

$$V_t^\pi = \int_0^t \pi_u dW_u, \tag{2.13}$$

where V_t^π is a σ-martingale bounded from below. Therefore, it is a local martingale under each $Q \in \mathcal{M}_{\text{loc}}(W)$.

For the remainder of the paper, let ν represent some fixed and constant (future) time. Let $\phi = (\Delta, \Xi^\nu)$ denote a payoff of an asset (or admissible trading strategy) where: (i) $\Delta = (\Delta_t)_{0 \le t \le \nu}$ is an arbitrary càdlàg nonnegative and nondecreasing semimartingale adapted to $\mathbb{F}$, which represents the asset's cumulative dividend process; and (ii) $\Xi^\nu \in \mathcal{F}_\nu$ is a nonnegative random variable, which represents the asset's terminal payoff at time ν.

Finally, let Φ_0 be the collection of all payoffs available in this form. Unfortunately, this set Φ_0 of asset payoffs is too large and lacks certain desirable properties. We, therefore, need to restrict our attention to the subset Φ of Φ_0 defined by

DEFINITION 2.8 (Set of Super-Replicated Cash Flows).

$$\text{Let } \Phi := \{\phi \in \Phi_0 : \exists \pi \in \mathcal{A}, a \in \mathbb{R}_+ \text{ such that } \Delta_\nu + \Xi^\nu \le a + V_\nu^\pi\}. \tag{2.14}$$

The set Φ represents those asset cash flows that can be super-replicated by trading in the risky asset and money market account. As seen later, it is the relevant set of cash flows for our no dominance assumption. We first show that this subset of asset cash flows is a convex cone.

LEMMA 2.9. *Φ is closed under addition and multiplication by positive scalars, i.e., it is a convex cone.*

Proof. Fix $\phi^1, \phi^2 \in \Phi$ and let $\phi = \phi^1 + \phi^2$, where $\phi^i = (\Delta^i, \Xi^{i,\nu^i})$ with maturity ν^i. Without loss of generality, we can take $\nu^1 \le \nu^2$. There exists π^1, π^2 such that

$$\Delta_t^i + \Xi^{i,\nu^i} \mathbf{1}_{\{\nu^i \le t\}} \le a^i + \int_0^t \pi_u^i \, dW_u, \quad i = 1, 2 \tag{2.15}$$

Let $\Delta_t = \Delta_t^1 + \Delta_t^2 + \Xi^{1,\nu^1} \mathbf{1}_{\{\nu^1 = t\}}$, $\nu = \nu^2$, and $\Xi^\nu = \Xi^{2,\nu^2}$.

$$\Delta_t + \Xi^\nu \mathbf{1}_{\{\nu = t\}} \le (a^1 + a^2) + \int_0^t (\pi_u^1 + \pi_u^2) \, dW_u = a + \int_0^t \pi_u \, dW_u, \tag{2.16}$$

where $a = a^1 + a^2$, $\pi = \pi^1 + \pi^2$. The proof for multiplication is trivial. □

If $\phi \in \Phi$ then for each $Q \in \mathcal{M}_{\text{loc}}(W)$,

$$E_Q[\Delta_\nu + \Xi^\nu] \le a + E_Q[V_\nu^\pi] \le a. \tag{2.17}$$

The first inequality follows because V^π is a wealth process of admissible trading strategies. The second inequality follows because V_ν^π is a nonnegative (because both Δ_ν and Ξ^ν are nonnegative) Q-local martingale bounded below, and hence a Q-supermartingale such that $V_0^\pi = 0$. Therefore, each asset $\phi \in \Phi$ is integrable under any ELMM. This is the reason for restricting our attention to the set of cash flows $\Phi \subset \Phi_0$.

This set Φ is large enough to contain many of the assets of interest in derivatives pricing. For example:

EXAMPLE 2.10 (Call Option). Consider a call option on S maturing at time T with strike price K. Assume that the stock does not pay dividends. Then, we can make the identification: $W = S$, $\Delta \equiv 0$, $\nu = T$, and $\Xi^\nu = (S_T - K)^+$.

It is easily seen that this claim is super-replicated by the trading strategy $\pi = (\mathbf{1}_{\{t \le T\}})_{t \ge 0}$ with $a = S_0$. Therefore, the payoff to this call option is in Φ.

To motivate the definition of no dominance, suppose that there are two different ways of obtaining a cash flow $\phi \in \Phi$. Assume that we can either buy an asset **A** which produces the cash flow ϕ, or that we can create an admissible trading strategy, a portfolio **B**, that also produces the cash flow ϕ. Further, suppose that the price of **A** is higher than the construction cost of **B**. In this illustration, portfolio **B** dominates asset **A**, because it has the same cash flows but a lower price.

At first glance, this situation would seem to generate a simple arbitrage trading strategy (i.e., violate NFLVR). Indeed, one would like to short asset **A** and long the trading strategy **B**. However, for many market economies, this combined trading strategy would not be admissible because of the short position in asset **A**. Hence, not all such "mispricings" are excluded by the NFLVR assumption (e.g., see Jarrow et al. 2006). To exclude such "mispricings," we need an additional assumption.

We note that if traders prefer more wealth to less, then no rational agent would ever buy **A** to hold in their optimal portfolio. If a trader wanted the cash flow ϕ, then they would hold the trading strategy **B** instead. This implies that a necessary condition for an economic equilibrium is that the price of A and the construction cost of B must coincide. Consequently, we would not expect to see any dominated assets or portfolios in a well-functioning market.

To formalize this idea, let us denote the *market price* of ϕ at time t by $\Lambda_t(\phi)$. Fix $\phi = (\Delta, \Xi^\nu) \in \Phi$. For a pair of stopping times $\sigma < \mu \leq \nu$, define the net gain $G_{\sigma,\mu}(\phi)$, by purchasing σ and selling at $\mu \leq \nu$, by

$$G_{\sigma,\mu}(\phi) = \Lambda_\mu(\phi) + \int_\sigma^\mu d\Delta_s + \Xi^\nu \mathbf{1}_{\{\nu=\mu\}} - \Lambda_\sigma(\phi). \tag{2.18}$$

DEFINITION 2.11 (Dominance). Let $\phi^1, \phi^2 \in \Phi$ be two assets. If there exists a pair of stopping time $\sigma < \mu \leq \nu$ such that

$$G_{\sigma,\mu}(\phi^2) \geq G_{\sigma,\mu}(\phi^1), \quad a.s.$$

and such that $E[\mathbf{1}_{\{G^2_{\sigma,\mu} > G^1_{\sigma,\mu}\}} | \mathcal{F}_\sigma] > 0$ a.s., then we say that asset 1 is dominated by asset 2 at time σ.

Finally, we impose the following assumption.

ASSUMPTION 2.12 (No Dominance). *Let the market price be represented by a function* $\Lambda_t : \Phi \to \mathbb{R}_+$ *such that there are no dominated assets in the market.*

This is Merton's (1973) no dominance assumption in modern mathematical terms. Note that this assumption consists of two parts. One, the fact that the market price is a function, implies that for each asset cash flow there is a unique market price. And, two, it implies that the market price must satisfy no dominance. In essence, it codifies the intuitively obvious idea that, all things being equal, financial agents prefer more to less. Different from Assumption 2.3, it does not require an admissible trading strategy to exploit any deviations. For an example which is consistent with NFLVR, but excluded by No Dominance, see Jarrow et al. (2006).

It is also important to note that no dominance also excludes suicide strategies (see Harrison and Pliska 1981 for a definition and related discussion). The notion of dominance is also related to the maximal elements used in the proof of the first fundamental theorem of asset pricing (see Delbaen and Schachermayer 1994, 1995).

3. THE FUNDAMENTAL PRICE AND BUBBLES

In the classical theory of mathematical finance, for a primary[8] asset trading in an arbitrage-free market, there is no difference between the market price, the arbitrage-free price, and the fundamental price, even if the market is incomplete (see Harrison and Kreps 1979; Harrison and Pliska 1981). This is true because the classical theory only considers finite horizon models with value processes that, under no-arbitrage, are Q-martingales for all EMM's Q. So, the traded asset's market price equals its arbitrage-free price which equals the conditional expectation of the asset's payoffs under any Q. Here (and to be made precise subsequently), the conditional expectation of the stock's payoffs is interpreted as the present value of the asset's future cash flows, called its *fundamental value*. Intuitively, defining a *bubble* as the difference between the asset's market and fundamental prices, we see that (by fiat) classical mathematical finance theory has no price bubbles!

In contrast, in the modern theory of mathematical finance (post Delbaen and Schachermayer 1994, 1998a,b), bubbles can exist. This is the local martingale approach for bubbles due to Loewenstein and Willard (2000a,b) and Cox and Hobson (2005). For a primary asset trading in a NFLVR market, although there is still no difference between the market and arbitrage-free prices, these need not equal the conditional expectation of the asset's payoffs—defined here as the fundamental price. Indeed, if for a given $Q \in \mathcal{M}_{\text{loc}}(W)$ the asset's price is a strict local martingale, then a bubble exists.

As shown by Jarrow et al. (2006), adding the assumption of no dominance to the above structure precludes the existence of bubbles in a complete market. Therefore, to study bubbles using the local martingale approach, one must investigate an incomplete market. A complication arises. In an incomplete market, by the Second Fundamental Theorem of Asset Pricing, there is a multiplicity of local martingale measures. To define the fundamental price, therefore, one of these measures must be selected. Using the same model structure, in conjunction with an arbitrary rule to choose a unique $Q \in \mathcal{M}_{\text{loc}}(W)$, generates a market with bubbles. But, unfortunately, this straightforward extension still retains the implication that bubbles cannot arise after the model starts. To obtain a theory that incorporates bubble "birth" in an incomplete market, we need to extend the standard local martingale approach as presented in Section 2. This is the purpose of the next section.

3.1. The Extended Economy

This section extends the economy of Section 2 to allow for the possibility of bubble "birth" after the model starts. A modification involves the market exhibiting different local martingale measures across time—a possibility not previously explored. Shifting local martingale measures corresponds to regime shifts in the underlying economy (in any of the economy's endowments, beliefs, risk aversion, institutional structures, or technologies). For pedagogical reasons we choose the simplest and most intuitive structure consistent with this extension. As indicated later, our extension could be easily generalized, but at a significant cost in terms of its mathematical complexity. We leave this generalization to future research.

To begin this extension, we need to define the regime shifting process. Let $(\sigma_i)_{i \geq 0}$ denote an increasing sequence of random times with $\sigma_0 = 0$. The random times $(\sigma_i)_{i \geq 0}$ represent

[8]By primary we mean not a derivative security on the asset.

the times of regime shifts in the economy. And, we let $(Y^i)_{i\geq 0}$ be a sequence of random variables characterizing the state of the economy at those times (the particular regime's characteristics) such that $(Y^i)_{i\geq 0}$ and $(\sigma)_{i\geq 0}$ are independent of each other. Moreover, we further assume that both $(Y^i)_{i\geq 0}$ and $(\sigma)_{i\geq 0}$ are also independent of the underlying filtration $\mathbb{F}$ to which the price process S is adapted.

Define two stochastic processes $(N_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ by

$$N_t = \sum_{i\geq 0} \mathbf{1}_{\{t\geq \sigma_i\}} \quad \text{and} \quad Y_t = \sum_{i\geq 0} Y^i \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}}. \tag{3.1}$$

N_t counts the number of regime shifts up to and including time t, whereas Y_t identifies the characteristics of the regime at time t. Let $\mathbb{H}$ be a natural filtration generated by N and Y and define the enlarged filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ (see Protter 2005, for a discussion of some of the general theory of filtration enlargement). By the definition of $\mathbb{G}$, $(\sigma_i)_{i\geq 0}$ is an increasing sequence of $\mathbb{G}$ stopping times.

