THE DYNAMICAL SYSTEMS APPROACH TO MACROECONOMICS

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By

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Abstract

The aim of this thesis is to provide mathematical tools for an alternative to the mainstream study of macroeconomics with a focus on debt-driven dynamics.

We start with a survey of the literature on formalizations of Minsky's Financial Instability Hypothesis in the context of stock-flow consistent models.

We then study a family of macro-economical models that date back to the Goodwin model. In particular, we propose a stochastic extension where noise is introduced in the productivity. Besides proving existence and uniqueness of solutions, we show that orbits must loop around a specific point indefinitely.

Subsequently, we analyze the Keen model, where private debt is introduced. We demonstrate that there are two key equilibrium points, intuitively denoted good and bad equilibria. Analytical stability analysis is followed by numerical study of the basin of attraction of the good equilibrium.

Assuming low interest rate levels, we derive an approximate solution through perturbation techniques, which can be solved analytically. The zero order solution, in particular, is shown to converge to a limit cycle. The first order solution, on the other hand, is shown to explode, rendering its use dubious for long term assessments.

Alternatively, we propose an extension of the Keen model that addresses the immediate completion time of investment projects. Using distributed time delays, we verify the existence of the key equilibrium points, good and bad, followed by their stability analysis. Through bifurcation theory, we verify the existence of limit cycles for certain mean completion times, which are absent in the original Keen model.

Finally, we examine the Keen model under government intervention, where we introduce a general form for the government policy. Besides performing stability analysis, we prove several results concerning the persistence of both profits and employment. In economical terms, we demonstrate that when the government is responsive enough, total economic meltdowns are avoidable.

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To my mother,

Angela dos Reis Carneiro

and all the fond memories.

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Chapter 1

Introduction

Neoclassical economics has been largely criticized over the past years, specially since the crisis that unfolded in 2007–2008. Be it for their unreasonable rational expectations assumption, or be it for the aggregate "Law of Demand", technically refuted by the Sonnenschein-Mantel-Debreu theorems [Son72], it is time to seek for an alternative. We will argue that even though stock flow consistent macroeconomic models represent a radical substitute [God04], they might be the ideal one. Rather than studying economies at the equilibrium, we suggest the very opposite, as economies rarely find themselves in equilibrium, yet their time evolution matters. Stock flow representation of the economy can be translated into a system of dynamical equations (either deterministic or stochastic) which will describe how the many different variables fluctuate over time.

Perhaps a deeper concern regarding neoclassical economics is its disregard for the financial system. As pointed out by Lavoie in [Lav08], the aforementioned crisis "is a reminder that macroeconomics cannot ignore financial relations, otherwise financial crises cannot be explained". For this reason, a great deal of this thesis is focused on a family of models that represent a mathematical formalization of Minsky's Financial Instability Hypothesis.

The structure of the thesis is as follows. We begin by introducing the concepts behind stock flow consistent models in Chapter 2. In Sections 2.2.1 and 2.2.2, we demonstrate that the Keen model both with (8.14) and without government intervention (5.7) conform to the stock-flow consistency requirements.

Next, Chapter 3 discusses the very basic foundation behind this entire work: the Goodwin model [Goo67]. Goodwin elegantly proposes one of the most prominent mathematical formulations of Marx's theory of class struggle. We show that the solutions satisfy a constant Lyapunov equation, regardless of the choice of the Phillips curve. In a later Section 6.3, we derive an analytical expression for the period of these solutions, which depends exclusively on the initial value of the Lyapunov function.

A stochastic extension, where productivity is allowed to drift randomly according to the level of employment, is then proposed in Chapter 4. In this chapter, based on joint work with A. N. Huu and M. R. Grasselli [CGH13], we first obtain sufficient conditions for existence and almost surely uniqueness of solutions. Next, we derive a probabilistic estimate for the time it takes for solutions to deviate sufficiently far from their initial orbit, in Lyapunov terms. More importantly, we show that solutions almost surely orbit around a pivot point, indefinitely and in finite time. We end the chapter by deriving a continuous extension of the proposed stochastic model that approximates it when the volatility term is negligible. Such extension can be fully solved analytically, as the sum of the solution to the Goodwin model plus a martingale term.

The financial sector is taken into account in Chapter 5, where we introduce the Keen model [Kee95] followed by a comprehensive analysis drawn from [GCL12], joint work with M. R. Grasselli. After verifying the existence of two key equilibria, coined "good" and "bad", for they represent states of prosperity and collapse, respectively, we perform local analysis. Necessary and sufficient conditions for their local stability are obtained, showing the both fixed points are simultaneously stable under usual conditions. Accordingly, solutions can converge to either equilibrium point, depending on their initial position. As an illustration, we numerically integrate solutions starting in a fine grid around the good equilibrium point, recording when they converge to this fixed point, roughly obtaining the basin of attraction of the good equilibrium.

Chapter 6 presents an extension of this model where we assume regimes of extremely low interest rate. There, we use perturbation techniques similar to those used in Section 4.3 to derive an approximate model which can be fully solved analytically. The corresponding zero-order solution resembles the Goodwin model (3.12), except for its non-linear investment function. Regardless, we show that the zero-order solution converges to a limit cycle, whose period is also analytically determined. The first-order solution, on the other hand, is shown to grow quadratically with time, thus spoiling the accuracy of this approximation for long-term investigations.

In Chapter 7, we propose a second extension of the Keen model (5.7), where we consider non-immediate capital project completion times. Usually, to study such implications, one would have to deploy delay-differential equations machinery. Through a clever mathematical device, commonly used in Mathematical Biology, we are able to avoid these complications, and continue in the realm of ordinary differential equations, albeit with higher-dimension. The intuition behind this technique is that we can approximate a discrete delay by the sum of exponentially distributed times, itself Erlang distributed. As the number of intermediate stages increases to infinity, this distribution converges to that of the Dirac delta, effectively representing the discrete delay we started with. After verifying the existence of corresponding "good" and "bad" equilibria, we show that the local stability of the first is only possible when the mean completion time is shorter than a certain threshold. We perform bifurcation analysis, which shows the existence of a super-critical Hopf bifurcation at that point. In other words, for mean completion times higher than the mentioned threshold, we have a limit cycle which attracts solutions which would otherwise converge to the "good" equilibrium point. Unsurprisingly, the period of this limit cycle grows as the mean completion time increases, until a second threshold is crossed, and the cycles cease to exist. Beyond this point, solutions seem to converge exclusively to the bad equilibrium point.

Chapter 8 is based on joint work with M. R. Grasselli, X-S. Wang, and J. Wu [CGWW13]. We propose an original extension of the Keen model where government intervention is introduced in a general setting, while respecting the stock flow consistent framework. First, we identify the "good" equilibrium, together with a variety of "bad" equilibria characterized by zero employment and negative exploding profit share, besides some extra finite equilibria associated to zero wage share, which are either unattainable or unstable under usual conditions. Next, we obtain necessary and sufficient conditions for their local stability, showing that all the "bad" equilibria can be successfully destabilized if the government subsidies are non-negative and responsive enough in a vicinity of zero employment. Meanwhile, the same cannot be said for a government operating under austerity. Moreover, and most importantly, we show that through a variety of reasonable conditions associated with a responsive government, we have that the model (8.14) is uniformly weakly persistent with respect to both the exponential of profit share and employment ratio. Ultimately, this result means that solutions can never remain trapped below arbitrarily low levels of profit share or employment, regardless of the specific initial conditions.

Chapter 2

Stock-Flow Consistent Minsky Models

There have been several attempts to model (Hyman) Minsky's Financial Instability Hypothesis (FIH henceforth), as surveyed by Lavoie in [Lav08], and Dos Santos in [DS05]. A short tour through the key aspects of the most influential of these models follows below.

One of the earliest attempts of formalizing the FIH was carried out by Taylor and O'Connell in their influential article [TO85], itself a variation of another paper by Taylor [Tay85]. Taylor and O'Connell [TO85] introduced innovative concepts, of which three deserve further discussion. First, they used the notion of a portfolio choice. Secondly, their investment function was modeled as a function of the spread between expected profit rate of firms (actual profit rate plus a confidence indicator) and the interest rate. Lastly, and perhaps most strikingly, they introduced cyclical dynamics through a differential equation linking the confidence indicator and the interest rate. Notwithstanding the remarkable impact of [TO85], their model has been criticized for numerous flaws. As Lavoie [Lav08] and Dos Santos [DS05] both point out, Taylor and O'Connell [TO85] model is not stock-flow consistent (SFC hereafter), as there are inconsistencies in how they treat the government debt. On top of that, the banking sector and the firms' leverage ratio do not play an explicit role in the model, while the supply of money is not endogenous. Relatedly, Taylor and O'Connell do not model firms' debt commitments, they can only finance themselves through equity issuance.

Another attempt of modeling Minsky's ideas was put forward by Franke and Semmler in [FS91]. They modified Taylor and O'Connell model incorporating the banking system. In addition, they introduced the firms' leverage ratio together with the interest rate to the dynamical equation driving the confidence indicator. Unfortunately, Frank and Semmler model shares most of the criticism targeted towards Taylor and O'Connell [TO85]. For instance, the fact that stock price is not determined by supply and demand, and the remaining problems with stock-flow consistency. In addition, Dos Santos [DS05] criticizes the (unrealistic) assumption that the whole stock of high-powered money (i.e. cash) is kept by banks as reserves. Still, its most celebrated contribution was to track the leverage ratio of the firms explicitly.

Yet another extension of Taylor and O'Connell [TO85] was developed by Radke in[Rad05]. He presented a rather complete model, which assumes that the confidence indicator is driven by the spread between the profit rate and the interest rate, and the firms' debt ratio. Simultaneously, he allowed for credit rationing, by assuming that banks can choose to grant a higher amount of loans to firms offering more collateral.

Delli Gatti and Gellagati published several papers in this topic by the 90s, for example [DGGG90] coauthored with Gardini,[DGGM98] coauthored with Minsky himself, and [DGG92]. The essence of their model is captured by the investment strategy, which is modeled as a function of Tobin's q ratio [BT68], retained earning and leverage over retained earning. The key insight is that when firms rely more heavily on external finance, instability ensues, despite any interest rate movement. Nonetheless, despite being faithful to Minsky's ideas, their model has suffered extensive criticism. As Lavoie [Lav08] observes, besides not being SFC, the model ignores the role interest payments from the firm debt have in the consumption function. Moreover, bank reserves seem to have no counterpart. In addition, as Dos Santos [DS05] points out, by assuming the supply of bank loans is a function of the interest earned alone, the authors are (perhaps not intentionally) assuming as well that either the amount of bank deposits or reserves do not fluctuate wildly, or that bank behaviour is not affected by such fluctuations. Lastly, the authors oversimplify the government sector when, as Dos Santos [DS05] words it, they "heroically assume that all taxes are zero".

Perhaps the earliest SFC formulations of the FIH was developed by Skott in [Sko81]. The model, further enhanced in an article [Sko88] and a book [Sko89a], introduces a firm that finances its investing through three sources: retained profits, equity, and debt. At the same time, households choose to consume based on their wealth, and decide their allocation on money or equities, while the government sector is completely absent. As Lavoie [Lav08] remarks, the money supply in this model is endogenous, while the stock price is determined by demand and supply, features not found in Franke and Semmler [FS91], or Taylor and O'Connell [TO85]. Lavoie continues to say that this model has had an "unfortunate lack of impact", which can be partially blamed on the fact that Skott leaves leverage ratios out of the picture. In a more recent publication [Sko94], Skott introduces the leverage ratio as an upper bound on the amount of loans supplied by banks, together with an investment function that depends on financial variables coined "fragility" – the sensitivity of investment to adverse shocks – and "tranquility" – the firms' ability to meet their financial obligations (as discussed in [DS05]).

None of the models mentioned up to this point consider household indebtedness. As Isenberg concluded in her study on the FIH in the 1920s [Ise88], "the production sector, non-financial firms, which is at the center of the financial instability hypothesis, did not exhibit a rising debt equity ratio". In a later article [Ise94], she explains that even though firms did not experience a rise in their their leverage ratios up to 1929, households did suffer from higher debt ratios. With this in mind, we bring our attention to Palley's work [Pal96], where he introduces two classes of households that borrow/lend from one another. By linking the debt-to-income ratio of borrowers positively with GDP, he is able to obtain Minsky's paradox of tranquility (Lavoie [Lav08]), along with an economic slowdown associated with higher levels of the interest rate and/or debt-to-income ratio. Just like many of the other models discussed here, this one is not SFC, as banks' reserves share no counterpart (Lavoie [Lav08]).

More recently, Lavoie and Godley [LG01] have designed a simple SFC model with only three sectors, households, banks, and firms, that has generated numerous extensions. For instance, Skott and Ryoo [SR08] made different behavioral assumptions, changing the arguments of the consumption function, which led to rather different results, indicating some sort of structural instability around the model specification. On a different note, Zezza and Dos Santos [ZDS04] extend the model by adding the central bank and a government sector, besides explicitly taking inflation in consideration. At a later stage, Zezza [Zez08] introduces two classes of households, the workers and the rich. While the former would rent houses, or rely on mortgages when purchasing real estate, the latter rather purchase houses without the need of financing, to collect rental income and/or capital gains (i.e. as an investment).

As an extreme example of how versatile this line of research can be, we turn to the model developed by Eatwell, Mouakil and Taylor [EMT08]. Not only they include a housing market, but they also split the financial sector in banks and their special purpose vehicles (SPV), which issue the mortgages and pack them into mortgage-back securities (MBS). Going further, repos are exchanged between the central bank and every other bank to meet their reserve requirements (therefore relaxing the treasure bills repurchase agreement mechanism). Furthermore, demand for houses is modeled as decreasing with housing prices, yet increasing with their rate of change, and decreasing with the mortgage rate and the leverage ratio of both households and banks.

To end this quick survey, we mention the work developed by Steve Keen. In [Kee95], he introduces a simple, yet beautiful extension of the popular, predator-prey like, Goodwin model [Goo67]. Without attempting to suit SFC requirements (even though such requirements can be met, as we will see), he directly derives the dynamics of macroeconomical variables (wage share, employment, and capitalist debt) by introducing a nonlinear investment function that depends solely on the profit share of the capitalists. Immediately after, he introduces an extension accounting for government intervention, whose policy includes both spending, when employment shrinks, and taxation, when profits soar. Its simplicity, and hence analytical tractability, allied with the fact that it captures the essence of the FIH, invites further investigation, turning it into the ideal candidate for a serious mathematical tour de force.

2.1 Constructing a stock-flow consistent model

We have thus far discussed several implementations of the FIH, pointing out which are SFC, and which are not, while leaving aside the description of this property. In this section, we will explore the basic ideas and concepts involved with a SFC model.

Schematically, SFC modeling involves three steps:

1. double-entry accounting: build the balance sheets, together with the transactions table and the flow of funds;

2. determine the behavioral assumptions, e.g. investment, consumption, financing flows;

3. perform "comparative dynamics" and/or prove desired analytical properties.

The output of the first two steps is a system of differential equations that seldom can be solved analytically. A numerical approximation is thus useful when assessing behaviour of a system under certain conditions. In addition, as explored throughout this thesis, one can prove various properties of the system analytically, e.g. determine its fixed points, together with their respective local/global stability, study bifurcations, and establish persistence of key variables.

Borrowing a few words from Dos Santos [DS05], "a fair depiction of Minsky's Wall Street paradigm requires an economy with households, firms, banks and a government (including a central bank)". Clearly, one can expand this list by including SPV's, along with different classes of households, as previously discussed in some of the surveyed models. These decisions are to be made by the researcher, having in mind the trade-off between tractability and explanatory power. In praise of parsimony, we shall stick with the first four sectors mentioned, plus the central bank. In terms of balance sheet items, a decent starting set is composed by cash, deposits, bank loans, government (treasury) bills, central bank advances and stock shares, besides, naturally, the capital goods.

Without further delay, let us introduce a somewhat general balance sheet, transactions and flow of funds. The entries in Table 2.1 are in real terms and mimic closely the framework developed by Godley and Lavoie [GL07] and Dos Santos [DS05].

A few assumptions go embedded in Table 2.1:

1. households can purchase stocks issued by both the firms and the banks (in line with [GL07]);

2. the central bank has zero net worth (in line with Dos Santos [DS05]);

3. the firms' current account distributes its profits to either the capital account, or its shareholders (households). In other words, the resulting sum of all the current account transactions is zero (in line with Godley and Lavoie [GL07] and Godley [God04]);

4. banks distribute a portion F_b of their profits to their shareholders, which includes, but is not limited to, the households (shares can be traded amongst other banks as well) – here we depart slightly from Godley and Lavoie [GL07], where they oddly assume that banks pay no dividends to the households;

5. banks do not pay taxes;

Table 2.1: Balance sheet, transactions and flow of funds for a general Minsky model.

Balance Sheet	Households	Firms current	Firms capital	Banks	Central Bank	Government	Sum
Cash	$+H_h$			$+H_b$	$-H$		0
Deposits	$+M_h$		$+M_f$	$-M$			0
Loans			$-L$	$+L$			0
Bills	$+B_h$			$+B_b$	$+B_c$	$-B$	0
Equities	$+E_f.p_f + E_b.p_b$		$-E_f.p_f$	$-E_b.p_b$			0
Central bank advances				$-A$	$+A$		0
Capital goods			$+K$				$+K$
Sum (net worth)	V_h	0	V_f	V_b	0	$-B$	K
Transactions							
Consumption	$-C$	$+C$					0
Investment		$+I$	$-I$				0
Government expenditures		$+G$				$-G$	0
Accounting memo [GDP]		$[Y]$					
Wages	$+W$	$-W$					0
Government subsidies		$+GS$				$-GS$	
Government taxes	$-T_h$	$-T_f$				$+T$	0
Interest on deposits	$+r_M.M_h$	$+r_M.M_f$		$-r_M.M$			0
Interest on loans		$-r_L.L$		$+r_L.L$			0
Interest on bills	$+r_B.B_h$			$+r_B.B_b$	$+r_B.B_c$	$-r_B.B$	0
Dividends	$+F_d + F_b$	$-F$	$+F_u$	$-F_b$	$-F_c$	$+F_c$	0
Sum	S_h	0	S_f	S_b	0	S_g	
Flow of Funds							
Cash	$+\dot{H}_h$			$+\dot{H}_b$	$-\dot{H}$		0
Deposits	$+\dot{M}_h$		$+\dot{M}_f$	$-\dot{M}$			0
Loans			$-\dot{L}$	$+\dot{L}$			0
Bills	$+\dot{B}_h$			$+\dot{B}_b$	$+\dot{B}_c$	$-\dot{B}$	0
Equities	$+\dot{E}_f.p_f + \dot{E}_b.p_b$		$-\dot{E}_f.p_f$	$-\dot{E}_b.p_b$			0
Central bank advances				$-\dot{A}$	$+\dot{A}$		0
Sum	S_h	0	S_f	S_b	0	S_g	0

6. the central bank sends its profits to the government, hence its zero net worth;

The final ingredient necessary is the set of behavioral assumptions. Naturally, a crucial hypothesis is the investment function, which directly assigns the dynamics of capital goods. Moreover, one needs to specify, among others, how households consume, what the government policies are in terms of spending and taxation, and how does the stock price evolve with time.

This crucial step is notably vulnerable to controversy and criticism. Once the "skeleton" of the model is laid down, there are numerous ways to close the system, each with its own advantages and disadvantages. Many of the models surveyed in the last section shared similar accounting structure, yet possessed distinct behavioral characteristics, rendering diverging results (e.g. Lavoie and Godley [LG01], and Skott and Ryoo [SR08]).