Since N and Y are independent of $\mathbb{F}$, every $(Q, \mathbb{F})$-local martingale is also a $(Q, \mathbb{G})$-local martingale. By this independence, changing the distribution of N and/or Y does not affect the martingale property of the wealth process W. Therefore, the set of ELMMs defined on $\mathcal{G}_\infty$ is *a priori* larger than the set of ELMMs defined on $\mathcal{F}_\infty$. We are not concerned with this enlarged set of ELMMs. We will, instead, focus our attention on the $\mathcal{F}_\infty$ ELMMs and sometimes write $\mathcal{M}^{\mathbb{F}}_{\text{loc}}(W)$ to explicitly recognize this restriction. With respect to this restricted set, given the Radon Nikodym derivative $Z_\infty = \frac{dQ}{dP}|_{\mathcal{F}_\infty}$, we define its density process by $Z_t = E[Z_\infty|\mathcal{F}_t]$. Of course, Z is an $\mathbb{F}$-adapted process. Note that this construction implies that the distribution of Y and N is invariant with respect to a change of ELMMs in $\mathcal{M}^{\mathbb{F}}_{\text{loc}}(W)$.

The independence of the filtration $\mathbb{H}$ from $\mathbb{F}$ gives this increased randomness in our economy the interpretation of being *extrinsic uncertainty*. It is well known that extrinsic uncertainty can affect economic equilibrium as in the sunspot equilibrium of Cass and Shell (1983). This form of our information enlargement, however, is not essential to our arguments. It could be relaxed, making both N and Y pairwise dependent, and dependent on the original filtration $\mathbb{F}$ as well. This generalization would allow bubble birth to depend on *intrinsic uncertainty* (see Froot and Obstfeld 1991, for a related discussion of intrinsic uncertainty). However, this generalization requires a significant extension in the mathematical complexity of the notation and proofs, so it is not emphasized in the text.

3.2. The Fundamental Price

This section makes precise our definition of the fundamental price. Consistent with the economics literature, we will define the fundamental price as the asset's discounted expected cash flows given a local martingale measure. The local-martingale measure Q selected from $\mathcal{M}_{\text{loc}}(W)$ for valuation will be that measure consistent with the market prices of the traded derivative securities. Schweizer and Wissel (2008) and Jacod and Protter (2009) show that if enough derivative securities trade (of a certain type), then Q is uniquely determined. These traded derivative securities effectively complete the market. We assume the Jacod and Protter (2009) conditions hold for the remainder of the paper.

Furthermore, we hypothesize that the measure selected can depend upon the current economic regime. As the regime shifts, so can the local martingale measure selected by the market. This selection process thereby determines the fundamental value and the existence or birth of price bubbles. More formally, we let the local martingale measure in our extended economy depend on the state of the economy at time t as represented by the original filtration $(\mathcal{F}_t)_{t\geq 0}$, the state variable(s) Y_t, and the number of regime shifts N_t that have occurred. Suppose $N_t = i$. Denote $Q^i \in \mathcal{M}_{\text{loc}}(W)$ as the ELMM "selected by the market" at time t given Y^i.

As in the earlier literature on bubbles, the fundamental price of an asset (or portfolio) represents the asset's expected discounted cash flows.

DEFINITION 3.1 (Fundamental Price). Let $\phi \in \Phi$ be an asset with maturity ν and payoff (Δ, Ξ^ν). The *fundamental price* $\Lambda_t^*(\phi)$ of asset ϕ is defined by

$$\Lambda_t^*(\phi) = \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\middle|\mathcal{F}_t\right]\mathbf{1}_{\{t<\nu\}\cap\{t\in[\sigma_i,\sigma_{i+1})\}} \tag{3.2}$$

$\forall t \in [0, \infty)$ where $\Lambda_\infty^*(\phi) = 0$.

In particular, the fundamental price of the risky asset S_t^* is given by

$$S_t^* = \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\tau dD_u + X_\tau \mathbf{1}_{\{\tau<\infty\}}\middle|\mathcal{F}_t\right]\mathbf{1}_{\{t<\tau\}\cap\{t\in[\sigma_i,\sigma_{i+1})\}}. \tag{3.3}$$

To understand this definition, let us focus on the risky asset's fundamental price. At any time $t < \tau$, given that we are in the ith regime $\{\sigma_i \leq t < \sigma_{i+1}\}$, the right side of expression (3.3) simplifies to

$$S_t^* = E_{Q^i}\left[\int_t^\tau dD_u + X_\tau \mathbf{1}_{\{\tau<\infty\}}\middle|\mathcal{F}_t\right].$$

Given the market's choice of the ELMM is $Q^i \in \mathcal{M}_{\text{loc}}(W)$ at time t, we see that the fundamental price equals its expected future cash flows. Note that the payoff of the asset at infinity, $X_\tau \mathbf{1}_{\{\tau=\infty\}}$, does not contribute to the fundamental price. This reflects the fact that agents cannot consume the payoff $X_\tau \mathbf{1}_{\{\tau=\infty\}}$.[9] Furthermore, note that at time $\tau = \infty$, the fundamental price $S_\tau^* = 0$. We emphasize that a fundamental price is not necessarily the same as the market price S_t. Under NFLVR and no dominance, the market price S_t equals the arbitrage-free price, but these need not equal the fundamental price S_t^*.

For notational simplicity, we can alternatively rewrite the fundamental price in terms of an equivalent probability measure, indexed by time t, that is not a local martingale measure because of this time dependence.

THEOREM 3.2. *There exists an equivalent probability measure Q^{t^*} such that*

$$\Lambda_t^*(\phi) = E_{Q^{t^*}}\left[\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\middle|\mathcal{F}_t\right]\mathbf{1}_{\{t<\nu\}}. \tag{3.4}$$

Proof. Let $Z^i \in \mathcal{F}_\infty$ be a Radon Nykodym derivative of Q^i with respect to P and $Z_t^i = E[Z^i|\mathcal{F}_t]$. Define

$$Z_\infty^{t^*} = \sum_{i=0}^{\infty} Z^i \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}. \tag{3.5}$$

[9]This convention is nonetheless somewhat arbitrary. The alternative convention is to include $X_\tau \mathbf{1}_{\{\tau=\infty\}}$ in the asset's cash flows. The consequence would be that there are no type 1 bubbles (as defined subsequently).

Then $Z_\infty^{t^*} > 0$ almost surely and

$$\begin{aligned} EZ_\infty^{t^*} &= E\left[\sum_{i=0}^{\infty} Z^i \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}\right] = \sum_{i=0}^{\infty} E[Z^i \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}] \\ &= \sum_{i=0}^{\infty} E[Z^i] E[\mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}] \\ &= \sum_{i=0}^{\infty} P(\sigma_i \le t < \sigma_{i+1}) \\ &= 1. \end{aligned} \tag{3.6}$$

Therefore, we can define an equivalent measure Q^{t^*} on $\mathcal{F}_\infty$ by $dQ^{t^*} = Z_\infty^{t^*} dP$. The Radon–Nykodim density $Z_t^{t^*}$ on $\mathcal{G}_t$ is

$$\begin{aligned} Z_t^{t^*} = \left.\frac{dQ^{t^*}}{dP}\right|_{\mathcal{G}_t} &= E[Z^{t^*}|\mathcal{F}_t] = \sum_{i=0}^{\infty} E[Z^i \mathbf{1}_{\{t\in[\sigma_i.\sigma_{i+1})\}}|\mathcal{G}_t] \\ &= \sum_{i=0}^{\infty} E[Z^i|\mathcal{G}_t]\mathbf{1}_{\{t\in[\sigma_i.\sigma_{i+1})\}}. \end{aligned} \tag{3.7}$$

Then

$$\begin{aligned} \Lambda_t^*(\phi) &= \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\middle| \mathcal{F}_t\right] \mathbf{1}_{\{t<\nu\}\cap\{t\in[\sigma_i,\sigma_{i+1})\}} \\ &= \sum_{i=0}^{\infty} E_{Q^i}\left[\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\middle| \mathcal{G}_t\right] \mathbf{1}_{\{t<\nu\}\cap\{t\in[\sigma_i,\sigma_{i+1})\}} \\ &= E\left[\left(\sum_{i=0}^{\infty} \frac{Z^i}{Z_t^i}\mathbf{1}_{\{t\in[\sigma,\sigma_{i+1})\}}\right)\left(\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\right)\middle| \mathcal{G}_t\right]\mathbf{1}_{\{t<\nu\}} \end{aligned} \tag{3.8}$$

and observing that

$$\frac{Z^i}{Z_t^i}\mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1}]\}} = \frac{Z^i \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}}{\sum_{i=0}^{\infty} Z_t^i \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}},$$

we can continue

$$\begin{aligned} &= E\left[\left(\frac{\sum_{i=0}^{\infty} Z^i \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}}{\sum_{i=0}^{\infty} Z_t^i \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}}\right)\left(\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\right)\middle| \mathcal{G}_t\right]\mathbf{1}_{\{t<\nu\}} \\ &= E\left[\left(\frac{Z_\infty^{t^*}}{Z_t}\right)\left(\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\right)\middle| \mathcal{G}_t\right]\mathbf{1}_{\{t<\nu\}} \\ &= E_{Q^{t^*}}\left[\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\middle| \mathcal{G}_t\right]\mathbf{1}_{\{t<\nu\}} \\ &= E_{Q^{t^*}}\left[\int_t^\nu d\Delta_u + \Xi^\nu \mathbf{1}_{\{\nu<\infty\}}\middle| \mathcal{F}_t\right]\mathbf{1}_{\{t<\nu\}}. \end{aligned} \tag{3.9}$$

□

We call Q^{t^*} the *valuation measure* at t, and the collection of valuation measures $(Q^{t^*})_{t\geq 0}$ the *valuation system*.

We next introduce the notions of static and dynamic markets. Static markets correspond to those markets considered in classical martingale pricing theory. Dynamic markets are a new concept.

DEFINITION 3.3 (Static and Dynamic Markets). If $N_t = 1$ for all t (no regime shifts), then

$$Q^{t^*}(A) = Q^0(A) \quad \forall A \in \mathcal{F}_\infty, t \geq 0. \tag{3.10}$$

In this case, we say the valuation system is *static*. By construction, in a static market, such a $Q^{t^*} \in \mathcal{M}_{\text{loc}}(W)$.

If the market is not static, we say that it is *dynamic*.

In general, markets are dynamic, although such markets are not studied in classical martingale pricing theory. The * superscript is used to emphasize that Q^{t^*} is the measure *chosen by the market*, and the superscript t is used to indicate that it is selected at time t. In the ith regime $\{\sigma_i \leq t < \sigma_{i+1}\}$, the valuation measure coincides with $Q^i \in \mathcal{M}_{\text{loc}}(W)$. As noted before, since Q^{t^*} is a family of ELMMs and not one that is fixed, $Q^{t^*} \notin \mathcal{M}_{\text{loc}}(W)$ in general, unless the system is static.[10]

Given the definition of an asset's fundamental price, we can now define the fundamental wealth process.

For subsequent usage, we see that the fundamental wealth process of the risky asset is given by

$$W_t^* = S_t^* \mathbf{1}_{\{t<\tau\}} + \int_0^{\tau\wedge t} dD_u + X_\tau \mathbf{1}_{\{\tau\leq t\}}. \tag{3.11}$$

Then,

$$W_t^* = \sum_{i=0}^{\infty} E_{Q^i}\left[\int_0^\tau dD_u + X_\tau \mathbf{1}_{\{\tau<\infty\}} \middle| \mathcal{F}_t\right] \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}} \tag{3.12}$$

$\forall t \in [0,\infty)$ and $W_\infty^* = \int_0^\tau dD_u + X_\tau \mathbf{1}_{\{\tau<\infty\}}$.

Alternatively, we can rewrite W_t^* by

$$W_t^* = \sum_{i=0}^{\infty} E_{Q^i}\left[W_\infty^* | \mathcal{F}_t\right] \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}} \quad \forall t \in [0,\infty). \tag{3.13}$$

In general, the choice of a particular ELMM affects fundamental values. But, for a certain class of ELMMs, when $\tau < \infty$ the fundamental values are invariant. This invariant class is characterized in the following lemma.

LEMMA 3.4. *Suppose $\tau < \infty$ almost surely. In the ith regime $\{\sigma_i \leq t < \sigma_{i+1}\}$, if the market chooses $Q^i \in \mathcal{M}_{UI}(W)$, then the fundamental price of the risky asset S_t^* and fundamental wealth W_t^* do not depend on the choice of the measure Q^i almost surely.*

[10]Although the definition of the fundamental price as given depends on the construction of the extended economy, one could have alternatively used expression (3.4) as the initial definition. This alternative approach relaxes the extrinsic uncertainty restriction explicit in our extended economy.

Proof. Fix Q^*, $R^* \in \mathcal{M}_{UI}(W)$. $\tau < \infty$ implies that $W_\infty = W^*_\infty$. Let $W_t^{Q^*}$ and $W_t^{R^*}$ be the fundamental prices on $\{\sigma_i \le t < \sigma_{i+1}\}$ when $Q^i = Q^*$ and R^*, respectively. Since W is uniformly integrable martingale under Q^* and R^*,

$$\begin{aligned} W_t^{Q^*} &= E_{Q^*}\left[W^*_\infty|\mathcal{F}_t\right] = E_{Q^*}\left[W_\infty|\mathcal{F}_t\right] \\ &= W_t = E_{R^*}\left[W_\infty|\mathcal{F}_t\right] \\ &= E_{R^*}\left[W^*_\infty|\mathcal{F}_t\right] \\ &= W_t^{R^*} \quad \text{a.s. on } \{\sigma_i \le t < \sigma_{i+1}\} \end{aligned} \tag{3.14}$$

The difference of $W_t^{Q^*}$ and $S_t^{Q^*}$ does not depend on the choice of measure. Therefore, $W_t^{Q^*} = W_t^{R^*}$ implies $S_t^{Q^*} = S_t^{R^*}$ on $\{\sigma_i \le t < \sigma_{i+1}\}$. □

This lemma applies to the risky asset only. If the measure shifts from $Q^i \in \mathcal{M}_{UI}(W)$ to $R^i \in \mathcal{M}_{UI}(W)$, then the fundamental price of other assets can in fact change.

The next lemma describes the relationship between the fundamental prices of the risky asset when two measures are involved, one being a measure $R^* \in \mathcal{M}_{\mathrm{NUI}}(W)$.

LEMMA 3.5. *Suppose $\tau < \infty$. In the ith regime $\{\sigma_i \le t < \sigma_{i+1}\}$, consider the case where $Q^i \in \mathcal{M}_{UI}(W)$ and $R^i \in \mathcal{M}_{NUI}(W)$. Then,*

$$W_t^{R^*} \le W_t^{Q^*}, \quad a.s. \quad on\ \{\sigma_i \le t < \sigma_{i+1}\}. \tag{3.15}$$

That is, the fundamental price based on a uniformly integrable martingale measure is greater than that based on a nonuniformly integrable martingale measure.

Proof. Pick $Q^* \in \mathcal{M}_{\mathrm{UI}}(W)$ and $R^* \in \mathcal{M}_{\mathrm{NUI}}(W)$. Since $\tau < \infty$ almost surely, $W_\infty = W^*_\infty$. Under R^*, W is not a uniformly integrable nonnegative martingale and $W_t \ge E_{R^*}[W_\infty|\mathcal{F}_t]$. Therefore,

$$\begin{aligned} W_t^{Q^*} - W_t^{R^*} &= E_{Q^*}\left[W^*_\infty|\mathcal{F}_t\right] - E_{R^*}\left[W^*_\infty|\mathcal{F}_t\right] \\ &= E_{Q^*}\left[W_\infty|\mathcal{F}_t\right] - E_{R^*}\left[W_\infty|\mathcal{F}_t\right] \\ &= W_t - E_{R^*}\left[W_\infty|\mathcal{F}_t\right] \\ &\ge 0. \end{aligned} \tag{3.16}$$

□

We can now finally define what we mean by a price bubble.

3.3. Bubbles

As standard in the economics literature,

DEFINITION 3.6 (Bubble). An asset price bubble β for S is defined by

$$\beta = S - S^*. \tag{3.17}$$

Recall that S_t is the market price and S_t^* is the fundamental value of the asset. Hence, a price bubble is defined as the difference in these two quantities.

4. A CHARACTERIZATION OF BUBBLES

This section characterizes all possible price bubbles in both static and dynamic models.

4.1. Static Markets

Static markets are the first natural generalization of a complete market. In a complete market, there is only one ELMM. In a static market, there is also only one ELMM across all times, although markets may be incomplete. Because the complete market case was studied in Jarrow et al. (2006), and the analysis is very similar, the reader is referred to the original paper for the relevant proofs.

By definition, in a static market, there exists a unique $Q^* \in \mathcal{M}_{\text{loc}}(W)$ such that $Q^{*t}(A) = Q^*(A)$ for all $t \geq 0$. Then, the fundamental wealth process W_t^* is given by

$$\begin{aligned} W_t^* &= E_{Q^*}\left[\int_t^\tau dD_u + X_\tau \mathbf{1}_{\{\tau<\infty\}} \middle| \mathcal{F}_t\right]\mathbf{1}_{\{t<\tau\}} \\ &\quad + \int_0^{t\wedge\tau} dD_u + X_\tau \mathbf{1}_{\{\tau\leq t\}} \\ &= E_{Q^*}\left[\int_0^\tau dD_u + X_\tau \mathbf{1}_{\{\tau<\infty\}} \middle| \mathcal{F}_t\right], \end{aligned}$$

which is a Q^*-uniformly integrable martingale. Since W is Q^*-local martingale, this implies that the price bubble β is a Q^*-local martingale.

Recall that the stopping time τ represents the maturity of our risky asset.

THEOREM 4.1. *If there exists a nontrivial bubble $\beta \not\equiv 0$, then we have three possibilities:*

(1) β is a local martingale (which could be a uniformly integrable martingale) if $P(\tau = \infty) > 0$.
(2) β is a local martingale but not a uniformly integrable martingale if is unbounded, but with $P(\tau < \infty) = 1$.
(3) β is a strict Q^*-local martingale,[11] *if τ is a bounded stopping time.*

As indicated, there are three types of bubbles that can be present in an asset's price. Type 1 bubbles occur when the asset has an infinite life with a payoff at $\{\tau = \infty\}$. Type 2 bubbles occur when the asset's life is finite, but unbounded. Type 3 bubbles are for assets whose lives are bounded.

The first question one considers when discussing bubbles is why arbitrage does not exclude bubbles in an NFLVR economy. To answer this question, let us consider the obvious candidate trading strategy for an arbitrage opportunity. This trading strategy is to short the risky asset during the bubble, and to cover the short after the bubble crashes. For type 1 and type 2 bubbles, this trading strategy fails to be an arbitrage because all trading strategies must terminate in finite time, and the bubble may outlast this trading strategy with positive probability. For type 3 bubbles, this trading strategy fails because of the admissibility condition. Admissibility requires the trading strategy's wealth to exceed some fixed lower bound almost surely. Unfortunately, with positive probability, a type 3 bubble can increase such that the short position's losses violate the lower bound. The admissibility condition is a type of short sale restriction, and these are well known to generate bubbles in equilibrium models (see Santos and Woodford 1997; Scheinkman and Xiong 2003a). For examples of bubbles in an NFLVR market, we refer the reader to Jarrow et al. (2006).

In a complete market, the addition of no dominance assumption excludes these bubbles due to the ability of an admissible trading strategy to generate a *long* position in the asset at

[11] A strict local martingale is a local martingale that is not a martingale.

a lower cost than purchasing the asset directly (due to the bubble). Note that synthetically creating a long position in the asset does not violate the NFLVR admissibility restriction. Because a static market need not be complete, bubbles are not excluded by the no dominance assumption (because the replicating strategy need not exist).

We can refine Theorem 4.1 to obtain a unique decomposition of an asset price bubble that yields some additional insights.

THEOREM 4.2. *S admits a unique (up to an evanescent set) decomposition*

$$S = S^* + \beta = S^* + (\beta^1 + \beta^2 + \beta^3), \tag{4.1}$$

where $\beta = (\beta_t)_{t\geq 0}$ is a càdlàg local martingale and

1. *β^1 is a càdlàg nonnegative uniformly integrable martingale with $\beta^1_t \to X_\infty$ almost surely,*
2. *β^2 is a càdlàg nonnegative nonuniformly integrable martingale with $\beta^2_t \to 0$ almost surely,*
3. *β^3 is a càdlàg nonnegative supermartingale (and strict local martingale) such that $E\beta^3_t \to 0$ and $\beta^3_t \to 0$ almost surely. That is, β^3 is a potential.*

Furthermore, $(S^* + \beta^1 + \beta^2)$ is the greatest submartingale bounded above by W.

As in the previous Theorem 4.1, β^1, β^2, β^3 correspond to the types 1, 2, and 3 bubbles, respectively. First, for type 1 bubbles with infinite maturity, we see that the β^1 bubble component converges to the asset's value at time ∞, X_∞. This time ∞ value X_∞ can be thought of as analogous to fiat money, embedded as part of the asset's price process. Indeed, it is a residual value to an asset that pays zero dividends for all finite times. Second, this decomposition also shows that for finite maturity assets, $\tau < \infty$, the critical threshold is that of uniform integrability. This is due to the fact that when $\tau < \infty$, the β^2, β^3 bubble components converge to 0 almost surely, while they need not converge in L^1. Finally, the β^3 bubble components are strict local martingales, and not martingales.

As a direct consequence of this theorem, we obtain the following corollary.

COROLLARY 4.3. *Any asset price bubble β has the following properties:*

1. $\beta \geq 0$,
2. $\beta_\tau \mathbf{1}_{\{\tau<\infty\}} = 0$,
3. *if $\beta_t = 0$ then $\beta_u = 0$ for all $u \geq t$, and*
4. *if no dividends, then $S_t = E_{Q^*}[S_T|\mathcal{F}_t] + \beta^3_t - E_{Q^*}[\beta^3_T|\mathcal{F}_t]$ for any $t \leq T \leq \tau$.*

Condition (1) states that bubbles are always nonnegative, i.e., the market price can never be less than the fundamental value. Condition (2) states that if the bubble's maturity is finite $\tau < \infty$, then the bubble must burst on or before τ. Finally, condition (3) states that if the bubble ever bursts before the asset's maturity, then it can never start again. Alternatively stated, condition (3) states that in the context of our model, bubbles must either exist at the start of the model or they never will exist. And, if they exist and burst, then they cannot start again (this corollary is well known in the empirical literature for discrete time economies, see, e.g., Diba and Grossman 1987; Weil 1990). Condition (4) shows why the market price S is not a Q^* martingale. The difference between the market price and its conditional expectation is due to the type 3 bubble component, because the fundamental value, the type 1, and the type 2 bubble components are themselves Q^* martingales.

4.2. Dynamic Markets

In a dynamic market, there is no single ELMM generating fundamental values across time. The valuation measures Q^{s^*} and Q^{t^*} at times $s < t$ are usually two different measures, and neither is an ELMM. It follows, therefore, that the local martingale property of a bubble β in a static market is no longer preserved.

The following is a trivial but important observation generalizing Corollary 4.3 to dynamic markets.