By way of example, in the next section, we will show that both the Keen model [Kee95], and its extension with government intervention, which will be explored later in Chapters 5, and 8, satisfy the SFC requirements.

2.2 Adapting Keen to the SFC framework

2.2.1 Standard Keen model

Our goal in this subsection is to derive the Keen model from the framework set in the previous section. To this end, we need to simplify and eliminate most of the items in Table 2.1, as the Keen model [Kee95] does not involve many of the sectors we just described. To begin with, we remove both the government and the central bank. As well, we get rid of stock shares, and cash (due to the absence of a central bank), while assuming that $r_M = r_L = r$.

The simplified "skeleton" is described in Table 2.2, from which we obtain the following

Table 2.2: Balance sheet, transactions and flow of funds for the standard Keen model.

Balance Sheet	Households	Firms current	Firms capital	Banks	Sum
Deposits	$+M_h$		$+M_f$	$-M$	0
Loans			$-L$	$+L$	0
Capital goods			$+K$		$+K$
Sum (net worth)	V_h	0	V_f	0	K
Transactions					
Consumption	$-C$	$+C$			0
Investment		$+I$	$-I$		0
Accounting memo [GDP]		[Y]			
Wages	$+W$	$-W$			0
Interest on deposits	$+r.M_h$	$+r.M_f$		$-r.M$	0
Interest on loans		$-r.L$		$+r.L$	0
Dividends		$-F$	$+F_u$		0
Sum	S_h	0	S_f	S_b	
Flow of Funds					
Deposits	$+\dot{M}_h$		$+\dot{M}_f$	$-\dot{M}$	0
Loans			$-\dot{L}$	$+\dot{L}$	0
Sum	S_h	0	S_f	S_b	0

$$\dot{M}_h = W - C + rM_h \tag{2.1}$$

$$\dot{M}_f - \dot{L} = F_u - I = C - W + rM_f - rL \tag{2.2}$$

$$\dot{L} - \dot{M} = rL - rM \tag{2.3}$$

with output

$$Y = C + I \tag{2.4}$$

Define now the firms' profit share of the output as

$$\begin{aligned}\pi &= \frac{F_u}{Y} = \frac{F}{Y} \\ &= Y^{-1}\left(Y - W - r(L - M_f)\right) \\ &= 1 - \omega - rd\end{aligned} \tag{2.5}$$

where $\omega = W/Y$, $D = L - M_f = M_h$, and $d = D/Y$. Just like Keen [Kee95], assume that investment is given by a general function of π

$$I = \kappa(\pi)Y \tag{2.6}$$

meaning that the change in capital stock is investment minus depreciation,

$$\dot{K} = I - \delta K = \kappa(\pi)Y - \delta K \tag{2.7}$$

The dynamics of debt can be derived through (2.2) and (2.4)

$$
\begin{aligned}
\dot{D} = \dot{L} - \dot{M}_f &= W - C + r(L - M_f) \\
&= \omega Y - (Y - I) + rD \\
&= Y\left(\omega - 1 + \kappa(\pi) + rD\right) \\
&= Y\left(\kappa(\pi) - \pi\right)
\end{aligned}
\tag{2.8}
$$

Following Goodwin's [Goo67] steps, diligently explained in Chapter 3, we assume that the capital to output ratio remains constant, $K = \nu Y$, which yields the following dynamics for Y

$$
\dot{Y} = Y\left(\frac{\kappa(\pi)}{\nu} - \delta\right) \tag{2.9}
$$

Next, we assume that productivity (in goods/workers) and the total labor force both grow exponentially

$$
\dot{a} = \alpha a \tag{2.10}
$$

$$
\dot{N} = \beta N \tag{2.11}
$$

Assuming full capital utilization, the employed labor force is then given by $L = Y/a$, while the employment ratio is then $\lambda = L/N$. We now make another behavioral assumption, that the bargaining equation for wages $\mathrm{w} = W/L$ follows from the Phillips curve depending solely on the employment level,

$$
\dot{\mathrm{w}} = \Phi(\lambda)\mathrm{w} \tag{2.12}
$$

As a result, we have the following three-dimensional model for ω, λ and d

$$\begin{aligned}\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right]\\ \dot{\lambda} &= \lambda\left[\frac{\kappa(\pi)}{\nu} - \alpha - \beta - \delta\right]\\ \dot{d} &= \kappa(\pi) - \pi - \left[\frac{\kappa(\pi)}{\nu} - \delta\right]\end{aligned} \tag{2.13}$$

and we have recovered the standard Keen model, just as defined in [Kee95], in full STC form. Observe that consumption is fully determined by the investment strategy of the capitalist sector,

$$C = Y - I = Y\left[1 - \kappa(\pi)\right] \tag{2.14}$$

This shortcoming can be avoided by assuming households can decide their savings policy, as done by Skott in [Sko89b], where the accommodating variable is the price of goods.

2.2.2 Keen model with government intervention

In [Kee95], Keen immediately extends the model previously introduced by adding the government sector. Table 2.3 shows the augmented balance sheet, transactions and flow of funds representing all the relevant exchanges. Notice that we added a row representing government subsidies, which add to the firms' current account, yet do not contribute to the economy output.

From the second and third columns of Table 2.3, we obtain

$$C + G_e + GS - W - T + r(M_f - L) = F_u - I = \dot{M}_f - \dot{L} \tag{2.15}$$

If we define the firms' profit share as

Table 2.3: Balance sheet, transactions and flow of funds for the Keen model with government intervention.

Balance Sheet	Households	Firms current	Firms capital	Banks	Government	Sum
Deposits	$+M_h$		$+M_f$	$-M$		0
Loans			$-L$	$+L$		0
Bills	$+B_h$			$+B_b$	$-B$	0
Capital goods			$+K$			$+K$
Sum (net worth)	V_h	0	V_f	0	$-B$	K
Transactions						
Consumption	$-C$	$+C$				0
Investment		$+I$	$-I$			0
Government expenditures		$+G_e$			$-G_e$	0
Accounting memo [non-government GDP]		$[Y]$				
Government subsidies		$+GS$			$-GS$	0
Wages	$+W$	$-W$				0
Government taxes	0	$-T_f$			$+T$	0
Interest on deposits	$+r.M_h$	$+r.M_f$		$-r.M$		0
Interest on loans		$-r.L$		$+r.L$		0
Interest on bills	$+r_g.B_h$			$+r_g.B_b$	$-r_g.B$	0
Dividends		$-F$	$+F_u$			0
Sum	S_h	0	S_f	S_b	S_g	
Flow of Funds						
Deposits	$+\dot{M}_h$		$+\dot{M}_f$	$-\dot{M}$		0
Loans			$-\dot{L}$	$+\dot{L}$		0
Bills	$+\dot{B}_h$			$+\dot{B}_b$	$-\dot{B}$	0
Sum	S_h	0	S_f	S_b		0

$$\pi = 1 - \omega - rd + g - \tau \tag{2.16}$$

while the firms debt is

$$D_k = L - M_f \tag{2.17}$$

along with $d_k = \frac{D_k}{Y}$, $g = \frac{GS}{Y}$, and $\tau = \frac{T_f}{Y}$. The dynamics of the firms debt D_k follows as

$$\begin{aligned} \dot{D}_k = \dot{L} - \dot{M}_f &= W + T_f + r(L - M_f) - C - G_e - GS \\ &= W + T + rD_k + I - Y - GS \\ &= Y(\kappa(\pi) - (1 - \omega - rd_k - \tau + g)) \\ &= Y(\kappa(\pi) - \pi) \end{aligned} \tag{2.18}$$

Observe that $Y = C + I + G_e$, excluding any form of subsidies, GS. Government debt dynamics is thus

$$\dot{B} = r_g B + GS + G_e - T = Y(r_g b + g - \tau + g_e) \tag{2.19}$$

where $d_g = B/Y$, and $g_e = G_e/Y$. In order to match the government debt's dynamics of [Kee95], we need to assume that $G_e = 0$, that is, government expenditures are zero. In Chapter 8 we will see another form of government expenditure that respects the non-negative consumption constraint. As before, consumption is implicitly determined, this time by investment and government expenditure,

$$C = Y - I - G_e = Y\left[1 - \kappa(\pi) - g_e\right] \tag{2.20}$$

Once more, we follow Goodwin's [Goo67] steps, and arrive at the following differential equations for ω, λ, and d_k:

$$
\begin{aligned}
\dot{\omega} &= \omega \left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda \left[\frac{\kappa(\pi)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d}_k &= \kappa(\pi) - \pi - d_k \left[\frac{\kappa(\pi)}{\nu} - \delta\right]
\end{aligned} \tag{2.21}
$$

We are left to specify the dynamics of g, and τ. One alternative is to follow Keen [Kee95], who proposes non-linear functions that drive GS and T proportionally to Y. In Chapter 8, we extend this idea by splitting GS and T into two variables each, one with growth proportional to Y, and another that varies proportionally to itself. Through this more general design, we are able to reproduce a plethora of behaviors, including what was originally proposed by Keen [Kee95].

In any case, it is straightforward to see that the interaction between the government debt d_g and the primary variables $(\omega, \lambda, d_k, g, \tau)$ is unilateral: one does not need d_g to determine the value of the remaining variables. For this reason, even if we consider that the government expenditure (excluding subsidies) is non-zero, the dynamics of $(\omega, \lambda, d_k, g, \tau)$ are not affected. On the other hand, determining d_g now requires the knowledge of g_e, which is specified in a way that guarantees non-negative consumption.

Chapter 3

Goodwin Model

Ever since its introduction in 1967, the model developed by Richard Goodwin in [Goo67] has been heavily studied and extended. Its importance is due perhaps to the fact that it pioneered the field of macroeconomics with endogenous economical cycles, whereas most models had so far relied on exogenous shocks to produce the same effect.

By adopting the Lotka-Volterra equations of population dynamics ([Lot25] and [Vol27]), Goodwin proposed one of the first and most elegant formulations of Marx's theory of class struggle. Indeed, the inspiration is evident: when profits are on the rise, investments will follow, adding more jobs to the economy and, in consequence, giving more bargaining power to the labor force. The workers will, in turn, demand higher wages, depleting the capitalists' surplus, which will lead to more frequent layoffs. The labor force will have less bargaining power, becoming more susceptible to reducing their wages, thus increasing profitability, and starting a new cycle.

In this chapter, we review the model under a minor modification, we introduce an exploding Phillips curve in order to bound the employment rate from above by unity. Moreover, we derive the solution in terms of a Lyapunov function, and illustrate the general behaviour through examples.

3.1 Mathematical formulation

We start with a model for wages and employment proposed by Goodwin [Goo67]. Consider the following Leontief production function for two homogeneous factors

$$Y(t) = \min\left\{\frac{K(t)}{\nu}, a(t)L(t)\right\}. \tag{3.1}$$

Here Y is the total yearly output, K is the stock of capital, ν is a constant capital–to–output ratio, L is the number of employed workers, and a is the labor productivity, that is to say, the number of units of output per worker per year. All quantities are assumed to be quoted in real rather than nominal terms, thereby already incorporating the effects of inflation, and are net quantities, meaning that intermediate revenues and expenditures are deducted from the final yearly output. Let the total labor force be given by

$$N(t) = N_0 e^{\beta t} \tag{3.2}$$

and define the employment rate by

$$\lambda(t) = \frac{L(t)}{N(t)} \tag{3.3}$$

Furthermore, let the labor productivity be

$$a(t) = a_0 e^{\alpha t}. \tag{3.4}$$

where α and β are constants. Finally, assume full capital utilization, so that

$$Y(t) = \frac{K(t)}{\nu} = a(t)L(t). \tag{3.5}$$

In addition to (3.1)–(3.5), Goodwin makes two key behavioral assumptions. The first is that the rate of change in real wages is a function of the employment rate.

Specifically, denoting real wages per unit of labor by w, Goodwin assumes that

$$\dot{\mathrm{w}} = \Phi(\lambda)\mathrm{w} \tag{3.6}$$

where $\Phi(\lambda)$ is an increasing function known as the Phillips curve. The second key assumption is known as Say's law and states that all wages are consumed and all profits are reinvested, so that the change in capital is given by

$$\dot{K} = (Y - \mathrm{w}L) - \delta K = (1 - \omega)Y - \delta K \tag{3.7}$$

where δ is a constant depreciation rate and ω is the wage share of the economy defined by

$$\omega(t) := \frac{\mathrm{w}(t)L(t)}{a(t)L(t)} = \frac{\mathrm{w}(t)}{a(t)}. \tag{3.8}$$

It then follows from (3.5) and (3.7) that the growth rate for the economy in this model is given by

$$\frac{\dot{Y}}{Y} = \frac{1-\omega}{\nu} - \delta := \mu(\omega). \tag{3.9}$$

Using (3.4), (3.6) and (3.8), we conclude that the wage share evolves according to

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{\mathrm{w}}}{\mathrm{w}} - \frac{\dot{a}}{a} = \Phi(\lambda) - \alpha. \tag{3.10}$$

Similarly, it follows from (3.2), (3.3), (3.4) and (3.9) that the dynamics for the employment rate is

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{Y}}{Y} - \frac{\dot{a}}{a} - \frac{\dot{N}}{N} = \frac{1-\omega}{\nu} - \alpha - \beta - \delta. \tag{3.11}$$

Combining (3.10) and (3.11) we arrive at the following two–dimensional system of differential equations:

$$\begin{aligned} \dot{\omega} &= \omega[\Phi(\lambda) - \alpha] \\ \dot{\lambda} &= \lambda\left[\frac{1-\omega}{\nu} - \alpha - \beta - \delta\right] \end{aligned} \tag{3.12}$$

3.2 Properties

In Goodwin's original article [Goo67] the Phillips curve is taken to be the linear relationship

$$\Phi(\lambda) = -\phi_0 + \phi_1\lambda, \tag{3.13}$$

for positive constants ϕ_0, ϕ_1, so that the system (3.12) reduces to the Lotka–Volterra equations describing the dynamics of a predator ω and a prey λ. Provided

$$\frac{1}{\nu} - \alpha - \beta - \delta > 0, \tag{3.14}$$

it is well known (see for example [HSD04]) that the trivial equilibrium $(\omega, \lambda) = (0, 0)$ is a saddle point, whereas the only non-trivial equilibrium

$$(\overline{\omega}, \overline{\lambda}) = \left(1 - \nu(\alpha + \beta + \delta), \frac{\alpha + \phi_0}{\phi_1}\right) \tag{3.15}$$

is non-hyperbolic. Moreover, solution curves with initial conditions in the positive quadrant are periodic orbits centered at $(\overline{\omega}, \overline{\lambda})$.

One obvious drawback of the model is that is does not constrain the variables ω and λ to remain in the unit square, as should be the case given their economic interpretation. At a later stage, we will drop Say's law as an assumption and replace (3.7) with a more general investment function allowing for external financing in the form of debt, giving rise to the Keen model [Kee95]. Accordingly, the wage share of economic output can exceed unity, so there is no need to impose a constraint on ω. The employment rate λ, however, still needs to satisfy $0 \leq \lambda(t) \leq 1$ for all times. As shown in [DHMP06] this can be achieved by taking the Phillips curve to be a continuously differentiable function Φ on $(0,1)$ satisfying

$$\Phi'(\lambda) > 0 \text{ on } (0,1) \tag{3.16}$$

$$\Phi(0) < \alpha \tag{3.17}$$

$$\lim_{\lambda\to 1^-} \Phi(\lambda) = \infty. \tag{3.18}$$

It can then be verified again that $(\omega,\lambda) = (0,0)$ is a saddle point and that the non–trivial equilibrium

$$(\overline{\omega},\overline{\lambda}) = \big(1-\nu(\alpha+\beta+\delta), \Phi^{-1}(\alpha)\big) \tag{3.19}$$

is non-hyperbolic. Using separation of variables and integrating the equation for $d\lambda/d\omega$, we find that the solution passing through the initial condition (ω_0,λ_0) satisfies the equation

$$\left(\frac{1}{\nu}-\alpha-\beta-\delta\right)\log\frac{\omega}{\omega_0} - \frac{1}{\nu}(\omega-\omega_0) = -\alpha\log\frac{\lambda}{\lambda_0} + \int_{\lambda_0}^{\lambda}\frac{\Phi(s)}{s}ds. \tag{3.20}$$

It follows that

$$V(\omega,\lambda) = \int_{\overline{\omega}}^{\omega}\frac{x-\overline{\omega}}{\nu x}\,dx + \int_{\overline{\lambda}}^{\lambda}\frac{\Phi(y)-\Phi(\overline{\lambda})}{y}\,dy \tag{3.21}$$

is a Lyapunov function associated to the system. In fact, $V(\omega_0,\lambda_0)$ is a constant of motion, since

$$\frac{dV}{dt} = \nabla V\cdot(\dot{\omega},\dot{\lambda}) = 0, \tag{3.22}$$

so that solutions starting at $(\omega_0,\lambda_0)\in(0,\infty)\times(0,1)$ remain bounded and satisfy $0 < \lambda < 1$ because of condition (3.18). Moreover, conditions (3.16) and (3.17) guarantee that the right–hand side of (3.20) has exactly one critical point at $\overline{\lambda} = \Phi^{-1}(\alpha)$ in $(0,1)$, so that any line of the form $\omega = p$ intersects it twice at most, which shows that the solution curves above do not spiral and are therefore closed bounded orbits around the equilibrium $(\overline{\omega},\overline{\lambda})$.

Unsurprisingly, the growth rate for the economy at the equilibrium point $(\overline{\omega},\overline{\lambda})$ is given by

$$\mu(\overline{\omega}) = \frac{1-\overline{\omega}}{\nu} - \delta = \alpha + \beta, \tag{3.23}$$

which is the sum of the population and productivity growth rates.

3.3 Example

We choose the fundamental economic constants to be

$$\alpha = 0.025, \quad \beta = 0.02, \quad \delta = 0.01, \quad \nu = 3 \tag{3.24}$$

and, following [Kee95], take the Phillips curve to be

$$\Phi(\lambda) = \frac{\phi_1}{(1-\lambda)^2} - \phi_0, \tag{3.25}$$

with parameters calibrated according to

$$\Phi(0) = -0.04 \qquad \Phi^{-1}(\alpha) = 0.96 \tag{3.26}$$

so that $\overline{\lambda} = 0.96$, and equations (3.16)–(3.18) are satisfied. It is then easy to see that the trajectories are the closed orbits given by

$$\left(\frac{1}{\nu}-\alpha-\beta-\delta\right)\log\frac{\omega}{\omega_0}-\frac{1}{\nu}(\omega-\omega_0) = (\phi_1-\phi_0-\alpha)\log\frac{\lambda}{\lambda_0}-\phi_1\log\frac{1-\lambda}{1-\lambda_0}+\phi_1\left(\frac{\lambda-\lambda_0}{(1-\lambda)(1-\lambda_0)}\right)$$

around the equilibrium point

$$(\overline{\omega}, \overline{\lambda}) = (0.8350, 0.96) \tag{3.27}$$

as shown in the phase portrait in Figure 3.1 for specific initial conditions (ω_0, λ_0). The cyclical behaviour of the model can be seen in Figure 3.2, where we also plot the total output Y as a function of time, showing a clear growth trend with rate

$$\mu(\overline{\omega}) = 0.045 \tag{3.28}$$

but subject to the underlying fluctuations in wages and employment.