THEOREM 4.4. *Bubbles are nonnegative. That is, if β denotes a bubble, then $\beta_t \geq 0$ for all $t \geq 0$.*

Proof. Fix $t \geq 0$. On $\{\sigma_i \leq t < \sigma_{i+1}\}$, the market chooses Q^i as a valuation measure and the fundamental price S_t^* is given by

$$\begin{aligned} S_t^* \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} &= E_{Q^i}\left[\int_t^\tau dD_u + X_\tau \mathbf{1}_{\{\tau<\infty\}} \mid \mathcal{F}_t\right] \mathbf{1}_{\{t<\tau\}} \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \\ &= S_t^{*i} \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}}, \end{aligned} \tag{4.2}$$

where S_t^{*i} denotes a fundamental price with valuation measure $Q^i \in \mathcal{M}_{\mathrm{loc}}(W)$ and

$$S_t^* = \sum_i S_t^{*i} \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}} \tag{4.3}$$

and

$$\beta_t^* = \sum_i \beta_{i,t} \mathbf{1}_{\{\sigma_i \leq t < \sigma_{i+1}\}}. \tag{4.4}$$

By Corollary 4.3, $\beta_i = S - S^{*i} \geq 0$ for each i and hence $\beta^* \geq 0$. □

Negative bubbles do not exist even in a dynamic market.

As shown in the previous section, bubble birth is not possible in a static market. In contrast, in a dynamic market, bubble birth is possible as the next example shows.

EXAMPLE 4.5. Suppose that the measure chosen by the market shifts at time σ_0 from $Q \in \mathcal{M}_{\mathrm{UI}}(W)$ to $R \in \mathcal{M}_{\mathrm{NUI}}(W)$. To avoid ambiguity, we denote a fundamental price based on valuation measures Q and R by W^{Q^*} and W^{R^*}, respectively. By Lemma 3.5, we can choose Q, R, and σ_0 such that the difference of fundamental prices based on these two measures,

$$W_{\sigma_0}^{Q^*} - W_{\sigma_0}^{R^*} \geq 0, \tag{4.5}$$

is strictly positive with positive probability. Then, the fundamental price and the bubble are given by

$$W_t^* = W_t^{Q^*} \mathbf{1}_{\{t<\sigma_0\}} + W_u^{R^*} \mathbf{1}_{\{\sigma_0 \leq t\}} \tag{4.6}$$

$$\beta_t = \beta_t^R \mathbf{1}_{\{\sigma_0 \leq t\}}. \tag{4.7}$$

And, a bubble is born at time σ_0.

As shown in Lemma 3.4, a switch from one measure Q to another measure Q' such that $Q, Q' \in \mathcal{M}_{\mathrm{UI}}(W)$ does not change the value of W^*. Therefore, if a bubble does not exist under Q, it also does not exist under Q'. Bubble birth occurs only when a valuation measure changes from a uniformly integrable martingale $Q \in \mathcal{M}_{\text{UI}}(W)$ to a nonuniformly integrable martingale $R \in \mathcal{M}_{\text{NUI}}(W)$.

5. EXAMPLES

In this section, we discuss several examples. Because Type 1 bubbles are simple and few assets have infinite lifetimes, we focus on assets with finite (but possibly unbounded) maturities.

5.1. Assets with Bounded Payoffs

We first consider those risky assets that have bounded payoffs.

THEOREM 5.1. *If $\int_0^\tau dD_u + X_\tau$ is bounded, then $S_t = S_t^*$ and the asset price does not have bubbles.*

Proof. By hypothesis, there exists $a \in \mathbb{R}_+$ such that $\int_0^\tau dD_u + X_\tau < a$. Then holding a units of the money market account dominates holding the risky asset. By No Dominance (Assumption 2.12)

$$S_t = \Lambda_t((D, X^\tau)) \le a. \tag{5.1}$$

Because a bounded local martingale is a uniformly integrable martingale, all ELMMs are in $\mathcal{M}_{\text{UI}}(W)$ and bubbles do not exist in S. □

Theorem 5.1 also holds for any arbitrary asset $\phi \in \Phi$ with bounded payoffs. We now provide some useful examples of assets with bounded payoffs.

EXAMPLE 5.2 (Arrow–Debreu Securities). Let ν be an $\mathbb{F}$-stopping time such that $\nu \le \tau$ almost surely and $A \in \mathcal{F}_\nu$. Consider an Arrow–Debreu security paying 1 at ν for $\nu \le \tau$ if event A happens, denoted by $\phi_A = (0, \mathbf{1}_A^\nu)$.[12]

Then, the market price of ϕ_A does not have a bubble, i.e.,

$$\Lambda_t(\phi_A) = \Lambda_t^*(\phi_A) = \sum_i E_{Q^i}[1_A|\mathcal{F}_t]\mathbf{1}_{\{t<\tau\}}\mathbf{1}_{\{\sigma_i \le t < \sigma_{i+1}\}}. \tag{5.2}$$

The market price of Arrow–Debreu securities equal the conditional valuation probability of $A \in \mathcal{F}_\nu$ implied by the market.

EXAMPLE 5.3 (Fixed Income Securities). Consider a default free coupon bond with coupons of C paid at times $t_1, \ldots, t_n = \nu \le \tau$ and a principal payment of P at time τ,[13] where τ is the maturity date of the bond. Then, letting $\Delta_t \equiv \sum_{i=1}^n C1_{\{t_i \le t\}}$ and $\Xi^\nu = P$, we have $\phi = (\Delta, \Xi^\nu)$ with $\Delta_\tau + \Xi^\nu$ bounded by the sum of all the coupons and principal

[12] Recall that we are using the money market account as the numeraire. A transformed analysis applies in the original (dollar) economy. Here, however, the payoff to the Arrow–Debreu security needs to be redefined to be 1 dollar at time ν. Letting A_ν denote the time ν market price of the money market account, the payoff to the Arrow–Debreu security in the numeraire is then $1/A_\nu$ units at time ν, and not 1 unit. This change has no affect on our analysis, because if the spot rate of interest $r \ge 0$ almost surely, with $A_0 = 1$, then $1/A_\nu \le 1$ almost surely.

[13] As with the Arrow–Debreu securities, these payoffs are in units of the money market account and they need to be appropriately transformed to get dollar prices in the original economy.

payments. Then, by Theorem 5.1, the default free bond price has no bubbles, i.e.,

$$\begin{aligned} \Lambda_t(\phi) &= \Lambda_t^*(\phi) \\ &= \sum_i E_{Q^i}\left[\sum_{i=1}^{n} C1_{\{t_i>t\}} + P1_{\{\nu>t\}}|\mathcal{F}_t\right]\mathbf{1}_{\{t<\tau\}}\mathbf{1}_{\{\sigma_i\le t<\sigma_{i+1}\}}. \end{aligned} \tag{5.3}$$

Although this example applies to default free bonds, the same logic can be used to show that credit risky bonds, credit default swaps, and collateralized default obligations (CDOs) exhibit no bubbles. This is because all of these fixed income securities' payoffs are bounded. For example, in the case of credit risky bonds, the cash flows are bounded by the sum of the promised payments. In the case of credit default swaps and CDOs, the maximum possible payments can be computed at origination of these contracts (see Lando 2004, for a description of these different instruments).

5.2. Black–Scholes Type Economies

It is interesting to study the standard Black–Scholes economy in both static and dynamic markets, yielding perhaps some unexpected, but new insights.

EXAMPLE 5.4 (Static Market, Finite Horizon). Fix $T \in \mathbb{R}_+$ and let S_t be a nondividend paying stock following a geometric Brownian motion, i.e.,

$$S_t = \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t\right\} \quad \forall t \in [0, T], \tag{5.4}$$

where $\mu, \sigma \in \mathbb{R}_+$, and B is a standard Brownian motion. Then, S is a Q-martingale, where Q is the probability measure on $\mathcal{F}_T$ defined by the Radon–Nikodym derivative $dQ/dP = \mathcal{E}(-(\mu/\sigma)B)_T$.

This is the standard Black–Scholes model, and we see by construction that there are no stock price bubbles.

EXAMPLE 5.5 (Static Market, Infinite Horizon). If we simply extend formula (5.4) from $[0, T]$ to $[0, \infty)$, then the situation changes dramatically. On an infinite horizon, S_t converges to 0 almost surely.[14] The fundamental value of the stock (recall that it pays no dividends over $[0, \infty)$), is $S_t^* = 0$. By definition, therefore,

$$\beta_t = S_t - S_t^* = S_t,$$

and the entire stock price is a bubble!

In this case, Q is not an EMM on $\mathcal{F}_\infty$. Indeed, P and Q are singular on $\mathcal{F}_\infty$. Hence, S is not a uniformly integrable martingale nor a (regular) martingale under the Q given earlier, but only a Q strict local martingale.

Although this example is plausible under NFLVR, when we also introduce the no dominance assumption 2.12, this example becomes problematic. Note that if the stock pays no dividends on $[0, \infty)$, then no dominance in conjunction with the Black–Scholes economy being complete implies that the asset has zero value, i.e., $S_t \equiv 0$. In this case, the model trivializes and becomes useless.

Therefore, if we want to use the Black–Scholes model in a static market, we need to restrict it to the finite horizon case. And, then one needs to interpret S_T as either:

[14]This follows due to the fact that $\mu - \frac{\sigma^2}{2} < 0$ because $\mu = 0$ in the normalized economy.

(i) a liquidating dividend (final cash flow) or (ii) the resale value at time T. In either case, because the Black–Scholes economy as given by expression (5.4) implies a complete market, we know that under both NFLVR and no dominance, there cannot be bubbles.

EXAMPLE 5.6 (Dynamic Market, Infinite Horizon). This example can be considered as an extension of the Black–Scholes formula, which is well defined on $[0, \infty)$. *It is also an example of a dynamic market in which bubble birth occurs.*

Let B^1, B^2 be two independent Q-Brownian motions. Fix $k > 1$. Let

$$\sigma = \inf\{\mathcal{E}(B^2)_t = k\}. \tag{5.5}$$

Define the processes Z and S by

$$Z_t = \mathcal{E}(B^2)_{t\wedge\sigma}, \qquad S_t = \mathcal{E}(B^1)_{t\wedge\sigma}. \tag{5.6}$$

We regard S as a stock process that pays no dividends $D \equiv 0$, and where the stock can default at time $\tau = \sigma$. If it defaults, it pays a final cash flow at the default time equal to $X_\tau = S_\sigma$.

The key difference of this example from the standard Black–Scholes model is the explicit introduction of a default time $\tau = \sigma$, so that S does not converge to 0 almost surely as $t \to \infty$. However, as in Lemma 2.6,

$$E_Q[S_\infty] = E_Q[\mathcal{E}(B^1)_\sigma \mathbf{1}_{\{\sigma<\infty\}}] = Q(\sigma < \infty) = \frac{1}{k}, \tag{5.7}$$

so S is a nonuniformly integrable martingale.

Let $R \in \mathcal{M}_{\text{loc}}(W)$ be the probability measure defined by $dR/dQ|_{\mathcal{F}_t} = Z_t$. As shown in Lemma 2.6, SZ is a Q-uniformly integrable martingale. It follows that S is an R-uniformly integrable martingale, since

$$E_R[S_\infty|\mathcal{F}_t] = \frac{E_Q[S_\infty Z_\infty|\mathcal{F}_t]}{Z_t} = \frac{Z_t S_t}{Z_t} = S_t \quad \text{a.s.} \tag{5.8}$$

Observe that S is a geometric Brownian motion stopped by σ under Q and R. Thus, S coincides with standard Black–Scholes model on $\{t < \sigma\}$.

Let us now introduce the regime shifting times σ_i, and suppose that at each of these times the market shifts from Q to R or vice-versa. *Then when shifting from R to Q, a bubble is born.* This is a Black–Scholes such as economy that is infinite horizon, but where the stock price process, prior to default, exhibits bubble birth and bubble disappearance.

5.3. Market Indices

Although the previous discussion concentrates on a single risky asset S, the theory remains unchanged if there are multiple risky assets and S represents a vector of risky asset price processes. It also applies to market indices. Let M denote the market price of an asset defined as an (weighted) average of (finitely many) individual risky assets trading in the market (e.g., Dow Jones Industrials, S&P 500 Index, etc.). Of course, the future cash flows associated with this portfolio are also a weighted average of the cash flows from the individual assets. As before, we can define the fundamental price of this index. If any asset in the market index has a bubble, then the market and the fundamental prices of this index differ, and a bubble exists.

EXAMPLE 5.7 (Bubbles in an Index Model). In portfolio theory, the return on an individual asset R_t is often modeled using an index model:

$$R_t = b R_t^M + \varepsilon_t, \tag{5.9}$$

where b is constant, R_t^M denotes the return on the index, and ε_t is a idiosyncratic return that is independent of R_t^M.

Taking the stochastic exponential of both sides of this expression, we obtain the stock price process S_t, i.e.,

$$S_t = \mathcal{E}(R)_t = \mathcal{E}(b R^M)_t \mathcal{E}(\varepsilon)_t. \tag{5.10}$$

If we assume, as is standard in the literature, that idiosyncratic risk earns no risk premium, then ε is a local martingale under both the physical and the valuation measures.

Let us consider a static market with the valuation measure $Q \in \mathcal{M}_{\text{loc}}(S)$. Since $\mathcal{E}(b R^M)$ and $\mathcal{E}(\varepsilon)$ are independent and b is a constant, the stock price process, $S = \mathcal{E}(R)$, is a Q-uniformly integrable martingale if and only if both $\mathcal{E}(R^M)$ and $\mathcal{E}(\varepsilon)$ are Q-uniformly integrable martingales. This implies that under the index model bubbles can exist in a stock because the bubble exists in a market index, or because it exists within the stock's idiosyncratic component itself.

6. DERIVATIVE SECURITIES

This section considers bubbles in derivative securities written on the risky asset. We focus on the standard derivatives: forward contracts, European and American call, and put options. We first need to formalize the definition of the fundamental price of a derivative security. To simplify the notation, we assume that the risky asset S pays no dividends over the time interval $(0, T]$, where $\tau > T$ almost surely. We define an arbitrary (European type) *derivative security* on the risky asset S to be a financial contract that has a random payoff at time T, where T is called the maturity date. The payoff is given by $H_T(S)$, where H_T is a functional on $(S_u)_{u \leq T}$. As is true in practice, our definition of a derivative security reflects the fact that the financial contract's payoffs are written on the *market price* of the risky asset, and not its fundamental value. This is a subtle and important observation.

We denote the time t market price of a derivative security H by Λ_t^H. We study derivative pricing in a dynamic market (hence a static market is a special case). Therefore, we assume that the market chooses a collection of ELMMs $(Q^i)_{i \geq 0} \in \mathcal{M}_{\text{loc}}(W)$ such that the derivative security's market price Λ_t^H is a Q^i-local martingale over the ith regime $\{\sigma_i \leq t < \sigma_{i+1}\}$.

Then, analogous to the risky asset, the *fundamental price of the derivative security* is defined to be the conditional expectation of the derivative's time T payoff using the valuation measure Q^{t^*} determined by $(Q^i)_{i \geq 0} \in \mathcal{M}_{\text{loc}}(W)$, i.e., $E_{Q^{t^*}}[H_T(S)|\mathcal{F}_t]$.

The *derivative security's price bubble* δ_t is defined as the difference between its market price and fundamental value,

$$\delta_t = \Lambda_t^H - E_{Q^{t^*}}[H_T(S)|\mathcal{F}_t].$$

The following lemma will prove useful in the subsequent analysis:

LEMMA 6.1. *Let H_T, H'_T be the payoffs of two derivative securities with the same maturity date.*

Let $\Lambda_t(H')$ *have no bubble, i.e.,*

$$\Lambda_t^{H'} = E_{Q^{t*}}[H'_T(S)|\mathcal{F}_t]. \tag{6.1}$$

If $H_T(S) \leq H'_T(S)$ *almost surely, then*

$$\Lambda_t^{H} = E_{Q^{t*}}[H_T(S)|\mathcal{F}_t]. \tag{6.2}$$

Proof. Because derivative securities have bounded maturities, we only need to consider type 3 bubbles. Let $\mathcal{L}$ be a collection of stopping times on $[0, T]$. Then for all $L \in \mathcal{L}$, $\Lambda_L(H) \leq \Lambda_L(H')$ by No Dominance (Assumption 2.12). Since $\Lambda(H')$ is a martingale on $[0, T]$, it is a uniformly integrable martingale and is in class (D) on $[0, T]$. Then $\Lambda(H)$ is also in class (D) and is a uniformly integrable martingale on $[0, T]$ (see Jacod and Shiryaev 2003, Definition 1.46, Proposition 1.47). Therefore, type 3 bubbles do not exist for this derivative security. □

This lemma states that if there is a derivative security with no bubble and whose payoff dominates another derivative security's payoff, then the dominated derivative security's market price will have no bubble as well. This, of course, is an extension of Theorem 5.1 to derivative securities.

6.1. European Call and Put Options

In this section, we consider three standard derivative securities: a forward contract, a European put option, and a European call option; all on the same risky asset. Each of these derivative securities are defined by their payoffs at their maturity dates. A *forward contract* on the risky asset with strike price K and maturity date T has a payoff $[S_t - K]$. We denote its time t market price as $V_t^f(K)$. A *European call option* on the risky asset with strike price K and maturity T has a payoff $[S_t - K]^+$, with time t market price denoted as $C_t(K)$. Finally, a *European put option* on the risky asset with strike price K and maturity T has a payoff $[K - S_t]^+$, with time t market price denoted as $P_t(K)$.[15] Finally, let $V_t^f(K)^*$, $C_t(K)^*$, and $P_t(K)^*$ be the fundamental prices of the forward contract, call option, and a put option, respectively.

A straightforward implication of the definitions is the following theorem.

THEOREM 6.2 (Put–Call Parity for Fundamental Prices).

$$C_t^*(K) - P_t^*(K) = V_t^{f^*}(K). \tag{6.3}$$

Proof. At maturity T,

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K \tag{6.4}$$

Because a fundamental price of a contingent claim with payoff function H is $E_{Q^{t*}}[H(S)_T|\mathcal{F}_t]$,

$$\begin{aligned} C_t^*(K) - P_t^*(K) &= E_{Q^{t*}}[(S_T - K)^+|\mathcal{F}_t] - E_{Q^{t*}}[(K - S_T)^+|\mathcal{F}_t] \\ &= E_{Q^{t*}}[S_T - K|\mathcal{F}_t] \\ &= V_t^{f^*}(K). \end{aligned} \tag{6.5}$$

□

[15]To be precise, we note that the strike price is quoted in units of the numeraire for all of these derivative securities.

Note that put–call parity for the fundamental price does not require the no dominance Assumption 2.12. It only requires that the asset's market price process satisfies NFLVR. Furthermore, put–call parity for the fundamental prices holds regardless of whether or not there are are bubbles in the asset's market price.

Perhaps surprisingly, put–call parity also holds for market prices, regardless of whether or not the underlying asset price has a bubble.

THEOREM 6.3 (Put–Call Parity for Market Prices).

$$C_t(K) - P_t(K) = V_t^f(K) = S_t - K. \tag{6.6}$$

Proof. This is a direct consequence of no dominance (Assumption 2.12). □

This theorem and proof are identical to that originally contained in Merton (1973). It depends crucially on the no dominance assumption. If only NFLVR holds, then put–call parity in market prices need not hold. For an example, see Jarrow et al. (2006). For related discussions of the economy without no dominance (Assumption 2.12), see also Cox and Hobson (2005) and Heston et al. (2007). Note that this theorem also values the forward contract.

THEOREM 6.4 (European Put Price). *For all* $K \geq 0$,

$$P_t(K) = P_t^*(K). \tag{6.7}$$

The proof of this theorem is trivial. Note that the payoff to the put option is bounded by K, hence by Theorem 5.1 the result follows. Hence, European put options always equal their fundamental values, regardless of whether or not the underlying asset's price has a bubble. We will revisit this observation when we discuss the empirical testing of bubbles in the paper's conclusion.

THEOREM 6.5 (European Call Price). *For all* $K \geq 0$,

$$C_t(K) - C_t^*(K) = S_t - E_{Q^{t*}}[S_T|\mathcal{F}_t]. \tag{6.8}$$

Proof.

$$\begin{aligned} V_t^f(K) &= S_t - K \\ &= (S_t - E_{Q^{t*}}[S_T|\mathcal{F}_t]) + (E_{Q^{t*}}[S_T|\mathcal{F}_t] - K) \\ &= V_t^{f^*}(K) + (S_t - E_{Q^{t*}}[S_T|\mathcal{F}_t]). \end{aligned} \tag{6.9}$$

Using put–call parity in fundamental prices:

$$C_t^*(K) - P_t^*(K) = V_t^{f^*}(K) \tag{6.10}$$

Using put–call parity in market prices,

$$C_t(K) - P_t(K) = V_t^f(K) \tag{6.11}$$

By subtracting equation (6.10) from equation (6.11),

$$\begin{aligned} &\left[C_t(K) - C_t^*(K)\right] - \left[P_t(K) - P_t^*(K)\right] = V_t^f(K) - V_t^{f^*}(K) \\ &= S_t - E_{Q^{t*}}[S_T|\mathcal{F}_t] \\ &= \delta_t, \end{aligned} \tag{6.12}$$

because the put option has a bounded payoff, $P_t(K) = P_t^*(K)$ and $C_t(K) - C_t^*(K) = \delta_t$. □

Because call options have finite maturity, call option bubbles must be of type 3, if they exist. The magnitude of such a bubble is independent of the strike price and it is related to the magnitude of the asset's price bubble. In a static market, corollary 4.3 shows that

$$S_t - E_{Q^*}[S_T|\mathcal{F}_t] = \beta_t^3 - E_{Q^*}\left[\beta_T^3|\mathcal{F}_t\right],$$

where β_t^3 is the type 3 bubble component in the underlying stock.[16] Here, the call option's bubble equals the difference between the type 3 bubble in the underlying stock less the expected type 3 bubble remaining at the option's maturity.

A second and important implication of this theorem is that even if the market satisfies NFLVR and no dominance, a stock price bubble implies that there exists no valuation measure Q^{t^*} such that the expected discounted value of the call option's payoffs equals its market price. Here, the standard martingale valuation methodology is not able to match market prices.

6.2. American Options

This section investigates the pricing of American options in a static market. Hence, there is a unique local martingale measure Q selected by the market. Because the time value of money plays an important role in analyzing the early exercise decision of American options, we need to modify the notation to make explicit the numeraire. In this regard, we denote the time t value of a money market account as

$$A_t = \exp\left(\int_0^t r_u du\right), \tag{6.13}$$

where r is the nonnegative adapted process representing the default free spot rate of interest. To simplify comparison with the previous sections, we still let S_t denote the risky asset's price in units of the numeraire.

DEFINITION 6.6 (The Fundamental Price of an American Option). The fundamental price $V_t^{A^*}(H)$ of an American option with payoff function H and maturity T is given by

$$V_t^{A^*}(H) = \sup_{\eta\in[t,T]} E_Q[H(S_\eta)|\mathcal{F}_t], \tag{6.14}$$

where η is a stopping time and the market selected $Q \in \mathcal{M}_{\text{loc}}(S)$.

This definition is a straightforward extension of the standard formula for the valuation of American options in the classical literature. It is also equivalent to the *fair price* as defined by Cox and Hobson (2005) when the market is complete. We apply this definition to a call option with strike price K and maturity T. Letting $C_t^{A^*}(K)$ denote the American call's fundamental value, the definition yields

$$C_t^{A^*}(K) = \sup_{\eta\in[t,T]} E_Q\left[\left(S_\eta - \frac{K}{A_\eta}\right)^+ \middle| \mathcal{F}_t\right]. \tag{6.15}$$

[16]In an analogous theorem in Jarrow et al. (2006), they used the implicit assumption that $T = \tau$ which would imply that $E_{Q^*}[\beta_T^3|\mathcal{F}_t] = 0$.