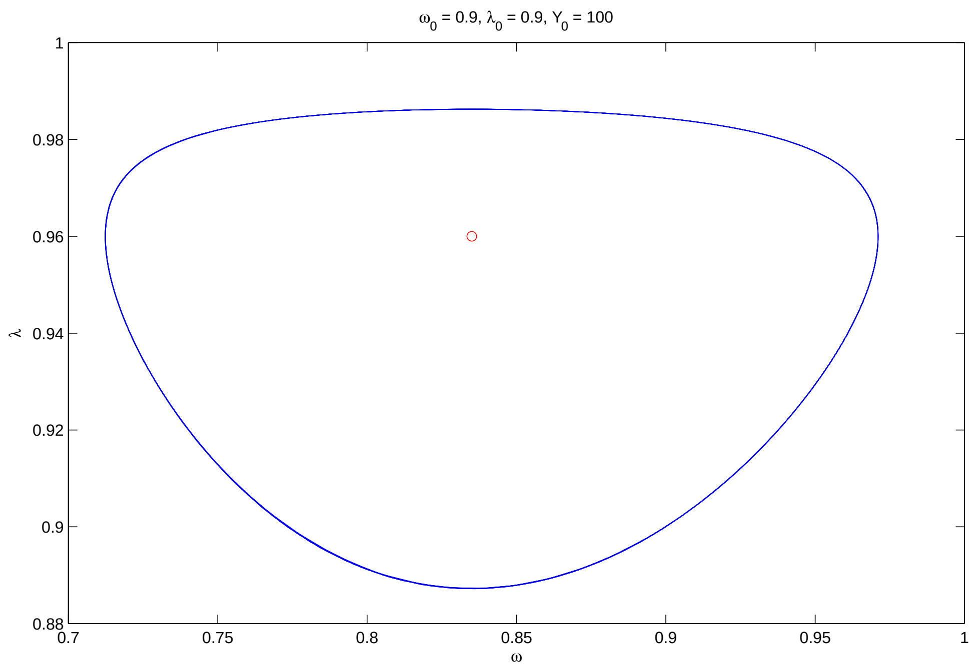

Figure 3.1: Employment rate versus wage share in the Goodwin model

3.4 Criticisms and extensions

The Goodwin model has been extensively criticized for its structural instability, in the sense that small perturbations of the vector field in (3.12) change the qualitative properties of its solution. Most of the literature proposes some extension producing a structurally stable limit cycle to address this issue, for example [Med79], [Sat85], [DN88], [Chi90], [FK92], [Spo95], [FM98] and [MF01]. In particular, Desai [Des73]

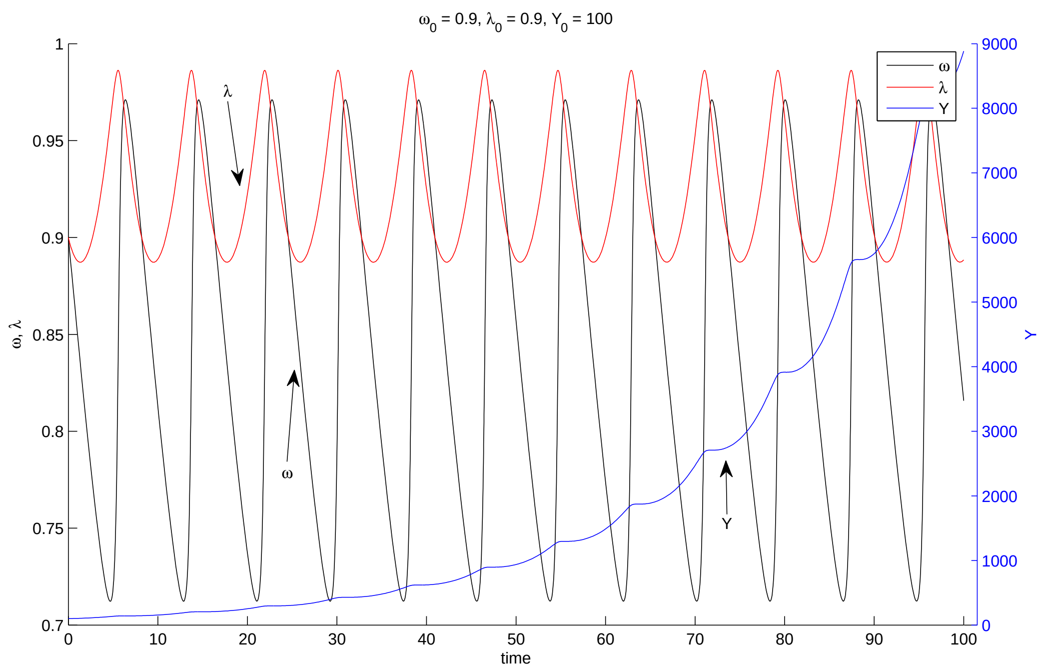

Figure 3.2: Employment rate, wages, and output over time in the Goodwin model

shows that expressing wages in nominal rather than real terms (as is common in the literature related to the Phillips curve) turns the non-hyperbolic equilibrium into a stable sink with trajectories spiraling towards it, whereas relaxing the assumption of a constant capital to output ratio ν leads to two non-trivial equilibrium points. In a different direction, van der Ploeg [vdP85] shows that introducing some degree of substitutability between labor and capital in the form of a more general constant elasticity of substitution (CES) production function also leads to a locally stable equilibrium, whereas Goodwin himself [Goo91] showed that allowing labor productivity to depend pro-cyclically on capital leads to unbounded oscillations, and the relative strength of both effects were analyzed by Aguiar–Conraria [AC08].

These and other extensions are reviewed by Veneziani and Mohun [VM06], where it is argued that instead of being a shortcoming, the structural instability of the Goodwin model can be used to analyze the "factors that determine the fragility of the basic mechanism". Not only they claim that "structurally stable models do not necessarily represent more satisfactory formalizations of Marx's theory of distributive conflict", but also continue "its structural instability is an extreme picture of the fragility of the structure of the symbiotic mechanism regulating distributive conflict".

Under this interpretation, the proposed extensions also help to explain the poor fitting of the original model to data as reported by Solow [Sol90] and Harvie [Har00], with structural changes being responsible for the observed long run phase portraits for (λ, ω), which shows an overall tendency towards cycles, but nothing resembling the closed trajectories implied by the Goodwin model.

Turning to the realm of stochastic dynamical systems, we introduce a stochastic extension of the Goodwin model in the next chapter. By adding a random noise to the productivity dynamics, we explore a variety of properties and concepts foreign to deterministic systems, and not discussed thus far.

Chapter 4

Stochastic Goodwin Model

In this Chapter, we will introduce a stochastic extension of the Goodwin model discussed in the previous Chapter. Rather than the economic interpretation of the proposed extension, we believe that the tools developed here are the major contributions to the field. Notions such as stochastic orbits and perturbation analysis will certainly be ingredients of a more ambitious project where one considers stochasticity in a fully developed Minsky model.

4.1 Introduction and mathematical formulation

Unlike what we had in the Goodwin model, let us consider a growth rate of productivity α which is not constant, but heterogeneous among the total labor force. We claim that the effective growth rate of the productivity should depend negatively on the level of employment. To understand why, imagine an economy close to full employment. The effective growth rate must be close to the average among the entire labor force, as there can only be a few workers left out. On the other hand, consider the opposite extreme: an economy where most of the labor force is unemployed. In such situation, the turnover of the employed workers must be at its maximum, thus the set of employed workers will compromise a different quality of workers at each instant of time. We can then expect to witness a wild fluctuation of the effective growth rate of productivity. In any situation in between, we should expect such variations to settle down as the employment rises. For this reason, we propose the following dynamics for productivity

$$da_t := a_t d\alpha_t = a_t \left(\alpha dt - v(\lambda_t) dW_t\right) \tag{4.1}$$

where $v(\lambda)$ is the volatility term, which we will assume to be a non-negative and decreasing function of λ satisfying $v(\lambda) = 0$. The negative sign accompanying the stochastic term was arbitrarily adopted simply for convenience. Here, W_t is a one dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions [Shr04].

Following the same definitions as in the Goodwin model, (3.1)–(3.3), (3.5)–(3.9), and using Itô's formula, we arrive at the following stochastic dynamical system

$$\begin{cases} d\omega_t = \omega_t \left[\left(\Phi(\lambda_t) - \alpha + v^2(\lambda_t)\right) dt + v(\lambda_t) dW_t\right] \\ d\lambda_t = \lambda_t \left[\left(\mu(\omega_t) - \alpha - \beta + v^2(\lambda_t)\right) dt + v(\lambda_t) dW_t\right] \end{cases} \tag{4.2}$$

where μ is defined just like (3.9) as $\mu(x) = \frac{1-x}{\nu} - \delta$, and represents the growth rate of the economy. Observe that the Dynkin operator associated to (4.2) applied to any function $f(t, \omega, \lambda) \in C^{2,1}(D \times \mathbb{R}_+)$ is

$$\begin{aligned} \mathcal{L}f = \frac{\partial f}{\partial t} &+ \omega \frac{\partial f}{\partial \omega}\left[\Phi(\lambda) - \alpha + v^2(\lambda)\right] + \lambda \frac{\partial f}{\partial \lambda}\left[\mu(\omega) - \alpha - \beta + v^2(\lambda)\right] \\ &+ \frac{1}{2} v^2(\lambda)\left[\omega^2 \frac{\partial^2 f}{\partial \omega^2} + \lambda^2 \frac{\partial^2 f}{\partial \lambda^2} + 2\omega\lambda \frac{\partial^2 f}{\partial \omega \partial \lambda}\right] \end{aligned} \tag{4.3}$$

We propose to study a specific case of this system by assuming a unique root to the deterministic part of the latter.

Assumption 1. *We assume a unique non-trivial equilibrium point* $(0, 0) \neq (\widetilde{\omega}, \widetilde{\lambda}) \in$

$D := (0, +\infty) \times (0, 1)$ *to the deterministic system*

$$\begin{cases} \dot{\omega}_t = \omega_t \left[\Phi(\lambda_t) - \alpha + v^2(\lambda_t)\right] \\ \dot{\lambda}_t = \lambda_t \left[\mu(\omega_t) - \alpha - \beta + v^2(\lambda_t)\right] \end{cases} \tag{4.4}$$

given by $\Phi(\widetilde{\lambda}) - \alpha + v^2(\widetilde{\lambda}) = 0$ *and* $\widetilde{\omega} := \mu^{-1}(\alpha + \beta - v^2(\widetilde{\lambda}))$.

Remark 4.1. If we take the function Φ as (3.25), and the function $v(\lambda)$ as

$$v(\lambda) = \sigma(1 - \lambda)^2 \tag{4.5}$$

then Assumption 1 translates into the following bound for σ

$$0 \leq \sigma < \min\left\{\sqrt{\alpha + \phi_0 - \phi_1}; \frac{\alpha + \phi_0}{2\sqrt{\phi_1}}\right\} \tag{4.6}$$

Based on Assumption 1, we define the potential function

$$\widetilde{V}(\omega, \lambda) := \widetilde{V}_1(\omega) + \widetilde{V}_2(\lambda) := \int_{\widetilde{\omega}}^{\omega} \frac{\mu(\widetilde{\omega}) - \mu(x)}{x}\, dx + \int_{\widetilde{\lambda}}^{\lambda} \frac{\Phi(y) - \Phi(\widetilde{\lambda})}{y}\, dy \tag{4.7}$$

as an extended version of (3.21) (if $v \equiv 0$, we recover the original Lyapunov function).

Remark 4.2. Taking the functions μ, Φ and v as they were defined in (3.9), (3.25) and (4.5), we obtain

$$\widetilde{V}_1(\omega) = \nu^{-1}(\omega - \widetilde{\omega}) - \left(\nu^{-1} - \mu(\widetilde{\omega})\right)\log(\omega/\widetilde{\omega}) \tag{4.8}$$

$$\widetilde{V}_2(\lambda) = \left[\phi_1 - \phi_0 - \Phi(\widetilde{\lambda})\right] \log\frac{\lambda}{\widetilde{\lambda}} - \phi_1 \log\frac{1-\lambda}{1-\widetilde{\lambda}} + \phi_1\left[\frac{1}{1-\lambda} - \frac{1}{1-\widetilde{\lambda}}\right] \tag{4.9}$$

According to [Kha12], a sufficient condition for our stochastic differential equation to have a unique global solution given any initial value is to require that the functions Φ and μ are globally Lipschitz. As our choice of Φ explodes when λ approaches 1, we need an alternative. This is the purpose of the next assumption.

Assumption 2. *There exists two constants k_1, k_2 such that, for all $(\omega, \lambda) \in D$,*

$$-\omega\mu'(\omega) - \mu(\omega) \leq k_1\widetilde{V}_1(\omega) + k_2 \ \text{ and } \ v^2(\lambda)\Phi'(\lambda) \leq k_1\widetilde{V}_2(\lambda) + k_2 \tag{4.10}$$

Remark 4.3. Notice that the inequality with μ is readily satisfied for the function (3.9). The other bound concerns both Φ and v, and ensures that $\lambda_t \leq 1$ almost surely.

We need one last assumption before we dive into the analytical results. It concerns the magnitude of the volatility term.

Assumption 3. *The volatility term is such that $v^2(\widetilde{\lambda}) < \frac{1}{2}v^2(0)$. This seemingly arbitrary assumption is easily achieved, and will be useful when determining the behaviour of the system close to the origin.*

Based on Lyapunov techniques, and flavors of Theorem 2.1 from [MMR02], we prove existence and uniqueness of a regular solution in D. For such, we require conditions similar to those of Theorem 3.4 from [Kha12], which are recalled in Appendix A for convenience.

Theorem 4.1. *Provided Assumptions 1 and 2 hold, and assuming that* (3.16)–(3.18)*, there exists a regular solution $(\omega_t, \lambda_t)_{t\geq 0}$ to system* (4.2) *starting at any point $(\omega_0, \lambda_0) \in D$. Moreover, the solution is unique up to $\mathbb{P}$-null sets, has the Markov property, and remains in D with probability one.*

Proof. The local Lipschitz growth and sub-linearity conditions of the coefficients of the system on every compact subset included in D follow from the continuous differentiability of the function Φ. We are left to check conditions (A.8) and (A.9) in

Theorem A.2 of Appendix A. Applying the Dynkin operator defined in (4.3) to $\widetilde{V}$ we get

$$\begin{aligned}\mathcal{L}\widetilde{V}(\omega,\lambda) =& \frac{1}{2}\left[v^2(\lambda)-v^2(\widetilde{\lambda})\right]\left[\Phi(\lambda)-\Phi(\widetilde{\lambda})\right]-\frac{1}{2}v^2(\widetilde{\lambda})\left[\Phi(\lambda)-\Phi(\widetilde{\lambda})\right]+\frac{1}{2}v^2(\lambda)\lambda\Phi'(\lambda)\\ &+[\mu(\widetilde{\omega})-\mu(\omega)]\left[\frac{1}{2}v^2(\lambda)-v^2(\widetilde{\lambda})\right]-\frac{1}{2}v^2(\lambda)\omega\mu'(\omega)\end{aligned}$$

The first term is non-positive, while the second term is bounded from above by $\frac{1}{2}v^2(\widetilde{\lambda})\left[\Phi(\widetilde{\lambda})-\Phi(0)\right]$ and decreases to $-\infty$ as λ goes to 1^-. Condition (4.10) provides the bound for third term since $|\lambda|<1$, and also the bound for the last line, as the function v is bounded. Altogether, there exist constants $k_1,k_2\in\mathbb{R}_+$ such that condition (A.8) holds for $\widetilde{V}$. It is also clear from separation of variables in $\widetilde{V}$ that

$$\inf_{\omega\in[0,+\infty)}\widetilde{V}(\omega,\lambda)=\widetilde{V}_2(\lambda)+\inf_{\omega\in[0,+\infty)}\widetilde{V}_1(\omega)=\widetilde{V}_2(\lambda) \tag{4.11}$$

and tends to infinity when λ goes to 1 or 0. We also have $\inf_{\lambda\in(0,1)}\widetilde{V}(\omega,\lambda)$ going to infinity as ω goes to 0 or $+\infty$. Condition (A.9) is then satisfied, which allows to apply Theorem A.2. □

The next theorem states the divergence of system (4.2) in the path-wise sense. It relies on another result of [Kha12], which is recalled in Appendix A as well.

Theorem 4.2. *Let (ω_t,λ_t) be a regular solution to* (4.2). *In addition to the requirements of Theorem 4.1, suppose that Assumption 3 holds as well. Then for any $\xi\in\Omega$, $(\omega_t,\lambda_t)(\xi)$ cannot converge with time to any point (ω,λ) in the closure of D, as $t\to\infty$.*

Proof. We will prove this result in three steps, for points in D first, then for the origin, and last for points belonging to its boundary.

1. For any point $(\omega,\lambda)\neq(\widetilde{\omega},\widetilde{\lambda})$ in D, we have either $\lambda\,[\mu(\omega)-\alpha-\beta+v^2(\lambda)]\neq 0$ or $\omega\,[\Phi(\lambda)-\alpha+v^2(\lambda)]\neq 0$. Then by the continuity of the functions μ and Φ, there must exist a neighborhood $B \subset D$ of (ω, λ) such that

$$\min(\max_B(\mathcal{L}\omega_t), \max_B(-\mathcal{L}\omega_t), \max_B(\mathcal{L}\lambda_t), \max_B(-\mathcal{L}\lambda_t)) < -\varepsilon \tag{4.12}$$

for some small $\varepsilon > 0$. We can then find function $W(x, y)$ defined in D (such as x, $K - x$, y or $K - y$), yet non-negative in B, such that $\mathcal{L}W < -\varepsilon$ in B.

We then apply Theorem A.3 of Appendix A and obtain that if $(\omega_t, \lambda_t) = (\omega, \lambda)$ at some arbitrary time $t \geq 0$, then it must almost surely leave the region B in finite time.

Otherwise, at the point $(\widetilde{\omega}, \widetilde{\lambda})$, we have

$$\mathcal{L}\widetilde{V}(\widetilde{\omega}, \widetilde{\lambda}) = \frac{1}{2}v^2(\widetilde{\lambda})\left[\widetilde{\lambda}\Phi'(\widetilde{\lambda}) - \widetilde{\omega}\mu'(\widetilde{\omega})\right] > 0 \tag{4.13}$$

Using the same argument, we get that the process exits a small region around $(\widetilde{\omega}, \widetilde{\lambda})$ in finite time almost surely.

2. Take now $(\omega, \lambda) = (0, 0)$. We shall demonstrate that there exists a ball around this point that must be exited in finite time almost surely. To see why, let's look at the Dynkin operator applied to $\widetilde{V}$ at $(0, 0)$

$$\mathcal{L}\widetilde{V}(0, 0) = \left[\frac{1}{2}v^2(0) - v^2(\widetilde{\lambda})\right]\left[\Phi(0) - \Phi(\widetilde{\lambda}) + \mu(\widetilde{\omega}) - \mu(0)\right] \tag{4.14}$$

which is strictly negative under Assumption 3. Continuity implies the existence of a ball around $(0, 0)$ with $\mathcal{L}\widetilde{V} < 0$ inside it, while Theorem A.3 provides the almost surely finiteness of its exit time.