Let $C^A(K)_t$ be the market price of this same option, and $C^E(K)_t$ the market price of an otherwise identical European call. Then, the following theorem is provable using standard techniques.

THEOREM 6.7. *Assume that the jump process of the asset's price, $\Delta S := (\Delta S_t)_{t\geq 0}$, where $\Delta S_t = S_t - S_{t-}$, satisfies the regularity conditions of Lemma A.1. Then, for all K*

$$C_t^E(K) = C_t^A(K) = C_t^{A^*}(K). \tag{6.16}$$

Proof.

(i) By Theorem A.2 with $G(x, u) = [x - K/A_u]^+$,

$$\begin{aligned} C^{A^*}(K)_t &= \sup_{t\leq\tau\leq T} E[(S_\tau - K/A_\tau)^+|\mathcal{F}_t] \\ &= E[(S_T - K/A_T)^+|\mathcal{F}_t] + (S_t - E[S_T|\mathcal{F}_t]) \\ &= C_t^{E^*}(K) + \beta_t^3 - E\big[\beta_T^3|\mathcal{F}_t\big] \\ &= C_t^E(K). \end{aligned} \tag{6.17}$$

The last equality is by Theorem 6.5. This equality implies, using Merton's original no dominance argument, that the American call option is not exercised early. The reason is that the European call's value is at least the value of a forward contract on the stock with delivery price K, and this exceeds the exercised value.

(ii) A unit of an American call option with arbitrary strike K is dominated by a unit of an underlying asset. Therefore, by No Dominance (Assumption 2.12),

$$C_t^A(K) \leq S_t. \tag{6.18}$$

Let $\gamma_t := C_t^A(K) - C_t^{A^*}(K)$ be a bubble of an American call option with strike K. Because American options have finite maturity, γ_t is of type 3 and is a strict local martingale. Then by (i) and a decomposition of S_t,

$$\begin{aligned} C_t^{E^*}(K) + \beta_t^3 - E\big[\beta_T^3|\mathcal{F}_t\big] + \gamma_t &= C_t^{A^*}(K) + \gamma_t \\ &= C_t^A(K) \leq S_t \\ &= S_t^* + \beta_t^1 + \beta_t^2 + \beta_t^3, \end{aligned} \tag{6.19}$$

and therefore

$$\gamma_t \leq \big[S_t^* - C_t^{E^*}(K) + \beta_t^1\big] + \beta_t^2 + E\big[\beta_T^3|\mathcal{F}_t\big]. \tag{6.20}$$

The right side of equation (6.20) is a uniformly integrable martingale on [0, T]. Hence γ is a nonnegative local martingale dominated by a uniformly integrable martingale. Therefore, $\gamma_t \equiv 0$. □

This theorem is the generalization of Merton's (1973) famous no early exercise theorem, i.e., given the underlying stock pays no dividends, otherwise identical American and European call options have identical prices. This extension is the first equality in expression (6.16), applied to the options' *market prices*. Just as in the classic theory, this implies that an American call option on a stock with no dividends is not exercised early.

The second equality implies that *American call option prices exhibit no bubbles, even if there is an asset price bubble!* This result follows because the stopping time associated with the American call's fundamental value (as distinct from the exercise strategy of the American call's market price) explicitly incorporates the price bubble into the supremum. Indeed, the fundamental value of the American call option is the minimal supermartingale dominating the value function. If there is a price bubble, then the stopping time associated with the American call option's fundamental value is stopped early with strictly positive probability. This is understood by examining the difference between the fundamental values of the European and American call. If stopping early had no value, then it must be true that $C_t^{A^*}(K) = C_t^{E^*}(K)$. However, by Theorem 6.5, an asset price bubble creates a difference between an American and European calls' *fundamental* prices, i.e.,

$$C_t^{A^*}(K) - C_t^{E^*}(K) = \beta_t^3 - E_Q\left[\beta_T^3 | \mathcal{F}_t\right] > 0.$$

The intuition for the possibility of stopping early is obtained by recognizing that the market price equals the fundamental value plus a price bubble. The price bubble is a nonnegative supermartingale that is expected to decline. Its effect on the market price of the stock is therefore equivalent to a continuous dividend payout. And, it is well known that continuous dividend payouts make early exercise of (the fundamental value of) an American call possible.

7. FORWARD AND FUTURES PRICES

This section studies both forward and futures prices trading in a static market. As in the previous section, there is a unique local martingale measure Q selected by the market. In the classical theory, differences between forward and futures prices can only arise in a stochastic interest rate economy. Consequently, we need to make explicit the money market account numeraire in the notation for the asset's price process. In this regard, we let S denote the *dollar* price of the risky asset, and S/A the price in units of the numeraire. Then, $Q \in \mathcal{M}_{\mathrm{loc}}(S)$ implies that S/A is a Q-local martingale. To simplify the presentation, we also assume that the risky asset pays no dividends over the time interval $(0, T]$, where $\tau > T$ almost surely.

For some key results, we need to introduce trading in default free zero-coupon bonds. In this regard, we let $p(t, T)$ be the time *t market price* of a sure dollar paid at time T. Because zero coupon bonds have bounded payoffs, by Theorem 5.1, we know that zero-coupon bonds have no bubbles, hence this market price also represents the fundamental price. However, this distinction will not be used later.

7.1. Forward Prices

Forward contracts were defined in Section 6. Recall that a forward contract on the risky asset S with strike price K and maturity T is defined by its time T payoff $[S_T - K]$. The time *t forward price* for this contract, denoted $f_{t,T}$, is defined to be that strike price K that gives the T-maturity forward contract zero *market* value at time t. Given these definitions, it is easy to prove the following theorem.

THEOREM 7.1.

$$f_{t,T} \times p(t, T) = S_t.$$

Proof. By No Dominance (Assumption 2.12), any two trading strategies yielding the same payoff have the same market price. Let Portfolio A be a unit of a long forward contract and $f_{t,T}$ units of a zero coupon bond maturity at time T. Let Portfolio B be a unit of the underlying asset. Let Λ^A and Λ^B denote market prices of each portfolio. Then

$$0 + f_{t,T}p(t,T) = \Lambda_t^A = \Lambda_t^B = S_t \tag{7.1}$$

because both portfolios have the same payoff S_T at maturity. □

This is an intuitive and well-known result which follows directly from the no dominance Assumption 2.12.

COROLLARY 7.2 (Forward Price Bubbles).

(1) $f_{t,T} \geq 0$.
(2) $\frac{f_{t,T} \cdot p(t,T)}{A_t}$ *is a Q-local martingale for each* $Q \in \mathcal{M}_{\mathrm{loc}}(W)$.
(3) $f_{t,T} \cdot p(t,T) = S_t^* + \beta_t$ *where* $\beta_t = S_t - S_t^*$.

Proof. The proof follows trivially because the risky asset's price has these properties and $p(t,T)$ is nonnegative. □

Thus, we see that discounted forward prices inherit the properties of the risky asset's price bubble. In fact, any bubble present in the discounted forward price for a risky asset must be equal to the bubble in the risky asset's market price.

7.2. Futures Prices

A futures contract is similar to a forward contract. It is a financial contract, written on the risky asset S, with a fixed maturity T. It represents the purchase of the risky asset at time T via a prearranged payment procedure. The prearranged payment procedure is called marking-to-market. Marking-to-market obligates the purchaser (long position) to accept a continuous cash flow stream[17] equal to the continuous changes in the futures prices for this contract. The time *t futures prices*, denoted $F_{t,T}$, are set (by market convention) such that newly issued futures contracts (at time t) on the same risky asset with the same maturity date T, have zero *market value*. Hence, futures contracts (by construction) have zero market value at all times, and a continuous cash flow stream equal to $dF_{t,T}$. At maturity, the last futures price must equal the asset's price $F_{T,T} = S_T$ (see Duffie 2001 or Shreve 2004, for further clarification).

With respect to futures contracts, in the existing finance literature, the characterization of a futures price implicitly (and sometimes explicitly) uses the existence of a given local martingale measure Q under which the futures price is a martingale (e.g., see Duffie 2001, p. 173 or Shreve 2004, p. 244). Because futures prices have bounded maturities, this excludes (by fiat), the existence of futures price bubbles. Thus, to study bubbles in futures prices, we first need to generalize the characterization of a futures price to remove this implicit (or explicit) restriction.

Let us construct a portfolio long one futures contract. The discounted wealth process of this portfolio, denoted V_t^F, is then given by

$$V_t^F = \int_0^t \frac{1}{A_u} dF_{u,T} = \left(\frac{F_{t,T}}{A_t} - F_{0,T}\right) + \int_0^t \frac{F_{u,T}}{A_u} r_u du, \tag{7.2}$$

where $A_t = \exp(\int_0^t r_u du)$ and the second equality is due to an integration by parts.

[17] For simplicity, we assume that futures contracts are marked-to-market continuously.

If $(V_t^F)_{t\geq 0}$ is not locally bounded from below, then buying a futures contract is not an admissible trading strategy. In the context of our model, this implies that futures contracts cannot trade. To avoid this contradiction, given that we already assume futures contracts trade, we assume (without further loss of generality) that V_t^F is locally bounded.

Let $(T_n)_{n\geq 1}$ be a sequence of stopping times such that $(V_{T_n\wedge t}^F)_{t\geq 0}$ is bounded from below for each n. Then, there exists a $Q \in \mathcal{M}_{\text{loc}}(W)$ such that V^F is a local martingale by applying the First Fundamental Theorem of asset pricing to the market with the assets $(A_t, V_t^F)_{t\geq 0}$, stopped at T_n, each n. Note that by stopping, V^F is locally a Q-local martingale, and hence a Q-local martingale.

DEFINITION 7.3. Semimartingales $(F_{t,T})_{0\leq t\leq T}$ satisfying the following properties are called *NFLVR futures price processes*.

(1) V_t^F is locally bounded from below, i.e., there exists a sequence of stopping times $(T_n)_{n\geq 1}$ such that $(V_{t\wedge T_n}^F)_{t\geq 0}$ is bounded from below for each n.
(2) There exists a $Q \in \mathcal{M}_{\text{loc}}(W)$ such that $(V_{t\wedge T_n}^F)_{t\geq 0}$ is a Q-local martingale, where $(T_n)_{n\geq 1}$ is the sequence of stopping times satisfying condition 1.
(3) $F_{T,T} = S_T$.

Let Φ^F denote the class of all NFLVR futures price processes. We also note that futures contracts are not replicable using an admissible trading strategy which uses only the risky asset, hence any NFLVR futures price process also satisfies the no dominance assumption.

Note that we do not require futures prices $(F_{t,T})_{t\geq 0}$ to be nonnegative.

With this definition, the following theorem immediately follows.

THEOREM 7.4. *Fix a* $Q \in \mathcal{M}_{\text{loc}}(W)$. *Assume that* S *is in* $L^1(dQ)$
Define $(F'_{t,T})_{t\geq 0} = (E_Q[S_T|\mathcal{F}_t])_{t\geq 0}$. *Then,* $(F'_{t,T})_{t\geq 0} \in \Phi^F$.

Proof. Since S_t is nonnegative, $F'_{t,T} = E_Q[S_T|\mathcal{F}_t] \geq 0$. By equation (7.2),

$$V_t^{F'} = \left(\frac{F'_{t,T}}{A_t} - F'_{0,T}\right) + \int_0^t \frac{F'_{u,T}}{A_u} r_u du \geq -F'_{0,T} \tag{7.3}$$

and $V_t^{F'}$ is admissible. $F'_{T,T} = S_T$ is trivial. Since $(F'_{t,T})_{t\geq 0}$ is a martingale and $1/A$ is continuous, $V^{F'}$ is a local martingale. □

As expected, the classical definition of a futures price (Duffie 2001, p. 173 or Shreve 2004, p. 244) gives an acceptable NFLVR futures price process. The classical futures price is a uniformly integrable martingale, and hence exhibits no bubbles. However, this is not the only possible NFLVR futures price process.