3. Finally, for some $d > 0$ smaller than the radius of the ball obtained in step 2, define the regions

$$R_B = [0, +\infty) \times [0, d] \tag{4.15}$$

$$R_L = [0, d] \times [0, 1] \tag{4.16}$$

$$R_U = [0, +\infty) \times [1 - d, 1] \tag{4.17}$$

with the corresponding stopping times

$$\tau_{R_B} = \inf \{t \geq 0 \ : \ (\omega_t, \lambda_t) \notin R_B\} \tag{4.18}$$

$$\tau_{R_L} = \inf \{t \geq 0 \ : \ (\omega_t, \lambda_t) \notin R_L\} \tag{4.19}$$

$$\tau_{R_U} = \inf \{t \geq 0 \ : \ (\omega_t, \lambda_t) \notin R_U\} \tag{4.20}$$

We wish to prove that each of these regions is exited in finite time almost surely, that is, the boundary of D cannot attract solutions indefinitely. Using the Markov property of the solution, we will consider a starting point in each of these regions and seek to prove that each corresponding stopping time is finite almost surely.

For the first region, we look at the Dynkin of $\omega \geq 0$

$$\mathcal{L}\omega = \omega \left[\Phi(\lambda) - \alpha + v^2(\lambda)\right] \leq -M\omega \leq 0 \tag{4.21}$$

where $M = \max_{[0,\varepsilon]} [\Phi(\lambda) - \alpha + v^2(\lambda)] < 0$. Therefore, if we restrict ourselves to $\omega > \varepsilon$, for any $\varepsilon > 0$, we have through Theorem A.3 that every region $R_B \cap \{\omega > \varepsilon\}$ must be exited in finite time almost surely. However, Doob's martingale convergence theorem (DMCT henceforward) on the non-negative super-martingale ω_t guarantees that $\omega_{t \wedge \tau_{R_B}}$ converges point-wise to some random variable ω_∞. Assuming by contradiction that τ_{R_B} is infinite in a set of positive measure, we can pick a $\xi \in \{\tau_{R_B} = +\infty\}$ and see that the solution linked to this random state must converge to a point in $\{0\} \times [0, d]$. This is, according to step 2, not possible. We have thus obtained our contradiction, which implies that $\tau_{R_B} < +\infty$ almost surely.

For the second region, use the function $1-\lambda$, with

$$\mathcal{L}(1-\lambda) = -\lambda\left[\mu(\omega) - \alpha - \beta + v^2(\lambda)\right] \leq -\lambda\left[\mu(d) - \alpha - \beta\right] \leq 0 \tag{4.22}$$

Restricting ourselves to $\lambda > \varepsilon$, we see that every region $R_L \cap \{\lambda > \varepsilon\}$ must be exited in finite time. With exactly the same argument as before, we find that τ_{R_L} must be finite almost surely.

Finally, for the third region, consider the function ω^{-2}, with

$$\mathcal{L}\omega^{-2} = -2\omega^{-2}\left[\Phi(\lambda) - \alpha - \frac{1}{2}v^2(\lambda)\right] \leq -2\omega^{-2}m \leq 0 \tag{4.23}$$

where $m = \Phi(1-d) - \alpha - \frac{1}{2}v^2(1-d) > 0$. Restricting ourselves to $\omega < 1/\varepsilon$ we obtain through the same argument as previously that either the region is exited in finite time almost surely or ω explodes to $+\infty$. As the latter would defy existence, we have finished the proof. □

Remark 4.4. Notice, quite importantly, that we do not prove that the region $R_B \cup R_L \cup R_U$ is exited in finite time almost surely. Instead, we show this property for each of these regions individually. Conveniently, this will be sufficient for the analysis that follows.

One could also be interested in the departure from the deterministic model (3.12). The next result provides a probabilistic estimate of the time it takes for the process $V(\omega_t, \lambda_t)$ to exit a ball around $V(\omega_0, \lambda_0)$, where V is the Lyapunov function defined in (3.21).

Theorem 4.3. *Let $(\omega_t, \lambda_t)_{t\geq 0}$ be a solution of system (4.2) with initial condition $(\omega_0, \lambda_0) \in D$. For any $\xi \in \Omega$ and $t \geq 0$, we define the quantity $V_0 := V(\omega_0, \lambda_0)$ and $e_t(\xi) := V(\omega_t(\xi), \lambda_t(\xi)) - V_0$, along with the stopping time*

$$\tau_c(\xi) := \inf\{t > 0 \ : \ |e_t(\xi)| \geq c\} \tag{4.24}$$

for any $0 < c < V_0$*. We then have the following estimate*

$$\mathbb{P}\left[\tau_c > B_{c,\zeta}^{-1}(c)\right] \geq \left(1 - \frac{I^2(V_0, c)}{\zeta^2}\right) \tag{4.25}$$

with $B_{c,\zeta}(t) := \frac{1}{2}R(V_0, c)t + \zeta\sqrt{t}$ *for two constants* R *and* I *depending only on* V_0 *and* c*.*

Proof. Applying Itô's formula to e_t, we find

$$\begin{aligned} de_t = &\frac{1}{2}v^2(\lambda)\left[\mu(\overline{\omega}) - \mu(\omega) + \Phi(\lambda) - \Phi(\overline{\lambda}) - \omega\mu'(\omega) + \lambda\Phi'(\lambda)\right] dt \\ &+ v(\lambda)\left[\mu(\overline{\omega}) - \mu(\omega) + \Phi(\lambda) - \Phi(\overline{\lambda})\right] dW_t \end{aligned} \tag{4.26}$$

Define the martingale by

$$M_t = \frac{1}{\sqrt{t}}\int_0^t v(\lambda_s)\left[\mu(\overline{\omega}) - \mu(\omega_s) + \Phi(\lambda_s) - \Phi(\overline{\lambda})\right] dWs \quad \text{for } t > 0 \tag{4.27}$$

with $M_0 = 0$, along with the $\mathcal{F}_{\tau_c}$-measurable set, for any $\zeta > 0$

$$A_\zeta = \left\{\xi \in \Omega : \sup_{0 \leq t \leq \tau_c} \left|M_t(\xi)\right| \leq \zeta\right\} \tag{4.28}$$

Notice that for t and $\xi \in \Omega$ such that $t < \tau_c(\xi)$, we have that $(\omega, \lambda) \in E(V_0, c) := \{(\omega, \lambda) \in D \ : \ |V(\omega, \lambda) - V_0| \leq c\}$. With this in mind, we define the finite quantities

$$R(V_0, c) := \max_{(\omega,\lambda)\in E(V_0,c)} v^2(y)\left[\mu(\overline{\omega}) - \mu(\omega) - \omega\mu'(\omega) + \lambda\Phi'(\lambda) + \Phi(\lambda) - \Phi(\overline{\lambda})\right] \tag{4.29}$$

$$I(V_0, c) := \max_{(\omega,\lambda)\in E(V_0,c)} v(\lambda)\left(\mu(\overline{\omega}) - \mu(\omega) + \Phi(\lambda) - \Phi(\overline{\lambda})\right) \tag{4.30}$$

We can find the probability of the event A_ζ by first using conditional probability and then following with Doob's martingale inequality for the absolute value of the càdlàg modification[1] of M_t, which is a sub-martingale that agrees with $|M_t|$ for all $t > 0$, namely

$$\begin{aligned}
\mathbb{P}\left[A_\zeta\right] &= \int_0^{+\infty} \mathbb{P}\left[A_\zeta \middle| \tau_c = t\right] f_{\tau_c}(t)\, dt \\
&\geq \int_0^{+\infty} \left(1 - \frac{1}{\zeta^2}\mathbb{E}\left[\frac{1}{t}\int_0^t v^2(\lambda_s)\left(\mu(\overline{x}) - \mu(\omega_s)\Phi(\lambda_s) - \Phi(\overline{\lambda})\right)^2 ds \middle| \tau_c = t\right]\right) f_{\tau_c}(t)\, dt \\
&\geq \int_0^{+\infty} \left(1 - \frac{I^2(V_0, c)}{\zeta^2}\right) f_{\tau_c}(t)\, dt \\
&= \left(1 - \frac{I^2(V_0, c)}{\zeta^2}\right)
\end{aligned} \tag{4.31}$$

where f_{τ_c} is the probability density function of the random variable τ_c. At last, we can integrate de_t to find the estimate, valid for any $\xi \in A_\zeta$ and $t < \tau_c(\xi)$

$$\begin{aligned}
|e_t(\xi)| &\leq \frac{1}{2}\int_0^t v^2(\lambda_s)\left|\mu(\overline{\omega}) - \mu(\omega_s) - \omega_s\mu'(\omega_s) + \lambda_s\Phi'(\lambda_s) + \Phi(\lambda_s) - \Phi(\overline{\lambda})\right| ds \\
&\quad + \left|\int_0^t v(\lambda_s)\left[\mu(\overline{\omega}) - \mu(\omega_s) + \Phi(\lambda_s) - \Phi(\overline{\lambda})\right] dW_s\right| \\
&\leq \frac{1}{2}R(V_0, c)t + \zeta\sqrt{t} = B_{c,\zeta}(t)
\end{aligned} \tag{4.32}$$

Finally, since $B_{c,\zeta}(t) \leq c$ for $t \leq B_{c,\zeta}^{-1}(c)$, we have that for $\xi \in A_\zeta$, $\tau_c(\xi) > B_{c,\zeta}^{-1}(c)$, which leads to

$$\mathbb{P}\left[\tau_c > B_{c,\zeta}^{-1}(c)\right] \geq \mathbb{P}\left[\tau_c > B_{c,\zeta}^{-1}(c) \middle| A_\mu\right]\mathbb{P}\left[A_\mu\right] \geq \left(1 - \frac{I^2(V_0, c)}{\zeta^2}\right) \tag{4.33}$$

[1] In spite of the process $|M_t|$ being discontinuous at $t = 0$, we can still verify through Chebyshev inequality that

$$\mathbb{P}\left[\lim_{t\to 0^+} |M_t| \geq \zeta\right] = \mathbb{P}\left[|Z| \geq \zeta/|h_0|\right] \leq Var\left[Z\right](h_0/\zeta)^2 = (|h_0|/\zeta)^2 \leq I^2(V_0, c)/\zeta^2$$

where h_s is in the integrand in (4.27), and $Z \sim N(0,1)$ is a Gaussian random variable.

finishing the proof. □

The interpretation of Theorem 4.3 is that we can, for a given confidence level, bound the growth of the solution in terms of the Lyapunov function (3.21).

4.2 Stochastic orbits with recurrent domains

Along the spirit of Theorem 4.2, we have no hopes of obtaining periodicity for the the stochastic system (4.2). In this section, we will address this issue by proposing a convenient and intuitive alternative, the notion that solutions must almost surely transition amongst regions that cover the domain D indefinitely.

Definition 1. *Let $\xi \in \Omega$ denote a random state, while $(\omega_{t_0}, \lambda_{t_0})$ represents a point on the line L connecting the origin and $(\widetilde{\omega}, \widetilde{\lambda})$, that is, $\omega_{t_0} = \frac{\widetilde{\omega}}{\widetilde{\lambda}}\lambda_{t_0}$ while $\lambda_{t_0} \in (0,1)$. As well, define $\theta_t(\xi)$ as the angle from the vector $\{\omega_{t_0} - \widetilde{\omega}; \lambda_{t_0} - \widetilde{\lambda}\}$ to the vector $\{\omega_t(\xi) - \widetilde{\omega}; \lambda_t(\xi) - \widetilde{\lambda}\}$, in clockwise direction. We define a **stochastic orbit** as the solution $(\omega_t(\xi), \lambda_t(\xi))_{t\in[t_0,t_0+T(\xi)]}$ of the system (4.2), where $T(\xi) = \inf\{t \geq t_0 : \theta_t(\xi) \geq 2\pi\}$. The stopping time $T(\xi)$ is called the period of the stochastic orbit.*

Observe that the locus of the starting point was specifically chosen for a reason. As we will see later in this section, solutions can only cross the line L in a specific direction. The next theorem asserts the almost surely finiteness of the period.

Theorem 4.4. *The period of any stochastic orbit is finite almost surely.*

The rest of the section is devoted to prove Theorem 4.4. In order to achieve that, we define regions $(R_i)_i$ of the domain $D := (0, +\infty) \times (0, 1)$, illustrated by Figure 4.1, and prove that the system exits each of them in finite time, transitioning in some appropriate direction.

Consider the concave decreasing function on $\mathbb{R}_+$

$$f(\omega) = \Phi^{-1}(\mu(\omega) - \beta) \tag{4.34}$$

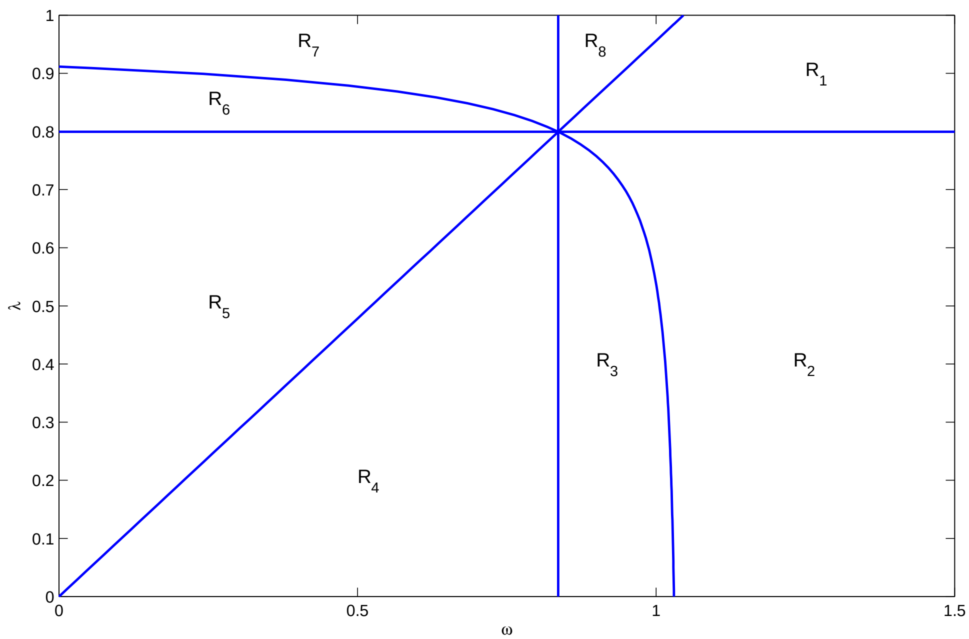

Figure 4.1: Domain D and its covering by $(R_i)_{i=1...8}$. Since $f(0) < 1$ and $y = 1$ is a vertical asymptote of Φ, this corresponds to the general case, where $x = 0$ is possible on R_7.

We also introduce the process $\rho := (\rho_t)_{t\geq 0}$ defined by $\rho_t := \lambda_t/\omega_t$, which is a finite variation, $\mathcal{F}$-adapted process with dynamics

$$d\rho_t = \rho_t \left(\mu(\omega) - \beta - \Phi(\lambda)\right) dt = \rho_t \left(\Phi(f(\omega)) - \Phi(\lambda)\right) dt \tag{4.35}$$

Define $\widetilde{\rho} := \widetilde{\lambda}/\widetilde{\omega}$. We divide the domain D in 8 sets:

$$D = \bigcup_{i=1}^{8} R_i \text{ with } \begin{cases} R_1 := \{(\omega,\lambda) \in D \;:\; \lambda \geq \widetilde{\lambda} \text{ and } \lambda/\omega \leq \widetilde{\rho}\} \\ R_2 := \{(\omega,\lambda) \in D \;:\; f(\omega) \leq \lambda \leq \widetilde{\lambda}\} \\ R_3 := \{(\omega,\lambda) \in D \;:\; \lambda \leq f(\omega) \text{ and } \omega \geq \widetilde{\omega}\} \\ R_4 := \{(\omega,\lambda) \in D \;:\; \omega \leq \widetilde{\omega} \text{ and } \lambda/\omega \leq \widetilde{\rho}\} \\ R_5 := \{(\omega,\lambda) \in D \;:\; \lambda \leq \widetilde{\lambda} \text{ and } \lambda/\omega \geq \widetilde{\rho}\} \\ R_6 := \{(\omega,\lambda) \in D \;:\; \widetilde{\lambda} \leq \lambda \leq f(\omega)\} \\ R_7 := \{(\omega,\lambda) \in D \;:\; \lambda \geq f(\omega) \text{ and } \omega \leq \widetilde{\omega}\} \\ R_8 := \{(\omega,\lambda) \in D \;:\; \omega \geq \widetilde{\omega} \text{ and } \lambda/\omega \geq \widetilde{\rho}\} \end{cases} \tag{4.36}$$

Remark 4.5. Notice that

$$\bigcap_{i=1}^{8} R_i = (\widetilde{\omega}, \widetilde{\lambda}) \text{ and } \bigcup_{i=1}^{8} R_i = D\ . \tag{4.37}$$

Moreover, $\widetilde{\lambda} = f(\widetilde{\omega})$. We also emphasize that $v(\widetilde{\lambda}) \neq 0$, so that around the point $(\widetilde{\omega}, \widetilde{\lambda})$, the system locally behaves like a diffusion on the plane, which implies, along Theorem 4.2, that

$$\mathbb{P}\left[(\omega_t, \lambda_t) = (\widetilde{\omega}, \widetilde{\lambda}),\ \text{for some } t > 0\right] = 0 \quad \forall (\omega_0, \lambda_0) \in D. \tag{4.38}$$

Accordingly, we have that this particular point is almost surely never reached, implying that, a priori, a solution can only leave a region R_i to one of its neighboring regions R_j, where

$$j \in \begin{cases} \{mod(i \pm 1, 8)\} & \text{if } i > 1 \\ \{2, 8\} & \text{if } i = 1 \end{cases} \tag{4.39}$$

According to Theorem A.2 any solution of system (4.2) is a Markovian process. We can thus equivalently treat a time translation as a different initial condition for the system. We define, for $(\omega_s, \lambda_s) \in D$ the stopping times

$$\tau_i(\omega_s, \lambda_s) := \inf\{t \geq s \ : \ (\omega_t, \lambda_t) \in R_i\}, \ i = 1, \dots, 8. \tag{4.40}$$

as the first-entry time to each region R_i. The dependence in s is implied in the above formulation, yet we shall be exempt of any ambiguity in the proofs below: $\tau_i(\omega, \lambda)$ is a short notation for $\tau_i(\omega_0, \lambda_0)$ with $(\omega_0, \lambda_0) = (\omega, \lambda)$. Our first proposition will be extremely useful in the proofs of the proceding results.

Proposition 4.1. *Define*

$$\varrho(\omega, \lambda) := \frac{\lambda_t - \widetilde{\lambda}}{\omega_t - \widetilde{\omega}}, \tag{4.41}$$

and

$$F_c(x) = \tan\left(\tan^{-1}(x) + \tan^{-1}(c)\right) \tag{4.42}$$

for any

$$c \in \left(0, \left(\widetilde{\rho} + \frac{M}{m}\right)^{-1}\right) \tag{4.43}$$

where

$$M := \max_{(\omega,\lambda)\in[\bar{\omega},\widetilde{\omega}]\times[\widetilde{\lambda},\bar{\lambda}]} \lambda(\omega - \widetilde{\omega})\left[\mu(\omega) - \alpha - \beta\right] - \omega(\lambda - \widetilde{\lambda})\left[\Phi(\lambda) - \alpha\right] \geq 0 \tag{4.44}$$

$$m := \min_{(\omega,\lambda)\in[\bar{\omega},\widetilde{\omega}]\times[\widetilde{\lambda},\bar{\lambda}]} \omega\widetilde{\omega}v^2(\lambda) \geq 0 \tag{4.45}$$

are finite constants. Then we have that the process

$$F_c \circ \varrho(\omega_t, \lambda_t) \tag{4.46}$$

is a super-martingale in $S_c := D \setminus \{\widetilde{\rho} \leq \varrho(\omega, \lambda) \leq \frac{1}{c}\}$

Proof. Denote $\varrho_t = \varrho(\omega_t, \lambda_t)$, $\Delta\omega_t := \omega_t - \widetilde{\omega}$, and $\Delta\lambda_t := \lambda_t - \widetilde{\lambda}$. Our goal is to prove that the Dynkin operator applied to the function $F_c \circ \varrho(\omega, \lambda)$ is non-positive in the region S_c. We can apply Itô's formula to ϱ_t and obtain

$$d\varrho_t = \frac{d\lambda_t}{\Delta\omega_t} - \frac{\Delta\lambda_t}{(\Delta\omega_t)^2} d\omega_t + \frac{v^2(\lambda_t)}{(\Delta\omega_t)^2}\left[\frac{\Delta\lambda_t}{\Delta\omega_t}\omega_t^2 - \omega_t\lambda_t\right] dt \ . \tag{4.47}$$

Then, noticing that $F_c(\varrho) = (\varrho + c)(1 - \varrho c)$, we get

$$dF_c(\varrho_t) = \frac{1+c^2}{(1-\varrho_t c)^2}\left(d\varrho_t + \frac{c}{1-\varrho_t c} d\left\langle \varrho_t \right\rangle\right) \tag{4.48}$$

Dropping the redundant t subscripts, we have explicitly

$$\begin{aligned}(\Delta\omega)^4 \frac{(1-\varrho c)^2}{1+c^2}\mathcal{L}F_c &= (\Delta\omega)^3\lambda\left[\mu(\omega) - \alpha - \beta + v^2(\lambda)\right] \\ &\quad - (\Delta\lambda)(\Delta\omega)^2\omega\left[\Phi(\lambda) - \alpha + v^2(\lambda)\right] \\ &\quad + v^2(\lambda)\left[\omega(\Delta\omega)\left(\lambda\widetilde{\omega} - \widetilde{\lambda}\omega\right) + \frac{c}{1-\varrho c}\left(\widetilde{\lambda}\omega - \lambda\widetilde{\omega}\right)^2\right]\end{aligned} \tag{4.49}$$

Besides, note that $x \mapsto 1/(x - \varrho)$ is monotonically decreasing for $x \in (-\infty, \varrho)$ and $x \in (\varrho, +\infty)$. Therefore, as $c^{-1} > \widetilde{\rho}$, and $\varrho \notin [\widetilde{\rho}, c^{-1}]$, we have

$$\frac{c}{1-\varrho c} = \frac{1}{c^{-1} - \varrho} < \frac{1}{\widetilde{\lambda}/\widetilde{\omega} - \varrho} = \frac{\widetilde{\omega}(\Delta\omega)}{\widetilde{\lambda}\omega - \widetilde{\omega}\lambda} \tag{4.50}$$

in the region S_c, so that we can further bound $\mathcal{L}F_c$ by

$$\begin{aligned}(\Delta\omega)^2 \frac{(1-\varrho c)^2}{1+c^2}\mathcal{L}F_c &\leq \lambda(\Delta\omega)\left[\mu(\omega) - \alpha - \beta + v^2(\lambda)\right] - \omega(\Delta\lambda)\left[\Phi(\lambda) - \alpha + v^2(\lambda)\right] \\ &\quad - v^2(\lambda)\left(\widetilde{\lambda}\omega - \lambda\widetilde{\omega}\right) \\ &= \lambda(\Delta\omega)\left[\mu(\omega) - \alpha - \beta\right] - \omega(\Delta\lambda)\left[\Phi(\lambda) - \alpha\right] \\ &\leq 0 \ \text{ in } D \setminus [\bar{\omega}, \widetilde{\omega}] \times [\widetilde{\lambda}, \bar{\lambda}]\end{aligned} \tag{4.51}$$

Which proves the desired result in the region $S_c \setminus [\bar{\omega}, \widetilde{\omega}] \times [\widetilde{\lambda}, \bar{\lambda}]$. We are left to show the same in the rectangle $[\bar{\omega}, \widetilde{\omega}] \times [\widetilde{\lambda}, \bar{\lambda}] \subset S_c$. In order to accomplish that, let us rewrite (4.49) as

$$
\begin{aligned}
(\Delta\omega)^2\frac{(1-\varrho c)^2}{1+c^2}\mathcal{L}F_c &= \lambda\Delta\omega\left[\mu(\omega)-\alpha-\beta+v^2(\lambda)\right]-\omega\Delta\lambda\left[\Phi(\lambda)-\alpha+v^2(\lambda)\right]\\
&+v^2(\lambda)\widetilde{\omega}\left[\omega\left(\varrho-\widetilde{\rho}\right)+\frac{\widetilde{\omega}\left(\varrho-\widetilde{\rho}\right)^2}{c^{-1}-\varrho}\right]\\
&=\lambda\Delta\omega\left[\mu(\omega)-\alpha-\beta+v^2(\lambda)\right]-\omega\Delta\lambda\left[\Phi(\lambda)-\alpha+v^2(\lambda)\right]\\
&+v^2(\lambda)\widetilde{\omega}\frac{\varrho-\widetilde{\rho}}{\varrho-c^{-1}}\left(\lambda-\omega/c\right)
\end{aligned}
\tag{4.52}
$$

Since $\varrho \le 0$ in the rectangle, $\frac{\varrho-\widetilde{\rho}}{\varrho-c^{-1}} \le 1$, thus

$$
\begin{aligned}
(\Delta\omega)^2\frac{(1-\varrho c)^2}{1+c^2}\mathcal{L}F_c &\le \lambda\Delta\omega\left[\mu(\omega)-\alpha-\beta\right]-\omega\Delta\lambda\left[\Phi(\lambda)-\alpha\right]\\
&+v^2(\lambda)\left[\lambda\Delta\omega-\omega\Delta\lambda+\widetilde{\omega}\left(\lambda-\omega\left(\frac{1}{c}-\widetilde{\omega}+\widetilde{\omega}\right)\right)\right]\\
&=\lambda\Delta\omega\left[\mu(\omega)-\alpha-\beta\right]-\omega\Delta\lambda\left[\Phi(\lambda)-\alpha\right]\\
&-\omega\widetilde{\omega}v^2(\lambda)\left(\frac{1}{c}-\widetilde{\rho}\right)\\
&\le M-m\left(\frac{1}{c}-\widetilde{\rho}\right)\le 0 \ \text{ in } [\bar{\omega},\widetilde{\omega}]\times[\widetilde{\lambda},\bar{\lambda}]
\end{aligned}
\tag{4.53}
$$

For any c satisfying (4.43) which finishes the proof. □

The next proposition provides the most straightforward result.

Proposition 4.2. *Take* $(\omega_0,\lambda_0)=(\omega,\lambda)\in R_1$. *Then,* $\mathbb{P}\left[\tau_8(\omega,\lambda)<\tau_7(\omega,\lambda)\right]=0$. *Similarly, for* $(\omega_0,\lambda_0)=(\omega,\lambda)\in R_5$, $\mathbb{P}\left[\tau_4(\omega,\lambda)<\tau_3(\omega,\lambda)\right]=0$.

Proof. This is a direct consequence of the absence of Brownian motion in ρ. Take $(\omega_0,\lambda_0)\in R_1$. Then on $[0,\tau_3(\omega_0,\lambda_0)]$, the process ρ is non increasing $\mathbb{P}-$a.s., meaning that R_8 cannot be reached without first crossing region R_7. The other side is similar.

□

Remark 4.6. Note that the whole proof works even if $\tau_i = +\infty$ for any i involved. Notice also that Proposition 4.2 implies that if $(\omega_0, \lambda_0) \in R_1$, then

$$\tau_2(\omega_0, \lambda_0) < \tau_3(\omega_0, \lambda_0) < \tau_4(\omega_0, \lambda_0) < \tau_5(\omega_0, \lambda_0) \ \mathbb{P} - \text{a.s.} \tag{4.54}$$

and that if $(\omega_0, \lambda_0) \in R_5$, $\tau_6(\omega_0, \lambda_0) < \tau_7(\omega_0, \lambda_0) < \tau_8(\omega_0, \lambda_0) < \tau_1(\omega_0, \lambda_0)$ $\mathbb{P} -$ a.s.. As well, observe that the definition of a stochastic orbit was inspired by this result.

In the following proposition, we use results of sections 3.7 and 3.8 in [Kha12] on recurrent domains.

Proposition 4.3. *Take* $(\omega_0, \lambda_0) \in R_1$. *Then* $\mathbb{P}\left[\tau_2(\omega_0, \lambda_0) < +\infty\right] = 1$.

Proof. To show that $\tau_2 < +\infty$ $\mathbb{P} -$ a.s., we apply Theorem A.3 to the function $\sqrt{\lambda}$, for which we have $\sqrt{\lambda} \geq \sqrt{\widetilde{\lambda}} > 0$, and

$$\begin{aligned} \mathcal{L}\sqrt{\lambda} = \frac{1}{2}\sqrt{\lambda}\left[\mu(\omega) - \alpha - \beta + \frac{3}{4}v^2(\lambda)\right] \leq -\frac{\sqrt{\lambda}v^2(\widetilde{\lambda})}{8} \\ \leq -\frac{\sqrt{\widetilde{\lambda}}v^2(\widetilde{\lambda})}{8} < 0 \quad \forall \omega \geq \widetilde{\omega}, \lambda \geq \widetilde{\lambda} \end{aligned} \tag{4.55}$$

The theorem guarantees that (ω_t, λ_t) leaves R_1 in finite time, and this is only possible via R_2. □

Proposition 4.4. *Take* $(\omega_0, \lambda_0) \in R_2 \cup R_3$. *Then* $\mathbb{P}\left[\tau_1(\omega_0, \lambda_0) \wedge \tau_4(\omega_0, \lambda_0) < +\infty\right] = 1$.

Proof. Applying Itô's formula to $\sqrt{\omega}$, we find that

$$\begin{aligned} \mathcal{L}\sqrt{\omega} = \frac{1}{2}\sqrt{\omega}\left[\Phi(\lambda) - \alpha + \frac{3}{4}v^2(\lambda)\right] \leq -\frac{\sqrt{\omega}v^2(\lambda)}{8} \\ \leq -\frac{\sqrt{\widetilde{\omega}}v^2(\widetilde{\lambda})}{8} < 0 \quad \forall \omega \geq \widetilde{\omega}, \lambda \leq \widetilde{\lambda} \end{aligned} \tag{4.56}$$

Resorting to Theorem A.3 once more, we obtain the desired result that the region

$R_2 \cup R_3$ is exited in finite time $\mathbb{P}-$ a.s., which translates into the statement of this proposition. $\square$

The next couple of propositions deal with the opposite situation, concerning regions on the left of $\lambda/\omega = \widetilde{\rho}$.

Proposition 4.5. *Take* $(\omega_0, \lambda_0) \in R_5$. *Then* $\mathbb{P}\left[\tau_6(\omega_0, \lambda_0) < +\infty\right] = 1$.

Proof. Define the sequence of regions $\{B_n\}_{n\in\mathbb{N}}$ through

$$B_n = R_5 \cap \{\lambda > k/n\} \tag{4.57}$$

where $k > 0$ is small enough such that $(\omega_0, \lambda_0) \in B_1$. Applying Itô's formula to the function $\sqrt{\widetilde{\lambda} - \lambda}$ gives us

$$\begin{aligned}
\mathcal{L}(\sqrt{\widetilde{\lambda} - \lambda}) &= -\frac{1}{2\sqrt{\widetilde{\lambda} - \lambda}}\left[\lambda\left[\mu(\omega) - \alpha - \beta + v^2(\lambda)\right] + \frac{1}{4}\frac{\lambda^2}{\widetilde{\lambda} - \lambda}v^2(\lambda)\right] \\
&\le -\frac{1}{8}\frac{\lambda^2}{(\widetilde{\lambda} - \lambda)^{3/2}}v^2(\lambda) \\
&\le -\frac{1}{8}\frac{k^2}{\sqrt{n}\left(n\widetilde{\lambda} - k\right)^{3/2}}v^2(\widetilde{\lambda}) < 0 \quad \forall(\omega, \lambda) \in B_n
\end{aligned} \tag{4.58}$$

while $\mathcal{L}(\sqrt{\widetilde{\lambda} - \lambda}) \le 0$ in R_5. DMCT implies that there exists the point-wise limit

$$\lim_t \left(\sqrt{\widetilde{\lambda} - \lambda}\right)_{t\wedge\tau_6}(\xi) =: \left(\sqrt{\widetilde{\lambda} - \lambda}\right)_{\infty}(\xi) \tag{4.59}$$

for all $\xi \in \Omega$. In addition, Theorem A.3 guarantees that every set B_n is exited in finite time $\mathbb{P}-$ a.s.. As consequence, if $\xi \in \{\tau_5 = +\infty\}$, we have that $\lambda_(\xi) \to 0$, and consequently $\omega_t(\xi) \to 0$ as well. As Theorem 4.2 states, however, this is a contradiction to the very existence of the solution. $\square$

Proposition 4.6. *If* $(\omega_0, \lambda_0) \in R_6 \cup R_7$, *then* $\mathbb{P}\left[\tau_5(\omega_0, \lambda_0) \wedge \tau_8(\omega_0, \lambda_0) < +\infty\right] = 1$.

Proof. Define the sequence of regions $\{B_n\}_{n\in\mathbb{N}}$ through

$$B_n = R_6 \cup R_7 \cap \{\lambda < 1 - k/n\} \cap \{\omega > k/n\} \tag{4.60}$$

where $k > 0$ is small enough such that $(\omega_0, \lambda_0) \in B_1$. Applying Itô's formula to $\sqrt{\widetilde{\omega} - \omega}$, we find that

$$\begin{aligned} \mathcal{L}(\sqrt{\widetilde{\omega} - \omega}) &= -\frac{1}{2\sqrt{\widetilde{\omega} - \omega}} \left[\omega \left[\Phi(\lambda) - \alpha + v^2(\lambda) \right] + \frac{1}{4} \frac{\omega^2}{\widetilde{\omega} - \omega} v^2(\lambda) \right] \\ &\leq -\frac{1}{8} \frac{\omega^2}{(\widetilde{\omega} - \omega)^{3/2}} v^2(\lambda) \\ &\leq -\frac{1}{8} \frac{k^2}{\sqrt{n}(n\widetilde{\omega} - k)^{3/2}} v^2(1 - k/n) < 0 \quad \forall (\omega, \lambda) \in B_n \end{aligned} \tag{4.61}$$

while $\mathcal{L}(\sqrt{\widetilde{\omega} - \omega}) \leq 0$ in $R_6 \cup R_7$. DMCT implies that there exists the point-wise limit

$$\lim_t \left(\sqrt{\widetilde{\omega} - \omega} \right)_{t \wedge \tau_{5,8}} (\xi) =: \left(\sqrt{\widetilde{\omega} - \omega} \right)_{\infty} (\xi) \tag{4.62}$$

for all $\xi \in \Omega$, where we use the notation $\tau_{5,8} = \tau_5(\omega_0, \lambda_0) \wedge \tau_8(\omega_0, \lambda_0)$. In addition, Theorem A.3 guarantees that every set B_n is exited in finite time $\mathbb{P}-$a.s.. As consequence, if $\xi \in \{\tau_{5,8} = +\infty\}$, we have that either $\omega_t(\xi) \to 0$ or $\lambda_t(\xi) \to 1$. Either way, according to Theorem 4.2, we have a contradiction. □

Remark 4.7. With exactly the same argument, yet applied to the function $\sqrt{\rho^{-1} - \omega}$, it is possible to show that the region $R_6 \cup R_7 \cup R_8$ is exited in finite time $\mathbb{P}-$a.s., which implies that $\tau_1 \wedge \tau_5 < +\infty \, \mathbb{P}-$a.s..

So far we have obtained that once in R_1 (or rather in R_5), one must exit to R_2 (R_6), and subsequently move to $R_1 \cup R_4$ (alternatively, $R_5 \cup R_8$) in finite time $\mathbb{P}-$a.s.. Hence, we cannot conclude thus far that the region R_4 (R_8, respectively) will be reached almost surely in finite time. To remediate this situation, we provide a direct proof of these statements inspired by ideas present in the proof of Theorem 3.9 of

[Kha12]. The key insight is that if the process alternates between regions R_1 and R_2 (or R_5 and R_7) indefinitely, then we arrive at a contradiction that defies existence.

Proposition 4.7. *Take* $(\omega_0, \lambda_0) \in R_1 \cup R_2$. *Then* $\mathbb{P}[\tau_3(\omega_0, \lambda_0) < +\infty] = 1$.

Proof. **1.** Let $(v_n)_{n\geq 0}$ be a sequence of stopping times defined by $v_0 = 0$ and

$$v_n := \inf\{t \geq v_{n-1} \ : \ \lambda_t = \widetilde{\lambda} \text{ or } (\omega_t, \lambda_t) \in R_3\}, \quad n \geq 1. \tag{4.63}$$

By construction, if for some $n \geq 1$,$(\omega_{v_n}, \lambda_{v_n}) \in R_3$, then $v_k = v_n$ for all $k > n$.

Following Remark 4.6, Proposition 4.3 and Proposition 4.4, v_n is almost surely finite for all $n \geq 1$ and $\{\tau_3(\omega_0, \lambda_0) = +\infty\} \subset \bigcap_{n\geq 1}\{\lambda_{v_n} = \widetilde{\lambda}\}$.

We make the following claim

$$\lim_{t\to\infty} \rho_t(\xi) = 0, \quad \forall \xi \in \{\tau_3(\omega_0, \lambda_0) = +\infty\} \tag{4.64}$$

to be proved in step 2 below. Assuming this holds, we immediately get

$$\mathbb{P}[\tau_3(\omega_0, \lambda_0) = +\infty] \leq \mathbb{P}\left[\lim_n \omega_{v_n} = +\infty\right] = 0 \tag{4.65}$$

2. Our goal is to show that for all $\xi \in \{\tau_3(\omega_0, \lambda_0) = +\infty\}$, $\rho_t(\xi)$ converges to 0 (path wise). From DMCT on the non-negative super martingale $\rho_{t\wedge\tau_3}$, we know that there exists the random variable $\rho_\infty(\xi) = \lim_{t\to+\infty} \rho_t(\xi)$ for $\xi \in \{\tau_3(\omega_0, \lambda_0) = +\infty\}$. Assume by contradiction that $\rho_\infty(\xi) > 0$ for some ξ in a subset $E \subset \{\tau_3(\omega_0, \lambda_0) = +\infty\}$.