THEOREM 7.5 (Futures Price Bubbles). *Let* β *be a local* Q*-martingale, locally bounded from below,*[18] *with* $\beta_T = 0$. *Also assume that* S *is in* $L^1(dQ)$.
Define $(F_{t,T})_{t\geq 0}$ *by*

$$F_{t,T} = E_Q[S_T|\mathcal{F}_t] + \beta_t. \tag{7.4}$$

Then, $(F_{t,T})_{t\geq 0} \in \Phi^F$.

Proof. Observe that $E_Q[S_T|\mathcal{F}_t] \geq 0$ for each t by the nonnegativity of S_T. Since β_t is locally bounded from below, $(F_{t,T})_{t\geq 0}$ is also locally bounded from below. Without loss

[18] We note that β is not restricted to being nonnegative.

of generality (by stopping) we assume that $(F_{t,T})_{t\geq 0}$ is bounded from below by $-C$ for some $C \geq 0$. Then

$$V_t^F \geq -F_{0,T} - \frac{C}{A_t} + K\left(\frac{1}{A_t} - 1\right) \geq -F_{0,T} - C. \tag{7.5}$$

Therefore $F_{t,T} \in \Phi^F$. □

We see that futures price bubbles are consistent with futures contracts trading in a market satisfying NFLVR and no dominance.

In the classical approach, futures prices are given by $F_{t,T} = E_Q[S_T|\mathcal{F}_t]$, which is a uniformly integrable martingale under Q. Since S_T is nonnegative, $F_{t,T}$ is nonnegative. However, in an economy which allows bubbles, as Theorem 7.5 shows, a bubble can be negative if $F_{t,T} < E_Q[S_T|\mathcal{F}_t]$.

In a world of deterministic interest rates, in the presence of bubbles, using the original argument of Merton (as referenced in Cox et al. 1981), one can generate the payoffs to a futures contract using a self-financing trading strategy involving forward contracts. No dominance, therefore, implies that forward prices equal futures prices. This extends the classical results of Jarrow and Oldfield (1981) and Cox et al. (1981) in this regard. Consequently, under deterministic interest rates, forward price bubbles must equal those in futures prices. Of course, this equivalence does not extend to the more realistic stochastic interest rate economies included under Theorem.7.5.

8. CHARGES

This section shows the equivalence between the *local martingale approach* (Loewenstein and Willard 2000a,b; Cox and Hobson 2005; Heston et al. 2007) and the *charges approach* (Gilles 1988; Gilles and Leroy 1992; Jarrow and Madan 2000) to bubbles. This correspondence is obtained via a generalization of the arbitrage-free price system used by Harrison and Kreps (1979) and Harrison and Pliska (1981).

8.1. Price Operators

This section introduces the concept of a price operator. We start with the price function $\Lambda_t : \Phi \to \mathbb{R}_+$ introduced in the no dominance Assumption 2.12 that gives for each $\phi \in \Phi$, its time t price $\Lambda_t(\phi)$. Let $\Phi_m \subset \Phi$ represent the set of traded assets. For our economy $\Phi_m = \{1, S\}$.

The no dominance assumption implies the following lemma.

LEMMA 8.1 (Positivity and Linearity on Φ). *Let "$\geq_t$" denote dominance in the sense of Assumption 2.12 at time t.*

(1) Let $\phi', \phi \in \Phi$. If $\phi' \geq_t \phi$ for all t, then $\Lambda_t(\phi') > \Lambda_t(\phi)$ for all t almost surely.
(2) Let $a, b \in \mathbb{R}_+$ and $\phi', \phi \in \Phi$. Then, $a\Lambda_t(\phi') + b\Lambda_t(\phi) = \Lambda_t(a\phi' + b\phi)$ for all t almost surely.

Proof. Condition (1) is the definition of no dominance restated, and condition (2) follows by assuming strict inequality (for each direction in turn), and obtaining a contradiction using condition (1). □

In particular, if $\int_{(t,\nu]} d\Delta_u + \Xi^\nu = 0$ almost surely for $\phi = (\Delta, \Xi^\nu)$, then $\Lambda_t(\phi) = 0$ almost surely. Linearity excludes liquidity impacts as in Çetin, Jarrow, and Protter (2004), and it implies that Λ_t is finitely additive on Φ.

By Lemma 2.3, we know that the market prices of the traded assets satisfy NFLVR. Thus, for each traded asset $\phi \in \Phi_m$, $\Lambda(\phi)$ is a Q^i -local martingale on the set $\{\sigma^i \le t < \sigma_{i+1}\}$ for each i. This implies by Theorem 4.2 that for $\phi \in \Phi_m$,

$$\Lambda_t(\phi) = \Lambda_t^*(\phi) + \delta_t(\phi),$$

where $\delta_t(\phi)$ is a nonnegative Q^i-local martingale. Of course, $\delta_t(\phi)$ is the traded asset's price bubble. To extend this property of Λ_t on the set Φ_m to all of Φ, we add the following assumption.

ASSUMPTION 8.2. *Let $\Lambda_t : \Phi \to \mathbb{R}_+$ be such that for each $\phi \in \Phi$, there exists a δ such that*

$$\begin{aligned}\Lambda_t(\phi) &= \mathbf{1}_{\{t<\nu\}} \sum_{i\ge 0} \left(E_{Q^{t^*}}\left[\int_t^\nu d\Delta_u + \Xi^\nu \middle| \mathcal{F}_t \right] + \delta_t^i(\phi) \right) \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}} \\ &= \left(E_{Q^{t^*}}\left[\int_t^\nu d\Delta_u + \Xi^\nu \middle| \mathcal{F}_t \right] + \delta_t(\phi) \right) \mathbf{1}_{\{t<\nu\}} \\ &= \Lambda_t^*(\phi) + \delta_t(\phi), \end{aligned} \tag{8.1}$$

where Q^{t^*} is a valuation measure, $\delta^i(\phi)$ is a nonnegative Q^i-local martingale such that $\delta_\nu(\phi) = 0$ and

$$\delta_t(\phi) = \sum_{i\ge 0} \delta_t^i(\phi) \mathbf{1}_{\{t\in[\sigma_i,\sigma_{i+1})\}}. \tag{8.2}$$

We call any Λ_t satisfying this assumption a *market price operator* and denote the collection $(\Lambda_t)_{t\ge 0}$ by Λ. We call (Λ, Φ) a *price system.*

The notion of a price system was proposed in the seminal papers of Harrison and Kreps (1979) and Harrison and Pliska (1981). In Harrison and Kreps (1979), the price system is first defined on a collection of securities in L^2, replicable by self-financing simple trading strategies and then extended to $L^2(\Omega, \mathcal{F}, P)$. More importantly, the model has a finite time horizon and every local martingale in their framework is a uniformly integrable martingale. One of the their key conclusions (Theorem 2) is that the market admits no simple free lunches if and only if the market price operator is given by an expectation with respect to an equivalent martingale measure.

This theorem characterizes the existence of equivalent martingale measures, and it is now known as the First Fundamental Theorem of asset pricing. As shown by Delbaen and Shachermayer (e.g., 1994, 1998a,b), this is true in a much more general setting, properly interpreted. Because every martingale on a finite time horizon is a uniformly integrable martingale and closable, once an EMM is identified, the price of the asset before maturity is given as a conditional expectation, which leads to their characterization of the market price operator. In a more general setting, when the market price process of ϕ is a strict Q^i-local martingale or if the maturity ν is unbounded and $\Lambda(\phi)$ is a nonuniformly integrable martingale, market prices can differ from the conditional expectation. The bubble component $\delta(\phi)$ in equation (8.1) represents this difference.

8.2. Bubbles

In the literature, an alternative approach to explain bubbles is to introduce charges (see Gilles 1988; Gilles and Leroy 1992; Jarrow and Madan 2000). The following theorem shows that the local martingale characterization of market prices has a finitely additive market price operator if and only if bubbles exist.

THEOREM 8.3. *Fix $t \in \mathbb{R}_+$. The market price operator Λ_t is countably additive if and only if bubbles do not exist.*

Proof. Fix $\phi \in \Phi$ where $\phi = (\Delta, \Xi^\nu)$. If $\nu \leq t$, then $S_t = S_t^* = 0$. Therefore, it suffices to consider the case when $t < \nu$. Define a sequence of stopping times $(\tau_n)_{n \geq 0}$ by $\tau_0 = t$ and

$$\tau_n = \inf \left\{ s \geq t : \int_t^{s \wedge \nu} d\Delta_u + \Xi^\nu \geq n \right\} \wedge \nu, \quad n \geq 1 \tag{8.3}$$

and define $\phi^n \in \Phi$ by $\phi^0 = (0, 0)$ and

$$\phi^n = \left(\Delta^{\tau_n -}, \Xi^\nu \mathbf{1}_{\{\nu < \tau_n\}}\right) - \left(\Delta^{\tau_{n-1} -}, \Xi^\nu \mathbf{1}_{\{\nu < \tau_{n-1}\}}\right), \quad \forall n \geq 1 \tag{8.4}$$

where $\Delta^{\tau_n -}$ is a process such that $\Delta_u^{\tau_n -} = \Delta_{\tau_n \wedge u} - \triangle\Delta_{\tau_n} \mathbf{1}_{\{\tau_n = u\}}$. Then for each n, ϕ^n is bounded by n and

$$\phi = \sum_{n=0}^{\infty} \phi_n. \tag{8.5}$$

Since ϕ_n is bounded,

$$\begin{aligned} \Lambda_t(\phi_n) &= \Lambda_t^*(\phi_n) \\ &= E_{Q^{t*}}[\Delta_{\tau_n -} - \Delta_{\tau_{n-1} -} + \Xi^\nu \mathbf{1}_{\nu \in [\tau_{n-1}, \tau_n)} | \mathcal{F}_t] \mathbf{1}_{\{t < \nu\}}. \end{aligned} \tag{8.6}$$

Assume that Λ_t is countably additive. Then

$$\begin{aligned} \Lambda_t(\phi) = \Lambda_t(\phi) &= \Lambda_t \left(\sum_n \phi_n \right) = \sum_n \Lambda_t(\phi_n) \\ &= \sum_n E_{Q^{t*}}[\Delta_{\tau_n -} - \Delta_{\tau_{n-1} -} + \Xi^\nu \mathbf{1}_{\nu \in [\tau_{n-1}, \tau_n)} | \mathcal{F}_t] \mathbf{1}_{\{t < \nu\}} \\ &= E_{Q^{t*}} \left[\sum_n \left\{ \Delta_{\tau_n -} - \Delta_{\tau_{n-1} -} + \Xi^\nu \mathbf{1}_{\nu \in [\tau_{n-1}, \tau_n)} \right\} \middle| \mathcal{F}_t \right] \mathbf{1}_{\{t < \nu\}} \\ &= E_{Q^{t*}} \left[\int_t^\nu d\Delta_u + \Xi^\nu | \mathcal{F}_t \right] \mathbf{1}_{\{t < \nu\}}, \end{aligned} \tag{8.7}$$

since $\Delta_{\nu -} = \Delta_\nu$. This implies that bubbles do not exist in the market price of ϕ. Since this is true for all $\phi \in \Phi$, bubbles do not exist. Conversely, if bubbles do not exist then the market price operator is given by a conditional expectation and countable additivity holds. □

This theorem shows that the characterization of bubbles as charges is an alternative perspective of our model based on the characterization of local martingales, but in essence is not different.