Define the $\mathcal{F}_\infty$-measurable random variable

$$\lim_t \int_0^t \mathbb{1}_{\{\mu(\omega_s(\xi))-\beta-\Phi(\lambda_s(\xi))<-\varepsilon\}}\, ds = C_\varepsilon(\xi) \tag{4.66}$$

which measures the amount of time the process (ω_t, λ_t) spends further than ε away from the boundary $R_2 \cap R_3$. Observe that we can control the growth of ρ_t through

$$d \log \rho_t(\xi) = (\mu(\omega_t(\xi)) - \beta - \Phi(\lambda_t(\xi)))\, dt < -\varepsilon \mathbb{1}_{\{\mu(\omega_t(\xi)) - \beta - \Phi(\lambda_t(\xi)) < -\varepsilon\}} dt \tag{4.67}$$

which gives us the following

$$\log \rho_t(\xi) < \log(\rho_0) - \varepsilon \int_0^t \mathbb{1}_{\{\mu(\omega_s(\xi)) - \beta - \Phi(\lambda_s(\xi)) < -\varepsilon\}}\, ds \tag{4.68}$$

In other words, for all $\xi \in E$, we have

$$C_\varepsilon(\xi) < \frac{\log(\rho_0) - \log(\rho_\infty(\xi))}{\varepsilon} < +\infty \tag{4.69}$$

Define now the following random times[2]

$$\zeta_{\varepsilon,n} := \inf \left\{ t \geq 0 \ : \ \int_0^t \mathbb{1}_{\{\mu(\omega_s(\xi)) - \beta - \Phi(\lambda_s(\xi)) < -\varepsilon\}} ds \geq C_\varepsilon(\xi) - \frac{1}{n} \right\} \tag{4.70}$$

together with the subindices $k_n := \inf\{m \ : \ v_m \geq \zeta_{\varepsilon,n}\}$ for $n = 1, 2, \cdots$.

By definition, once $\zeta_{\varepsilon,n}$ has elapsed, the process (ω_t, λ_t) cannot be arbitrarily away from the boundary $R_2 \cap R_3$ for more than $1/n$ time units. Consequently, there must exist yet another sequence of random times $s_n \in (v_{k_n}, v_{k_n} + 1/n)$ such that

$$-\varepsilon < \mu(\omega_{s_n}) - \beta - \Phi(\lambda_{s_n}) < 0 \tag{4.71}$$

This immediately implies that $\lim_n (s_n - v_{k_n})(\xi) = 0$ for all $\xi \in E$. Reminding ourselves that $\lambda_{v_{k_n}} = \widetilde{\lambda}$ in E, and invoking the continuity of the process $(\lambda_t)_{t \geq 0}$, we have

$$\lim_n \lambda_{s_n}(\xi) = \widetilde{\lambda}, \quad \forall \xi \in E \subset \{\tau_3(\omega_0, \lambda_0) = +\infty\} \tag{4.72}$$

This contradicts (4.71), if we choose ε sufficiently small, proving the desired result

[2]We refrain from using the nomenclature stopping time, as it is not applicable here. Indeed, these random times are not $\mathcal{F}$-adapted.

that E is a $\mathbb{P}$-null set. $\square$

Proposition 4.8. *Take* $(\omega_0, \lambda_0) \in R_5 \cup R_6$. *Then* $\mathbb{P}[\tau_7(\omega_0, \lambda_0) < +\infty] = 1$.

Proof. **1.** Define the sequence of regions $\{B_n\}_{n\in\mathbb{N}}$ through

$$B_n = R_5 \cup R_6 \cap \{\Phi(\lambda) + \beta - \mu(\omega) < -k/n\} \cap \{\rho < kn\} \tag{4.73}$$

where $k > 0$ big enough such that $(\omega_0, \lambda_0) \in B_1$. One can verify that the Dynkin operator applied to the function ω/λ satisfies

$$\begin{aligned} \mathcal{L}(\omega/\lambda) &= (\omega/\lambda)\left[\Phi(\lambda) - \beta - \mu(\omega)\right] \\ &\leq -\frac{1}{n^2} \quad \forall(\omega, \lambda) \in B_n \end{aligned} \tag{4.74}$$

while $\mathcal{L}(\omega/\lambda) \leq 0$ in $R_5 \cup R_6$. DMCT implies that there exists the point-wise limit

$$\lim_t (\omega/\lambda)_{t\wedge\tau_7}(\xi) =: (\omega/\lambda)_\infty(\xi) \tag{4.75}$$

for every $\xi \in \Omega$. Moreover, Theorem A.3 guarantees that every set B_n is exited in finite time $\mathbb{P}$ – a.s.. As consequence, if $\xi \in \{\tau_7(\omega_0, \lambda_0) = +\infty\}$, we have that either $(\omega_t, \lambda_t)(\xi)$ converges to the set $(R_6 \cap R_7) \bigcup \{0\} \times (0, \Phi^{-1}(\mu(0) - \beta))$, while the ratio $(\omega_t/\lambda_t)(\xi)$ tends to $(\omega/\lambda)_\infty(\xi)$. In other words, the solution $(\omega_t, \lambda_t)(\xi)$ converges either to some point in $R_6 \cap R_7$ (which contradicts Theorem 4.2), or to the segment $\{0\} \times (0, \Phi^{-1}(\mu(0) - \beta))$, in which case $\omega_t(\xi) \to 0$. $\square$

Alternative proof. Let $(v_n)_{n\geq 0}$ be a sequence of stopping times defined by $v_0 = 0$ and

$$v_n := \inf\{t \geq v_{n-1} \;:\; \lambda_t = \widetilde{\lambda} \text{ or } (\omega_t, \lambda_t) \in R_7\}, \quad n \geq 1. \tag{4.76}$$

By construction, if for some $n \geq 1$,$(\omega_{v_n}, \lambda_{v_n}) \in R_7$, then $v_k = v_n$ for all $k > n$.

Following Remark 4.6, Proposition 4.5 and Proposition 4.6, v_n is almost surely finite for all $n \geq 1$ and $\{\tau_7(\omega_0, \lambda_0) = +\infty\} \subset \bigcap_{n\geq 1}\{\lambda_{v_n} = \widetilde{\lambda}\}$.

We make the following claim

$$\lim_{t\to\infty} \rho_t(\xi) = +\infty, \quad \forall \xi \in \{\tau_7(\omega_0, \lambda_0) = +\infty\} \tag{4.77}$$

to be proved in step 2 below.

Assuming this holds, we directly obtain

$$\mathbb{P}\left[\tau_7(\omega_0, \lambda_0) = +\infty\right] \le \mathbb{P}\left[\lim_n \omega_{v_n} = 0\right] = 0 \tag{4.78}$$

2. Our goal is to show that for all $\xi \in \{\tau_7(\omega_0, \lambda_0) = +\infty\}$, $\rho_t^{-1}(\xi)$ converges to 0 (path-wise). From DMCT on the non-negative super martingale $\rho_{t\wedge\tau_7}^{-1}$, we know that there exists the random variable $\rho_\infty^{-1}(\xi) = \lim_{t\to+\infty} \rho_t^{-1}(\xi)$ for $\xi \in \{\tau_7(\omega_0, \lambda_0) = +\infty\}$. Assume by contradiction that $\rho_\infty^{-1}(\xi) > 0$ for some ξ in a subset $E \subset \{\tau_7(\omega_0, \lambda_0) = +\infty\}$.

Define the $\mathcal{F}_\infty$-measurable random variable

$$\lim_t \int_0^t \mathbb{1}_{\{\mu(\omega_s(\xi))-\beta-\Phi(\lambda_s(\xi))>\varepsilon\}}\, ds = C_\varepsilon(\xi) \tag{4.79}$$

which measures the amount of time the process (ω_t, λ_t) spends further than ε away from the boundary $R_6 \cap R_7$. Observe that we can control the growth of ρ_t through

$$d \log \rho_t^{-1}(\xi) = -\left(\mu(\omega_t(\xi)) - \beta - \Phi(\lambda_t(\xi))\right) dt < -\varepsilon \mathbb{1}_{\{\mu(\omega_t(\xi))-\beta-\Phi(\lambda_t(\xi))>\varepsilon\}} dt \tag{4.80}$$

from where we can derive the following

$$\log \rho_t^{-1}(\xi) < \log(\rho_0^{-1}) - \varepsilon \int_0^t \mathbb{1}_{\{\mu(\omega_s(\xi))-\beta-\Phi(\lambda_s(\xi))>\varepsilon\}}\, ds \tag{4.81}$$

In other words, for all $\xi \in E$, we have

$$C_\varepsilon(\xi) < \frac{\log(\rho_0^{-1}) - \log(\rho_\infty^{-1}(\xi))}{\varepsilon} < +\infty \tag{4.82}$$

Define now the following random times

$$\zeta_{\varepsilon,n} := \inf\left\{ t \geq 0 \ : \ \int_0^t \mathbb{1}_{\{\mu(\omega_s(\xi))-\beta-\Phi(\lambda_s(\xi))>\varepsilon\}} ds \geq C_\varepsilon(\xi) - \frac{1}{n} \right\} \tag{4.83}$$

together with the subindices $k_n := \inf\{m \ : \ v_m \geq \zeta_{\varepsilon,n}\}$ for $n = 1, 2, \cdots$.

By definition, once $\zeta_{\varepsilon,n}$ has elapsed, the process (ω_t, λ_t) cannot be arbitrarily away from the boundary $R_6 \cap R_7$ for more than $1/n$ time units. Consequently, there must exist yet another sequence of random times $s_n \in (v_{k_n}, v_{k_n} + 1/n)$ such that

$$0 < \mu(\omega_{s_n}) - \beta - \Phi(\lambda_{s_n}) < \varepsilon \tag{4.84}$$

This immediately implies that $\lim_n (s_n - v_{k_n})(\xi) = 0$ for all $\xi \in E$. Reminding ourselves that $\lambda_{v_{k_n}} = \widetilde{\lambda}$ in E, and invoking the continuity of the process $(\lambda_t)_{t\geq 0}$, we have

$$\lim_n \lambda_{s_n}(\xi) = \widetilde{\lambda}, \quad \forall \xi \in E \subset \{\tau_7(\omega_0, \lambda_0) = +\infty\} \tag{4.85}$$

This contradicts (4.84), if we choose ε sufficiently small, proving the desired result that E is a $\mathbb{P}$-null set. □

The next proposition addresses the transition from the boundary $\Phi(\lambda) = \mu(\omega) - \beta$ to the regions R_4 and R_8, and relies on a different proof for recurrence. The idea in the following is to provide a lower bound for the probability to reach the desired region R_4 (or R_8) given a specific locus for the starting point.

Proposition 4.9. *Take* $(\omega_0, \lambda_0) \in R_2 \cap R_3$. *Then* $\mathbb{P}[\tau_4(\omega_0, \lambda_0) < +\infty] = 1$. *Similarly, if* $(\omega_0, \lambda_0) \in R_6 \cap R_7$, *we can conclude that* $\mathbb{P}[\tau_8(\omega_0, \lambda_0) < +\infty] = 1$.

Proof. **1.** The proof for both statements are fairly similar. We will design the arguments together, pointing out where the differences lie. Consider the process $F_c \circ \varrho(\omega_t, \lambda_t)$, as defined in (4.46), which, according to Proposition 4.1, is a super-martingale in a set containing $\bigcup_{i=1}^3 R_i \cup \bigcup_{i=5}^7 R_i$ for some appropriate c.

Denoting $\tau_{1,4} := \tau_1(\omega_0, \lambda_0) \wedge \tau_4(\omega_0, \lambda_0)$ (alternatively $\tau_{5,8} := \tau_5(\omega_0, \lambda_0) \wedge \tau_8(\omega_0, \lambda_0)$), we have that $F_c \circ \varrho(\omega_{t\wedge\tau_{1,4}}, \lambda_{t\wedge\tau_{1,4}})$ ($F_c \circ \varrho(\omega_{t\wedge\tau_{5,8}}, \lambda_{t\wedge\tau_{5,8}})$) is still a super-martingale in the previous region. Also, optional sampling theorem, assisted by Proposition 4.4, yields

$$\begin{aligned} F_c(\varrho(\omega_0, \lambda_0)) &\geq \mathbb{E}\left[F_c(\varrho(\omega_{\tau_{1,4}}, \lambda_{\tau_{1,4}}))\right] \\ &= c\mathbb{P}\left[\tau_1(\omega_0, \lambda_0) < \tau_4(\omega_0, \lambda_0)\right] - \frac{1}{c}\mathbb{P}\left[\tau_4(\omega_0, \lambda_0) < \tau_1(\omega_0, \lambda_0)\right] \end{aligned} \tag{4.86}$$

Alternatively,

$$\begin{aligned} F_c(\varrho(\omega_0, \lambda_0)) &\geq \mathbb{E}\left[F_c(\varrho(\omega_{\tau_{5,8}}, \lambda_{\tau_{5,8}}))\right] \\ &= c\mathbb{P}\left[\tau_5(\omega_0, \lambda_0) < \tau_8(\omega_0, \lambda_0)\right] - \frac{1}{c}\mathbb{P}\left[\tau_8(\omega_0, \lambda_0) < \tau_5(\omega_0, \lambda_0)\right] \end{aligned} \tag{4.87}$$

Define $M := \max\{F_c(\varrho(\omega, \lambda)) : (\omega, \lambda) \in (R_2 \cap R_3) \cup (R_6 \cap R_7)\}$, which is negative for c small enough, to obtain a uniform bound

$$\mathbb{P}\left[\tau_4(\omega_0, \lambda_0) < \tau_1(\omega_0, \lambda_0)\right] \geq \frac{c(c-M)}{c^2+1} > 0,\ \forall (\omega, \lambda) \in R_2 \cap R_3 \tag{4.88}$$

Rather,

$$\mathbb{P}\left[\tau_8(\omega_0, \lambda_0) < \tau_5(\omega_0, \lambda_0)\right] \geq \frac{c(c-M)}{c^2+1} > 0,\ \forall (\omega, \lambda) \in R_6 \cap R_7 \tag{4.89}$$

2. According to Propositions 4.4 and 4.6, $\tau_{1,4} < +\infty$ $\mathbb{P}-$a.s. for $(\omega_0, \lambda_0) \in R_2 \cap R_3$, as well as $\tau_{5,8} < +\infty$ $\mathbb{P}-$ a.s. for $(\omega_0, \lambda_0) \in R_6 \cap R_7$. Additionally, Propositions 4.7 and 4.8 state that $\tau_3(\omega, \lambda)$ and $\tau_5(\omega, \lambda)$ are finite almost surely for $(\omega, \lambda) \in R_1$ and $(\omega, \lambda) \in R_5$, respectively. Taking $(\omega_0, \lambda_0) \in R_2 \cap R_3$ (or in $R_6 \cap R_7$), we define the sequence of stopping times $0 = u_0 \leq v_0 \leq u_1 \leq v_1 \leq u_2 \leq v_2 \leq \cdots \leq u_n \leq v_n \cdots$

through

$$\begin{cases} v_n := & \inf\{t \geq u_n \ : \ (\omega_t, \lambda_t) \in R_1 \cup R_4 \cup R_5 \cup R_8\} \\ u_{n+1} := & \inf\{t \geq v_n \ : \ (\omega_t, \lambda_t) \in (R_2 \cap R_3) \cup (R_6 \cap R_7) \text{ or } (\omega_t, \lambda_t) \in R_4 \cup R_8\} \end{cases} \tag{4.90}$$

for $n \geq 1$. By construction, we have that once $\omega_{v_n} = \widetilde{\omega}$, then the process has reached $R_4 \cup R_8$ and $u_m = v_m = v_n$ for all $m > n$. Similarly, if $\omega_{u_n} = \widetilde{\omega}$, then $u_m = v_m = u_n$ for all $m \geq n$. Accordingly, $\{\tau_4(\omega, \lambda) = +\infty\} = \cap_{n\geq 1}\{\omega_{v_n} \neq \widetilde{\omega}\}$ for $(\omega, \lambda) \in R_1 \cup R_2 \cup R_3$. Alternatively, $\{\tau_8(\omega, \lambda) = +\infty\} = \cap_{n\geq 1}\{\omega_{v_n} \neq \widetilde{\omega}\}$ for $(\omega, \lambda) \in R_5 \cup R_6 \cup R_7$. The sequence $(\{\omega_{v_n} \neq \widetilde{\omega}\})_{n\geq 1}$ is decreasing in the sense of inclusion (up to sets of measure zero[3]), so that

$$\begin{aligned} \mathbb{P}\left[\tau_4(\omega, \lambda) = +\infty\right] &= \lim_n \mathbb{P}\left[\omega_{v_n} \neq \widetilde{\omega}\right] \quad \forall (\omega, \lambda) \in R_2 \cap R_3, \text{ or} \\ \mathbb{P}\left[\tau_8(\omega, \lambda) = +\infty\right] &= \lim_n \mathbb{P}\left[\omega_{v_n} \neq \widetilde{\omega}\right] \quad \forall (\omega, \lambda) \in R_6 \cap R_7 \end{aligned} \tag{4.91}$$

Using Bayes' rule, allied with the fact that up to sets of measure zero, $\{\omega_{u_n} \neq \widetilde{\omega}\} \subset \{\omega_{v_{n-1}} \neq \widetilde{\omega}\}$, we have

$$\mathbb{P}\left[\omega_{v_n} \neq \widetilde{\omega}\right] = \mathbb{P}\left[\omega_{v_0} \neq \widetilde{\omega}\right] \prod_{k=1}^{n} \mathbb{P}\left[\omega_{v_k} \neq \widetilde{\omega} | \omega_{v_{k-1}} \neq \widetilde{\omega}\right] \leq \prod_{k=1}^{n} \mathbb{P}\left[\omega_{v_k} \neq \widetilde{\omega} | \omega_{u_k} \neq \widetilde{\omega}\right] \tag{4.92}$$

[3]Strictly speaking, $\omega = \widetilde{\omega}$ can also happen if the solution crosses the point $(\widetilde{\omega}, \widetilde{\lambda})$, but that happens with zero probability.

Using step 1 of the present proof,

$$
\begin{aligned}
\mathbb{P}\left[\omega_{v_n} \neq \widetilde{\omega}\right] &\leq \prod_{k=1}^{n} \mathbb{P}\left[\tau_1(\omega_{u_k}, \lambda_{u_k}) < \tau_4(\omega_{u_k}, \lambda_{u_k})\right] \\
&\leq \prod_{k=1}^{n} \left(1 - \frac{c(c-M)}{c^2+1}\right) \forall(\omega_0, \lambda_0) \in R_2 \cap R_3, \text{ or} \\
\mathbb{P}\left[\omega_{v_n} \neq \widetilde{\omega}\right] &\leq \prod_{k=1}^{n} \mathbb{P}\left[\tau_5(\omega_{u_k}, \lambda_{u_k}) < \tau_8(\omega_{u_k}, \lambda_{u_k})\right] \\
&\leq \prod_{k=1}^{n} \left(1 - \frac{c(c-M)}{c^2+1}\right) \forall(\omega_0, \lambda_0) \in R_6 \cap R_7
\end{aligned}
\tag{4.93}
$$

Substituting this back into (4.91), we obtain that $\mathbb{P}\left[\tau_4(\omega_0, \lambda_0) = +\infty\right] \leq \lim_n (1 - \frac{c(c-M)}{c^2+1})^n = 0$ (or alternatively that $\mathbb{P}\left[\tau_8(\omega_0, \lambda_0) < +\infty\right] \leq 0$), which concludes the proof. □

The following couple of propositions finish the loop of transitions. As we unfortunately do not have a uniform bound on the partial transition probabilities to work with, we develop a new technique that uses the weaker notion of convergence in probability to achieve the desired result.

Proposition 4.10. *Take $\omega_0 = \widetilde{\omega}$ and $\lambda_0 < \widetilde{\lambda}$. Then $\mathbb{P}\left[\tau_5(\omega_0, \lambda_0) < +\infty\right] = 1$.*

Proof. **1.** We first claim that $\tau_2(\omega_0, \lambda_0) \wedge \tau_5(\omega_0, \lambda_0)$ is finite almost surely for $(\omega_0, \lambda_0) \in R_3 \cap R_4$. To show this, consider the non-negative process $h(\omega) = \sqrt{\omega}$, with dynamics

$$
\begin{aligned}
dh(\omega) &= \frac{1}{2} h(\omega) \left[\left(\Phi(\lambda) - \alpha + \frac{3}{4} v^2(\lambda)\right) dt + v(\lambda) dW_t\right] \\
&\leq \frac{1}{2} h(\omega) \left[v(\lambda) dW_t - \frac{1}{4} v^2(\widetilde{\lambda}) dt\right] \quad \forall \lambda < \widetilde{\lambda}
\end{aligned}
\tag{4.94}
$$

DMCT implies that the stopped process $h(\omega_{t \wedge \tau_2 \wedge \tau_5})(\xi)$ converges point-wise to some $h_\infty(\xi)$ for every $\xi \in \Omega$.

Moreover, consider $\xi \in \{\tau_2(\omega_0, \lambda_0) \wedge \tau_5(\omega_0, \lambda_0) = +\infty\}$ and, for any small $\varepsilon > 0$, define the region $R_\varepsilon = (\varepsilon, +\infty) \times (0, \widetilde{\lambda}]$. We can further bound the Dynkin operator applied to the function $h(\omega)$ in R_ε by

$$\mathcal{L}h(\omega) \leq -\frac{1}{8}\sqrt{\varepsilon}v^2(\widetilde{\lambda}) \tag{4.95}$$

Using Theorem A.3, we obtain that the region R_ε is exited in finite time $\mathbb{P}-$ a.s.. Since the process starting at (ω_0, λ_0) given by the random realization ξ cannot exit to neither regions R_2 nor R_5 in finite time, we have that there must exist some $t < +\infty$ for which $\omega_t = \varepsilon$ and $\lambda_t < \varepsilon\widetilde{\lambda}/\widetilde{\omega}$.

We now claim that $h_\infty = 0$. To understand why, suppose by contradiction that $h_\infty > 0$. We could then choose $\varepsilon = h_\infty/2$ and obtain that there will always exist some future $t > 0$ for which $\omega_t = \varepsilon < h_\infty$, contradicting the fact that $\lim_t \omega_t = h_\infty$. As consequence, since h is a continuous bijection, and the solution must remain confined to the region $R_3 \cup R_4$, we obtain that $(\omega_t(\xi), \lambda_t(\xi)) \to (0, 0)$. According to Theorem 4.2, this is a contradiction, from which we obtain that $\tau_2(\omega_0, \lambda_0) \wedge \tau_5(\omega_0, \lambda_0) < +\infty \quad \mathbb{P}-$ a.s..

2. Next, we show that if it is possible that solutions never make it to R_5, then we can successfully establish the convergence of a certain transition probability.

If we take $(\omega_0, \lambda_0) \in R_2 \cap R_3$, then Proposition 4.9 guarantees that the solution reaches R_4 in finite time $\mathbb{P}-$ a.s.. On the other hand, if $(\omega_0, \lambda_0) \in R_3 \cap R_4$, by step 1 we obtain that the process leaves $R_3 \cup R_4$ in finite time $\mathbb{P}-$ a.s.. With this in mind, we define the following sequence of stopping times $0 = u_0 \leq v_0 \leq u_1 \leq v_1 \leq u_2 \leq v_2 \leq \cdots$ through

$$\begin{cases} u_{n+1} := & \inf\{t \geq v_n \;:\; \omega_t = \widetilde{\omega} \text{ or } (\omega_t, \lambda_t) \in R_5\} \\ v_n := & \inf\{t \geq u_n \;:\; (\omega_t, \lambda_t) \in R_2 \cup R_5\} \end{cases}, \ \forall n \geq 0 \tag{4.96}$$

By step 1 and Proposition 4.9, we have that $\mathbb{P}[u_n < +\infty] = \mathbb{P}[v_n < +\infty] = 1$. Moreover, we have the following chain of relations

$$\{\omega_{v_n} > \widetilde{\omega}\} = \{(\omega_{v_n}, \lambda_{v_n}) \in R_2\} \subset \{\omega_{u_n} = \widetilde{\omega}\} = \{(\omega_{v_{n-1}}, \lambda_{v_{n-1}}) \in R_2\}, \ \forall n \geq 1 \tag{4.97}$$

Therefore, $\{\tau_5(\omega_0, \omega_0) = +\infty\} = \bigcap_{n\geq 0}\{\omega_{v_n} \in R_2\}$, while $(\{\omega_{v_n} \in R_2\})_{n\geq 0}$ is a decreasing sequence of sets in the sense of inclusion. Altogether we get

$$\mathbb{P}\left[\tau_5(\omega_0, \lambda_0) = +\infty\right] = \lim_n \mathbb{P}\left[\omega_{v_n} > \widetilde{\omega}\right] \tag{4.98}$$

Using Bayes' formula and (4.97), we finally obtain for every $n \geq 1$

$$\mathbb{P}\left[\omega_{v_n} > \widetilde{\omega}\right] = \mathbb{P}\left[\omega_{v_0} > \widetilde{\omega}\right] \prod_{k=1}^{n} \mathbb{P}\left[\omega_{v_k} > \widetilde{\omega} | \omega_{v_{k-1}} > \widetilde{\omega}\right] = \mathbb{P}\left[\omega_{v_0} > \widetilde{\omega}\right] \prod_{k=1}^{n} \mathbb{P}\left[\omega_{v_k} > \widetilde{\omega} | \omega_{u_k} = \widetilde{\omega}\right] \tag{4.99}$$

Bringing (4.98) and (4.99) together, $\mathbb{P}\left[\tau_5(\omega_0, \lambda_0) = +\infty\right] > 0$ implies that

$$\lim_n \mathbb{P}\left[\omega_{v_n} > \widetilde{\omega} | \omega_{u_n} = \widetilde{\omega}\right] = 1 \tag{4.100}$$

3. To conclude, we apply Itô's formula to the function $x_t := e^{\frac{1}{8}v^2(\widetilde{\lambda})t}\sqrt{\omega}$, obtaining

$$\begin{aligned} dx_t &= e^{\frac{1}{8}v^2(\widetilde{\lambda})t}\sqrt{\omega}\left[\frac{1}{8}\left(v^2(\widetilde{\lambda}) - v^2(\lambda)\right)dt + \frac{1}{2}\left(\Phi(\lambda) - \alpha + v^2(\lambda)\right)dt + \frac{1}{2}v(\lambda)dW_t\right] \\ &\leq \frac{1}{2}e^{kt}\sqrt{\omega}v(\lambda)dW_t \quad \forall \lambda < \widetilde{\lambda} \end{aligned} \tag{4.101}$$

where $k = \frac{1}{8}v^2(\widetilde{\lambda})$, showing that x_t is a non-negative super-martingale for $t \in [u_n, v_n]$. Hence, Optional Sampling Theorem gives

$$\mathbb{E}\left[x_{t\wedge v_n} | \mathcal{F}_{t\wedge u_n}\right] \leq x_{t\wedge u_n} \tag{4.102}$$

As u_n, v_n are finite almost surely, we have through Fatou's Lemma that

$$\mathbb{E}\left[x_{v_n} | \mathcal{F}_{u_n}\right] = \mathbb{E}\left[\liminf_t x_{t\wedge v_n} | \mathcal{F}_{t\wedge u_n}\right] \leq \liminf_t \mathbb{E}\left[x_{t\wedge v_n} | \mathcal{F}_{t\wedge u_n}\right] \leq \liminf_t x_{t\wedge u_n} = x_{u_n} \tag{4.103}$$

or

$$\mathbb{E}\left[\sqrt{\omega_{v_n}}e^{k(v_n-u_n)}|\mathcal{F}_{u_n}\right] \leq \sqrt{\omega_{u_n}} \tag{4.104}$$

Denote the sigma algebra generated by ω_{u_n} by $\mathcal{G}_{u_n} \subset \mathcal{F}_{u_n}$, so that iterated conditioning yields

$$\begin{aligned}\mathbb{E}\left[\sqrt{\omega_{v_n}}e^{k(v_n-u_n)}|\mathcal{G}_{u_n}\right] &= \mathbb{E}\left[\mathbb{E}\left[\sqrt{\omega_{v_n}}e^{k(v_n-u_n)}|\mathcal{F}_{u_n}\right]|\mathcal{G}_{u_n}\right] \\ &\leq \mathbb{E}\left[\sqrt{\omega_{u_n}}|\mathcal{G}_{u_n}\right] = \sqrt{\omega_{u_n}}\end{aligned} \tag{4.105}$$

and thus by picking $\xi \in \{\omega_{u_n} = \widetilde{\omega}\}$, we find that

$$\mathbb{E}\left[e^{k[v_n-u_n]}\sqrt{\omega_{v_n}}(\mathbb{1}_{\{\omega_{v_n}\leq\widetilde{\omega}\}} + \mathbb{1}_{\{\omega_{v_n}>\widetilde{\omega}\}})\middle|\omega_{u_n} = \widetilde{\omega}\right] \leq \sqrt{\widetilde{\omega}}. \tag{4.106}$$

Since $\sqrt{\omega_{v_n}}\mathbb{1}_{\{\omega_{v_n}\leq\widetilde{\omega}\}} \geq 0$ and $\sqrt{\omega_{v_n}}\mathbb{1}_{\{\omega_{v_n}>\widetilde{\omega}\}} \geq \sqrt{\widetilde{\omega}}\mathbb{1}_{\{\omega(v_n)>\widetilde{\omega}\}}$ $\mathbb{P}-$a.s. for all $n \geq 1$, equation (4.106) implies

$$\mathbb{E}\left[e^{k[v_n-u_n]}\mathbb{1}_{\{\omega_{v_n}>\widetilde{\omega}\}}\middle|\omega_{u_n} = \widetilde{\omega}\right] \leq 1 \tag{4.107}$$

By adding and subtracting one to $e^{k[v_n-u_n]}$, we arrive at

$$\mathbb{E}\left[\left(e^{k[v_n-u_n]} - 1\right)\mathbb{1}_{\{\omega_{v_n}>\widetilde{\omega}\}}\middle|\omega_{u_n} = \widetilde{\omega}\right] \leq 1 - \mathbb{P}\left[\omega_{v_n} > \widetilde{\omega}|\omega_{u_n} = \widetilde{\omega}\right] \tag{4.108}$$

If $\omega_{u_n} = \widetilde{\omega}$ then $\lambda_{u_n} < \widetilde{\lambda}$ and by continuity, $\{v_n > u_n\} \supset \{\omega_{u_n} = \widetilde{\omega}\}$, implying

$$e^{k[v_n(\omega)-u_n(\xi)]} > 1,\ \forall \xi \in \{\omega_{u_n} = \widetilde{\omega}\} \tag{4.109}$$

Assuming that $\mathbb{P}\left[\tau_5(\omega_0, \lambda_0) = +\infty\right] > 0$, we have that (4.100) holds, implying

$$0 \leq \mathbb{E}\left[\left(e^{k[v_n-u_n]} - 1\right)\mathbb{1}_{\{\omega_{v_n}>\widetilde{\omega}\}}\middle|\omega_{u_n} = \widetilde{\omega}\right] \to 0 \text{ as } n \to +\infty \tag{4.110}$$

By the Markov inequality, we can obtain convergence in probability from the convergence in mean above, that is,

$$\mathbb{P}\left[\left(e^{k[v_n-u_n]}-1\right)\mathbb{1}_{\{\omega_{v_n}>\widetilde{\omega}\}}>\varepsilon|\omega_{u_n}=\widetilde{\omega}\right]\to 0 \quad \forall\varepsilon>0 \tag{4.111}$$

Bayes' rule gives us that the LHS above is bigger than or equal to

$$\begin{aligned}&\mathbb{P}\left[\left(e^{k[v_n-u_n]}-1\right)>\varepsilon|\omega_{v_n}>\widetilde{\omega},\omega_{u_n}=\widetilde{\omega}\right]\mathbb{P}\left[\omega_{v_n}>\widetilde{\omega}|\omega_{u_n}=\widetilde{\omega}\right]\\&=\mathbb{P}\left[v_n-u_n>\varepsilon'|\omega_{v_n}>\widetilde{\omega},\omega_{u_n}=\widetilde{\omega}\right]\mathbb{P}\left[\omega_{v_n}>\widetilde{\omega}|\omega_{u_n}=\widetilde{\omega}\right]\end{aligned} \tag{4.112}$$

where $\varepsilon'=k^{-1}\log(1+\varepsilon)$. Making use once more of the result obtained in step 2, we derive the following convergence in probability of the stopping times

$$\mathbb{P}\left[v_n-u_n>\varepsilon|\omega_{v_n}>\widetilde{\omega},\omega_{u_n}=\widetilde{\omega}\right]\to 0 \quad \forall\varepsilon>0 \tag{4.113}$$

Observe now that if $\{\tau_5=+\infty\}=\bigcap_{n\geq 1}\{\omega_{v_n}>\widetilde{\omega}\}\cap\{\omega_{u_n}=\widetilde{\omega}\}$. For $\xi\in\{\tau_5=+\infty\}$, by the continuous mapping theorem, we must have that the solutions at consecutive stopping times converge in probability, and that can only happen if the system overall converges to the point $(\widetilde{\omega},\widetilde{\lambda})$, which is a contradiction according to Theorem 4.2. Thus, we must have that $\mathbb{P}[\tau_5=+\infty]=0$. $\square$

Proposition 4.11. *Take $\omega_0=\widetilde{\omega}$ and $\lambda_0>\widetilde{\lambda}$. Then $\mathbb{P}[\tau_1(\omega_0,\lambda_0)<+\infty]=1$.*

Proof. **1.** First, Remark 4.7 implies that $\tau_1(\omega_0,\lambda_0)\wedge\tau_6(\omega_0,\lambda_0)<+\infty$ almost surely for $(\omega_0,\lambda_0)\in R_7\cup R_8$. That being said, we can define a sequence of almost surely finite stopping times $0=u_0\leq v_0\leq u_1\leq v_1\leq u_2\leq v_2\leq\cdots$ through

$$\begin{cases} v_n & :=\inf\{t\geq u_n\ :\ (\omega_t,\lambda_t)\in R_1\cup R_6\}\\ u_{n+1} & :=\inf\{t\geq v_n\ :\ \omega_t=\widetilde{\omega}\text{ or }(\omega_t,\lambda_t)\in R_1\}\end{cases},\forall n\geq 1 \tag{4.114}$$

satisfying the following

$$\{\omega_{v_n} < \widetilde{\omega}\} = \{(\omega_{v_n}, \lambda_{v_n}) \in R_6\} \subset \{\omega_{u_n} = \widetilde{\omega}\} = \{(\omega_{v_{n-1}}, \lambda_{v_{n-1}}) \in R_6\},\ \forall n \geq 1 \tag{4.115}$$

Meaning that if the process belongs to R_6 at time v_n, then, at the previous time u_n, it must have crossed the line $\omega = \widetilde{\omega}$. In turn, this implies that the process must have been in R_6 previously at v_{n-1} as well.

Therefore, if the region R_1 is never reached, is must be that at every instant v_n, we find ourselves in R_6. In more specific terms, $\{\tau_1(\omega_0, \omega_0) = +\infty\} = \bigcap_{n\geq 0}\{\omega_{v_n} \in R_6\}$, while $(\{\omega_{v_n} \in R_6\})_{n\geq 0}$ is a decreasing sequence of sets in the sense of inclusion. Altogether we get

$$\mathbb{P}\left[\tau_1(\omega_0, \lambda_0) = +\infty\right] = \lim_n \mathbb{P}\left[\omega_{v_n} < \widetilde{\omega}\right] \tag{4.116}$$

Using Bayes' formula and (4.115), we finally obtain for every $n \geq 1$

$$\mathbb{P}\left[\omega_{v_n} < \widetilde{\omega}\right] = \mathbb{P}\left[\omega_{v_0} < \widetilde{\omega}\right] \prod_{k=1}^{n} \mathbb{P}\left[\omega_{v_k} < \widetilde{\omega} | \omega_{v_{k-1}} < \widetilde{\omega}\right] = \mathbb{P}\left[\omega_{v_0} < \widetilde{\omega}\right] \prod_{k=1}^{n} \mathbb{P}\left[\omega_{v_k} < \widetilde{\omega} | \omega_{u_k} = \widetilde{\omega}\right] \tag{4.117}$$

Bringing (4.116) and (4.117) together, $\mathbb{P}\left[\tau_1(\omega_0, \lambda_0) = +\infty\right] > 0$ implies that

$$\lim_n \mathbb{P}\left[\omega_{v_n} < \widetilde{\omega} | \omega_{u_n} = \widetilde{\omega}\right] = 1 \tag{4.118}$$

2. To conclude, we apply Itô's formula to ω^2, finding that

$$\mathcal{L}(\omega^2) = 2h(\omega)(\Phi(\lambda) - \alpha + 3v^2(\lambda)/2) \tag{4.119}$$

Since $\Phi(\lambda) - \alpha + v^2(\lambda) \geq 0$ for $\lambda \geq \widetilde{\lambda}$, zero only if $\lambda = \widetilde{\lambda}$, whereas $v^2(\lambda) = 0$ only if $\lambda = 1$, we have that $\mathcal{L}(\omega^2) \geq m\omega^2$ for $\lambda \geq \widetilde{\lambda}$ where

$$m := \inf_{\lambda \in [\widetilde{\lambda}, 1)} 2\left(\Phi(\lambda) - \alpha\right) + 3v^2(\lambda) \tag{4.120}$$

The optional sampling theorem on $x_t := e^{-mt}\omega_t^2$ gives us that

$$\mathbb{E}\left[x_{t\wedge v_n}|\mathcal{F}_{t\wedge u_n}\right] \geq x_{t\wedge u_n} \tag{4.121}$$

Since $x_{t\wedge v_n} \leq \widetilde{\rho}^{-2}$, we can apply the dominated convergence theorem to obtain

$$\mathbb{E}\left[x_{v_n}|\mathcal{F}_{u_n}\right] \geq x_{u_n} \tag{4.122}$$

in other words

$$\mathbb{E}\left[\omega_{v_n}^2 e^{-m(v_n-u_n)}|\mathcal{F}_{u_n}\right] \geq \omega_{u_n}^2 \tag{4.123}$$

Denoting the sigma-algebra generated by ω_{u_n} as $\mathcal{G}_{u_n} \subset \mathcal{F}_{u_n}$, we have by iterated conditioning

$$\begin{aligned}\mathbb{E}\left[\omega_{v_n}^2 e^{-m(v_n-u_n)}|\mathcal{G}_{u_n}\right] &= \mathbb{E}\left[\mathbb{E}\left[\omega_{v_n}^2 e^{-m(v_n-u_n)}|\mathcal{F}_{u_n}\right]|\mathcal{G}_{u_n}\right] \geq \mathbb{E}\left[\omega_{u_n}^2|\mathcal{G}_{u_n}\right] \\ &= \omega_{u_n}^2\end{aligned} \tag{4.124}$$

that is, by choosing $\xi \in \{\omega_{u_n} = \widetilde{\omega}\}$, we have

$$\begin{aligned}\widetilde{\omega}^2 &\leq \mathbb{E}\left[\omega_{v_n}^2 e^{-m(v_n-u_n)}|\omega_{u_n} = \widetilde{\omega}\right] \\ &\leq \widetilde{\omega}^2\mathbb{E}\left[e^{-m(v_n-u_n)}\mathbb{1}_{\{\omega_{v_n}<\widetilde{\omega}\}}|\omega_{u_n} = \widetilde{\omega}\right] + \widetilde{\rho}^{-2}\mathbb{E}\left[\mathbb{1}_{\{\omega_{v_n}\geq\widetilde{\omega}\}}|\omega_{u_n} = \widetilde{\omega}\right]\end{aligned} \tag{4.125}$$