9. CONCLUSION

This section concludes the paper with a brief discussion of the existing empirical literature testing for bubbles, followed by some suggestions for future research. As mentioned in the introduction, there is a vast empirical literature with respect to bubbles, studying different markets over different time periods, including:

(1) the Dutch tulipmania 1634–1637 (see Garber 1989, 1990),
(2) the Mississippi bubble 1719–1720 (Garber 1990),
(3) the South Sea bubble of 1720 (Garber 1990; Temin and Voth 2004),
(4) foreign currency exchange rates (Evans 1986; Meese 1986),
(5) with respect to German hyperinflation in the early 1920s (Flood and Garber 1980),
(6) U.S. stock prices over the 20th century (West 1987, 1988; Diba and Grossman 1988; Dezhbakhsh and Demirguc-Kunt 1990; Froot and Obstfeld 1991; McQueen and Thorley 1994; Koustas and Serletis 2005),
(7) the 1929 U.S. stock price crash (White 1990; De Long and Shleifer 1991; Rappoport and White 1993; Donaldson and Kamstra 1996),
(8) land and stock prices in Japan 1980–1992 (Stone and Ziemba 1993),
(9) U.S. housing prices 2000–2003 (Case and Shiller 2003), and finally
(10) the NASDAQ 1998–2000 internet stock price peak (Ofek and Richardson 2003; Brunnermeier and Nagel 2004; Cunando et al. 2005; Battalio and Schultz 2006; Pastor and Veronesi 2006).

The majority of these empirical studies are based on models in discrete time with infinite horizons where there exists a martingale measure Q, and the traded assets have no terminal payoffs at $\tau = \infty$. By our Theorem 4.2, this last observation excludes type 1 bubbles. In discrete time models, when the current stock price is known, there are no local martingales. Hence, by construction these models exclude type 3 bubbles as well. Hence, the models in the existing literature have really only investigated the existence of type 2 bubbles (i.e., Is Q a uniformly integrable martingale measure or not?). As one might expect from such a vast literature, the evidence is inconclusive.

This empirical indeterminacy is due to the fact that to test

$$\beta_t = S_t - E_Q\left[\int_t^\infty dD_u \,\middle|\, \mathcal{F}_t\right] \neq 0,$$

one must assume a particular model for $E_Q[\int_t^\infty dD_u|\mathcal{F}_t]$. As such, these empirical tests involve a *joint hypothesis*: the assumed model and the null hypothesis $\beta_t \neq 0$. Different studies use different models with different conclusions obtained.

To our knowledge (as just mentioned) there appears to be no empirical study testing for type 3 bubbles. This is an open empirical question. Theorems 6.4 and 6.5 provide a plausible procedure for implementing such a test, assuming the market is incomplete, of course. Using the insights from Jacod and Protter (2009), if enough European put options trade, then we can infer the market selected ELMM Q from the put option market prices. Next, given Q, we can compute the fundamental prices of the traded European call options, and compare them to the calls' market prices. If they differ, a type 3 bubble exists. And, the magnitude of the bubble is related to the magnitude of the type 3 bubble in the asset's market price—providing the test for a type 3 asset price bubble.

This proposed testing procedure, however, does not test for either type 1 or type 2 asset price bubbles. To do this, it seems as if there is no choice other than to assume a particular model for the stock's fundamental price. We look forward to the continued empirical search for bubbles, and we hope that some of the theorems we have generated herein will be useful in that regard.

APPENDIX

This appendix proves some lemmas and theorems used in the American option pricing section of the text.

LEMMA A.1. *Let M_u be a nonnegative càdlàg local martingale. Assume that there exists some function f and a uniformly integrable martingale X such that*

$$\triangle M_u \leq f\left(\sup_{t\leq r<u} M_r\right)(1+X_u), \tag{A.1}$$

where $\triangle M_u = M_u - M_{u-}$. Then for $S_m = \inf\{u > t : M_u \geq x_m\}$,

$$\lim_{m\to\infty} E_Q\left[M_{S_m} 1_{\{S_m\in(t,T)\}}|\mathcal{F}_t\right] = M_t - E_Q[M_T|\mathcal{F}_t]. \tag{A.2}$$

Proof. To simplify the notation, we omit the Q subscript on the expectations operator. Let T_n be a fundamental sequence of M_t. Then $M_t^{T_n} = E[M_T^{T_n}|\mathcal{F}_t]$ and hence

$$M_t^{T_n} = M_t^{T_n} 1_{\{S_m=t\}} + E\left[M_{S_m}^{T_n} 1_{\{S_m\in(t,T)\}}|\mathcal{F}_t\right] + E\left[M_T^{T_n} 1_{\{S_m=T\}}|\mathcal{F}_t\right] \tag{A.3}$$

By hypothesis $M_{S_m}^{T_n} \leq x_m + f(x_m)(1+\triangle X_{S_m})$ and $M_T^{T_n} \leq x_m + f(x_m)(1+X_T)$. By the bounded convergence theorem,

$$M_t = \lim_{n\to\infty} M_t^{T_n} = M_t 1_{\{S_m=t\}} + E[M_{S_m} 1_{\{S_m\in(t,T)\}}|\mathcal{F}_t] + E[M_T 1_{\{S_m=T\}}|\mathcal{F}_t]. \tag{A.4}$$

Since X is a uniformly integrable martingale, it is in class D and $\{X^\tau\}_{\tau:\text{stopping times}}$ is uniformly integrable. Fix m. Then $M_T^{T_n}$, $M_{S_m}^{T_n}$ are bounded by a sequence of uniformly integrable martingales. Therefore, taking the limit with respect to n and interchanging the limit with the expectation yields:

$$M_t = \lim_{m\to\infty} E[M_{S_m} 1_{\{S_m\in(t,T)\}}|\mathcal{F}_t] + E[M_T|\mathcal{F}_t]. \tag{A.5}$$

□

THEOREM A.2. *Let M be a nonnegative local martingale with respect to $\mathbb{F}$ such that $\triangle M$ satisfies a condition specified in Lemma A.1. Let $G(x,t) : \mathbb{R}_+ \times [0,T] \to \mathbb{R}_+$ be a function such that*

- *$G(x,s) \leq G(x,t)$ for all $0 \leq s \leq t \leq T$,*
- *For all $t \in [0,T]$, $G(x,t)$ is convex with respect to x.*
- *$\lim_{x\to\infty} \frac{G(x,t)}{x} = c$ for all $t \in [0,T]$,*

then

$$\sup_{\tau\in[t,T]} E_Q[G(M_\tau,\tau)|\mathcal{F}_t] = E_Q[G(M_T,T)|\mathcal{F}_t] + (c\vee 0)(M_t - E_Q[M_T|\mathcal{F}_t]). \tag{A.6}$$

Proof. To simplify the notation, we omit the Q subscript on the expectations operator. Suppose $c \leq 0$. Then by monotonicity with respect to t and Jensen's inequality applied to a convex function G and a nonnegative local martingale M,

$$\begin{aligned}\sup_{\tau\in[t,T]} E[G(M_\tau,\tau)|\mathcal{F}_t] &\le \sup_{\tau\in[t,T]} E[G(M_\tau,T)|\mathcal{F}_t]\\ &\le E[G(M_T,T)|\mathcal{F}_t]\\ &\le \sup_{\tau\in[t,T]} E[G(M_\tau,\tau)|\mathcal{F}_t] \end{aligned} \tag{A.7}$$

and

$$\sup_{\tau\in[t,T]} E[G(M_\tau,\tau)|\mathcal{F}_t] = E[G(M_T,T)|\mathcal{F}_t]. \tag{A.8}$$

Suppose $c > 0$. Fix $\varepsilon > 0$. Then there exists $\xi > 0$ such that $\varepsilon > 0$, and $\exists\xi > 0$ such that $\forall x > \xi$, $\frac{G(x,0)}{x} > c-\varepsilon$ and hence $\frac{G(x,u)}{x} > c-\varepsilon$ for all $u \in [0,T]$. Let $\{x_n\}_{n\ge1}$ be a sequence in (ξ,∞) such that $x_n \uparrow \infty$. Let

$$S_n = \inf\{u > t : M_u \ge x_n\} \wedge T. \tag{A.9}$$

Without loss of generality we can assume that $M_t < x_n$. Since $G(\cdot,t)$ is increasing in t,

$$\begin{aligned}\sup_{\tau\in[t,T]} E[G(M_\tau,\tau)|\mathcal{F}_t] &\ge E[G(M_{S_n},S_n)|\mathcal{F}_t]\\ &= E[G(M_T,T)1_{\{S_n=T\}}|\mathcal{F}_t] + E[G(M_{S_n},S_n)1_{\{S_n<T\}}|\mathcal{F}_t]\\ &\ge E[G(M_T,T)1_{\{S_n=T\}}|\mathcal{F}_t] + E[G(M_{S_n},0)1_{\{S_n<T\}}|\mathcal{F}_t]. \end{aligned} \tag{A.10}$$

Since $M_{S_n} \ge x_n > \xi$, $G(M_{S_n},0) \ge (c-\varepsilon)M_{S_n}$. Next, let us take take a limit of $n\to\infty$. By Lemma A.1 applied with $\{S_n\}$ and the monotone convergence theorem,

$$\begin{aligned}&\lim_{n\to\infty}\sup_{\tau\in[t,T]} E[G(M_\tau,\tau)|\mathcal{F}_t]\\ &\quad\ge \lim_{n\to\infty}\left\{E[G(M_T,T)1_{\{S_n=T\}}|\mathcal{F}_t] + (c-\varepsilon)E[M_{S_n}1_{\{S_n<T\}}|\mathcal{F}_t]\right\}\\ &\quad\ge E[(G(M_T,T)|\mathcal{F}_t] + (c-\varepsilon)(M_t - E[M_T|\mathcal{F}_t]). \end{aligned} \tag{A.11}$$

Letting $\varepsilon \to 0$,

$$\sup_{\tau\in[t,T]} E[G(M_\tau,\tau)|\mathcal{F}_t] \ge E[G(M_T,T)|\mathcal{F}_t] + c\beta_t \tag{A.12}$$

To show the other direction, let $G^c(x,u) = cx - G(x,u)$. $G^c(x,\cdot)$ is a nonpositive increasing concave function w.r.t. x such that

$$\lim_{x\to\infty}\frac{G^c(\cdot,x)}{x} = 0. \tag{A.13}$$

By Jensen's inequality,

$$E[G^c(M_T,u)|\mathcal{F}_u] \le G^c(E[M_T|\mathcal{F}_u],u) \le G^c(M_u,u) \tag{A.14}$$

Therefore,

$$\begin{aligned}G(M_u,u) &\le c(M_u - E[G^c(M_T,u)|\mathcal{F}_u])\\ &= c\beta_u + E[G(M_T,u)|\mathcal{F}_u]\\ &\le c\beta_u + E[G(M_T,T)|\mathcal{F}_u]. \end{aligned} \tag{A.15}$$

Since this is true for all $u \in [t, T]$, $G(M_\tau, \tau) \le c\beta_\tau + E[G(M_T, T)|\mathcal{F}_\tau]$ for all $\tau \in [t, T]$. By the tower property of martingales, and a supermartingale property,

$$E[G(M_\tau, \tau)|\mathcal{F}_t] \le E[c\beta_\tau + E[G(M_T, T)|\mathcal{F}_\tau]|\mathcal{F}_t] \le E[G(M_T, T)|\mathcal{F}_t] + c\beta_t. \tag{A.16}$$

Therefore,

$$\sup_{\tau\in[t,T]} E[G(M_\tau, \tau)|\mathcal{F}_t] = E[G(M_T, T)|\mathcal{F}_t] + c\beta_t. \tag{A.17}$$ □

This theorem is an extension of Theorem B.2 in Cox and Hobson in two important ways. First, we relax the assumption that a martingale M_t be continuous. Second, the payoff function $G(\cdot, x)$ allows a more general form and, in particular, it allows an analysis of an American option in an economy with a non-zero interest rate.

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