Which implies that

$$\begin{aligned}0 &\leq \mathbb{E}\left[\left(1 - e^{-m(v_n-u_n)}\right)\mathbb{1}_{\{\omega_{v_n}<\widetilde{\omega}\}}|\omega_{u_n} = \widetilde{\omega}\right] \\ &\leq \left(\widetilde{\lambda}^{-1} - 1\right)\left[1 - \mathbb{P}\left[\omega_{v_n} < \widetilde{\omega}|\omega_{u_n} = \widetilde{\omega}\right]\right] \to 0 \quad \text{as } n \to +\infty\end{aligned} \tag{4.126}$$

where we have used the result obtained in step 1. We can derive convergence in probability from the convergence in mean, that is,

$$\mathbb{P}\left[\left(1 - e^{-m(v_n-u_n)}\right)\mathbb{1}_{\{\omega_{v_n}<\widetilde{\omega}\}} > \varepsilon|\omega_{u_n} = \widetilde{\omega}\right] \to 0 \quad \forall\varepsilon > 0 \tag{4.127}$$

Bayes' rule gives us that the LHS above is bigger than or equal to

$$\begin{aligned}&\mathbb{P}\left[\left(1-e^{-m(v_n-u_n)}\right)>\varepsilon|\omega_{v_n}<\widetilde{\omega},\omega_{u_n}=\widetilde{\omega}\right]\mathbb{P}\left[\omega_{v_n}<\widetilde{\omega}|\omega_{u_n}=\widetilde{\omega}\right]\\&=\mathbb{P}\left[v_n-u_n>\varepsilon'|\omega_{v_n}>\widetilde{\omega},\omega_{u_n}=\widetilde{\omega}\right]\mathbb{P}\left[\omega_{v_n}<\widetilde{\omega}|\omega_{u_n}=\widetilde{\omega}\right]\end{aligned} \tag{4.128}$$

where $\varepsilon' = -m^{-1}\log(1-\varepsilon)$. Applying once more the result obtained in step 1, we derive the following convergence in probability of the stopping times

$$\mathbb{P}\left[v_n-u_n>\varepsilon|\omega_{v_n}<\widetilde{\omega},\omega_{u_n}=\widetilde{\omega}\right]\to 0 \quad \forall\varepsilon>0 \tag{4.129}$$

Observe now that if $\{\tau_1 = +\infty\} = \bigcap_{n\geq 1}\{\omega_{v_n} < \widetilde{\omega}\}\cap\{\omega_{u_n} = \widetilde{\omega}\}$. For $\xi \in \{\tau_1 = +\infty\}$, by the continuous mapping theorem, we must have that the solutions at consecutive stopping times converge in probability, and that can only happen if the system overall converges to the point $(\widetilde{\omega},\widetilde{\lambda})$, which is a contradiction according to Theorem 4.2. Thus, we must have that $\mathbb{P}[\tau_1 = +\infty] = 0$. $\square$

4.3 Small volatility approximation

This section is dedicated to an in-depth investigation of the stochastic Goodwin model when the volatility term is small. Through perturbation theory techniques, we propose an approximation to the solution of (4.2), which can be fully solved analytically. Our hope is that this will shed some light on the model first, before embarking in numerical simulations.

We start by assuming that the volatility function takes small values and can be expressed as $v(\lambda) := \varepsilon\varsigma(\lambda)$ for $\varepsilon > 0$ a small constant. We then look for solutions of system (4.2) of the form

$$\begin{cases}\omega_t = \omega_0(t)+\varepsilon\omega_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\\ \lambda_t = \lambda_0(t)+\varepsilon\lambda_\varepsilon(t)+\mathcal{O}(\varepsilon^2)\end{cases} \tag{4.130}$$

Observe that we can approximate the non-linear functions Φ and $\varsigma(\lambda)$ as

$$\begin{aligned}\Phi(\lambda) &= \Phi(\lambda_0 + \lambda_\varepsilon + \mathcal{O}(\varepsilon^2) \\ &= \Phi(\lambda_0) + \varepsilon\lambda_\varepsilon\Phi'(\lambda_0) + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{4.131}$$

$$\begin{aligned}\varsigma(\lambda) &= \varsigma\left(\lambda_0 + \lambda_\varepsilon + \mathcal{O}(\varepsilon^2)\right) \\ &= \varsigma(\lambda_0) + \varepsilon\lambda_\varepsilon\varsigma'(\lambda_0) + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{4.132}$$

while the function μ is affine, so that

$$\begin{aligned}\mu(\omega) &= \mu(\omega_0) + \varepsilon\omega_\varepsilon\mu'(\omega_0) + \mathcal{O}(\varepsilon^2) \\ &= \mu(\omega_0) - \varepsilon\omega_\varepsilon/\nu + \mathcal{O}(\varepsilon^2)\end{aligned} \tag{4.133}$$

Using Itô's formula in (4.130), we obtain the following equation for $d\omega$

$$\begin{aligned}d(\omega_0 + \varepsilon\omega_\varepsilon) &= (\omega_0 + \varepsilon\omega_\varepsilon)\left[(\Phi(\lambda_0) + \varepsilon\lambda_\varepsilon\Phi'(\lambda_0) - \alpha)\,dt + \varepsilon\varsigma(\lambda_0)dW_t\right] + O(\varepsilon^2) \\ &= \omega_0\left[\Phi(\lambda_0) - \alpha\right]dt + \varepsilon\left\{\left[\omega_\varepsilon\left(\Phi(\lambda_0) - \alpha\right) + \lambda_\varepsilon\omega_0\Phi'(\lambda_0)\right]dt + \omega_0\varsigma(\lambda_0)dW_t\right\} + O(\varepsilon^2)\end{aligned} \tag{4.134}$$

and $d\lambda$

$$\begin{aligned}d(\lambda_0 + \varepsilon\lambda_\varepsilon) &= (\lambda_0 + \varepsilon\lambda_\varepsilon)\left[(\mu(\omega_0) - \alpha - \beta + \varepsilon\omega_\varepsilon\mu'(\omega_0))\,dt + \varepsilon\varsigma(\lambda_0)dW_t\right] + \mathcal{O}(\varepsilon^2) \\ &= \lambda_0\left[\mu(\omega_0) - \alpha - \beta\right]dt + \varepsilon\left\{\left[\lambda_\varepsilon\left(\mu(\omega_0) - \alpha - \beta\right) + \omega_\varepsilon\lambda_0\mu'(\omega_0)\right]dt \right. \\ &\left. +\lambda_0\varsigma(\lambda_0)dW_t\right\} + O(\varepsilon^2)\end{aligned} \tag{4.135}$$

The fundamental theorem of perturbation theory [SMJ98] allows us to group the terms accompanying each power of ε from both sides and match them. Accordingly, one finds that ω_0 and λ_0 solve the deterministic Goodwin model (3.12), while $\omega_\varepsilon, \lambda_\varepsilon$

solve the following system

$$\begin{cases} d\omega_\varepsilon &= \left[\omega_\varepsilon \left(\Phi(\lambda_0) - \alpha\right) + \omega_0 \lambda_\varepsilon \Phi'(\lambda_0)\right] dt + \omega_0 \varsigma(\lambda_0) dW_t \\ d\lambda_\varepsilon &= \left[\lambda_\varepsilon \left(\mu(\omega_0) - \alpha - \beta\right) + \omega_\varepsilon \lambda_0 \mu'(\omega_0)\right] dt + \lambda_0 \varsigma(\lambda_0) dW_t \end{cases} \tag{4.136}$$

which is a linear stochastic dynamical system with periodic coefficients, through the function $\omega_0(t), \lambda_0(t)$. Define $\vec{v}_\varepsilon := [\omega_\varepsilon, \lambda_\varepsilon]^\intercal$, we can rewrite (4.136) in vector form as

$$d\vec{v}_\varepsilon(t) = A(t)\vec{v}_\varepsilon(t)dt + b(t)dW(t) \tag{4.137}$$

where

$$A(t) := \begin{bmatrix} p(t) & \omega_0(t)\Phi'(\lambda_0(t)) \\ \lambda_0(t)\mu'(\omega_0(t)) & \mu(\omega_0(t)) - \alpha - \beta \end{bmatrix} \quad \text{and} \quad b(t) := \begin{bmatrix} \omega_0(t)\varsigma(\lambda_0(t)) \\ \lambda_0(t)\varsigma(\lambda_0(t)) \end{bmatrix}. \tag{4.138}$$

In a later section, we compute the period of a generalized version of the Lotka-Volterra model. Such result, stated as Theorem 6.1, can be used to obtain, in particular, the period of the functions $\omega_0(t)$ and $\lambda_0(t)$, which we shall denote by T. Consequently, we have that $A(t)$ and $b(t)$ are T-periodic as well. The initial conditions are given by $(\omega_0(0), \lambda_0(0)) = (\omega_0, \lambda_0)$, while $(\omega_\varepsilon(0), \lambda_\varepsilon(0)) = (0, 0)$. We will also assume that $(\omega_0, \lambda_0) \neq (\widetilde{\omega}, \widetilde{\lambda})$ for the rest of this section.

The next result provides a closed-form solution to (4.136).

Proposition 4.12. *Let*

$$p(t) := \Phi(\lambda_0(t)) - \alpha \tag{4.139}$$

$$q(t) := \mu(\omega_0(t)) - \alpha - \beta \tag{4.140}$$

$$G(t) := \begin{bmatrix} \omega_0(t) & 0 \\ 0 & \lambda_0(t) \end{bmatrix} \tag{4.141}$$

where the pair $(\omega_0(t), \lambda_0(t))$ *solves the system* (3.12) *with initial condition* $(\omega_0(0), \lambda_0(0)) \neq$

$(\overline{\omega}, \overline{\lambda})$. *The solution* $(\vec{v}_\varepsilon(t))_{t\geq 0}$ *of the system* (4.137) *is then*

$$\begin{aligned}\vec{v}_\varepsilon(t) &= G(t)\Psi(t)\left[G^{-1}(0)\vec{v}_\varepsilon(0) + \int_0^t \Psi^{-1}(s)\mathbf{1}\varsigma(\lambda_0(s))\,dW_s\right] \\ &= G(t)\Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{1}\varsigma(\lambda_0(s))\,dW_s \quad \textit{for } \vec{v}_\varepsilon(0) = [0,0]^\intercal\end{aligned} \tag{4.142}$$

with $\mathbf{1} = [1,1]^\intercal$ *and* I_2 *the identity matrix of* $\mathbb{M}_{2\times 2}(\mathbb{R})$, *and*

$$\Psi(t) := \begin{bmatrix} \frac{p(t)}{p(0)} - q(0)\Gamma_x(t) & p(0)\Gamma_x(t) \\ q(0)\Gamma_y(t) & \frac{q(t)}{q(0)} - p(0)\Gamma_y(t) \end{bmatrix} \tag{4.143}$$

$$\begin{aligned}\Gamma_x(t) &:= p(t)\int_0^t \frac{\lambda_0(s)\Phi'(\lambda_0(s))}{p(s)^2}\,ds \\ \Gamma_y(t)) &:= q(t)\int_0^t \frac{\omega_0(s)\mu'(\omega_0(s))}{q(s)^2}\,ds\end{aligned} \tag{4.144}$$

Proof. For $(\omega_0(0), \lambda_0(0)) \neq (\overline{\omega}, \overline{\lambda})$ in the domain D, we can define the process $\vec{z}(t) := [x(t), y(t)]^\intercal := [\omega_\varepsilon(t)/\omega_0(t), \lambda_\varepsilon(t)/\lambda_0(t)]^\intercal$. Through Itô's formula, we obtain that

$$d\vec{z}(t) = A_z(t)\vec{z}(t)dt + b_z(t)dW(t) \tag{4.145}$$

with

$$A_z(t) := \begin{bmatrix} 0 & \lambda_0(t)\Phi'(\lambda_0(t)) \\ \omega_0(t)\mu'(\omega_0(t)) & 0 \end{bmatrix} \quad \text{and} \quad b_z(t) := \varsigma(\lambda_0(t))\mathbf{1} \tag{4.146}$$

As the above elements are all functions of ω_0 and λ_0, we have that A_z and b_z are both T-periodic. First we claim that we can solve the homogeneous deterministic problem

$$d\vec{z}_H(t) = A_z(t)\vec{z}_H(t)dt \tag{4.147}$$

using the state-density matrix $\Psi(t)$

$$\vec{z}_H(t) = \Psi(t)\vec{z}_H(0) \tag{4.148}$$

To show this, we will build the state-density matrix and show that it can be written as (4.143) . One can easily verify that $\vec{z}_1(t) := [p(t), q(t)]^\intercal$ is a solution of (4.147). Denote by $\vec{z}_2 := [x_2(t), y_2(t)]^\intercal$ another solution, linearly independent from $\vec{z}_1$. A fundamental matrix can be constructed with these two linearly independent solutions. Abel's formula, together with the fact that $tr(A') = 0$, implies that for any $t \geq 0$

$$c := y_2(0)p(0) - x_2(0)q(0) = y_2(t)p(t) - x_2(t)q(t) \tag{4.149}$$

From which we see that if $\lambda_0(t) \neq \overline{\lambda}$ then $x_2(t) = -c\,[\mu(\omega_0(t) - \alpha - \beta]^{-1}$, which is well defined[4]. Conversely, if $\omega \neq \overline{\omega}$, then $y_2(t) = c\,[\Phi(\lambda_0(0) - \alpha]^{-1}$, also well defined. Otherwise, if neither $\lambda_0(t) = \overline{\lambda}$ nor $\omega_0(t) = \overline{\omega}$, we can isolate either $x_2(t)$ or $y_2(t)$ and substitute in (4.147) to obtain uncoupled one-dimensional ODEs

$$\begin{aligned} \frac{dx_2(t)}{dt} &= \frac{\lambda_0(t)\Phi'(\lambda_0(t))}{p(t)}\left(c + x_2(t)q(q)\right) \\ \frac{dy_2(t)}{dt} &= \frac{\omega_0(t)\kappa'(\omega_0(t))}{q(t)}\left(c + y_2(t)p(t)\right) \end{aligned} \tag{4.150}$$

A bit of calculus simplifies it to

$$\begin{aligned} \frac{d}{dt}\frac{x_2(t)}{p(t)} &= c\frac{\lambda_0(t)\Phi'(\lambda_0(t))}{p(t)^2} \\ \frac{d}{dt}\frac{y_2(t)}{q(t)} &= c\frac{\omega_0(t)\mu'(\omega_0(t))}{q(t)^2} \end{aligned} \tag{4.151}$$

Solving these ODEs is trivial, and one immediately finds the solution to (4.147) as

$$\begin{aligned} x_2(t) &= \frac{p(t)}{p(0)}x_2(0) + c\Gamma_x(t) \\ y_2(t) &= \frac{q(t)}{q(0)}y_2(0) + c\Gamma_y(t) \end{aligned} \tag{4.152}$$

[4]From Chapter 3, we know that if $(\omega_0(0), \lambda_0(0)) \neq (\overline{\omega}, \overline{\lambda})$, then $(\omega_0(t), \lambda_0(t))$ will belong to closed orbit that does not contain $(\overline{\omega}, \overline{\lambda})$

Replacing c by its value in terms of $x_2(0)$ and $y_2(0)$ provides $x_2(t)$ and $y_2(t)$ in terms of $x_2(0)$ and $y_2(0)$, which leads to the state-density matrix (4.143). Since $\Psi(t)$ is also a solution to (4.147), we have

$$\frac{d}{dt}\left(\Psi(t)\right)^{-1} = -\Psi^{-1}(t)\frac{d}{dt}\left(\Psi(t)\right)\Psi^{-1}(t) = -\Psi^{-1}(t)A_z(t) \tag{4.153}$$

We can then use Itô's formula on $\Psi^{-1}\vec{z}$ and find

$$\begin{aligned} d\left(\Psi^{-1}\vec{z}\right) &= -\Psi^{-1}A_z\vec{z}dt + \Psi^{-1}\left[A_z\vec{z}dt + b_z dW_t\right] \\ &= \Psi^{-1}b_z dW_t \end{aligned} \tag{4.154}$$

Taking into account that $\Psi(0) = I_2$, the identity matrix, we have the following solution for $\vec{z}(t)$

$$\vec{z}(t) = \Psi(t)\left[\vec{z}(0) + \int_0^t \Psi^{-1}(s)b_z(s)\,dW_s\right] \tag{4.155}$$

which leads us to the desired result below

$$\begin{aligned} \vec{v}_\varepsilon(t) &= G(t)\vec{z}(t) \\ &= G(t)\Psi(t)\left[G^{-1}(0)\vec{v}_\varepsilon(0) + \int_0^t \Psi^{-1}(s)b_z(s)\,dW_s\right] \\ &= G(t)\Psi(t)\int_0^t \Psi^{-1}(s)b_z(s)\,dW_s \quad \text{ for } \vec{v}_\varepsilon(0) = [0,0]^\intercal \end{aligned} \tag{4.156}$$

$\square$

Remark 4.8. Notice that integration by parts on Γ_x and Γ_y yields

$$q(t)\Gamma_x(t) + p(t)\Gamma_y(t) = \frac{q(t)p(t)}{q(0)p(0)} - 1 \tag{4.157}$$

which holds true whenever $p(t) \neq 0$ or $q(t) \neq 0$. As well, by the periodicity of the integrands we see in (6.34), we find

$$\Gamma_x(nT+t) = \Gamma_x(t) + n\frac{p(t)}{p(0)}\Gamma_x(T) \text{ and } \Gamma_y(nT+t) = \Gamma_y(t) + n\frac{q(t)}{q(0)}\Gamma_y(T)\ . \tag{4.158}$$

It is then easy to see that $\det \Psi(t) = 1$, and thus $\Psi^{-1}(t)$ can be determined to be

$$\Psi^{-1}(t) = \begin{bmatrix} \frac{q(t)}{q(0)} - p(0)\Gamma_y(t) & -p(0)\Gamma_x(t) \\ -q(0)\Gamma_y(t) & \frac{p(t)}{p(0)} - q(0)\Gamma_x(t) \end{bmatrix}. \tag{4.159}$$

Moreover, for $t = T$, the period of the zero-order model, we find that

$$q(0)\Gamma_x(T) = -p(0)\Gamma_y(T) =: \bar{\Gamma} \tag{4.160}$$

which leads to the following monodromy matrix

$$\Psi(T) = \begin{bmatrix} 1 - \bar{\Gamma} & \frac{p(0)}{q(0)}\bar{\Gamma} \\ -\frac{q(0)}{p(0)}\bar{\Gamma} & 1 + \bar{\Gamma} \end{bmatrix}. \tag{4.161}$$

with inverse

$$\Psi^{-1}(T) = \begin{bmatrix} 1 + \bar{\Gamma} & -\frac{p(0)}{q(0)}\bar{\Gamma} \\ \frac{q(0)}{p(0)}\bar{\Gamma} & 1 - \bar{\Gamma} \end{bmatrix} \tag{4.162}$$

This monodromy matrix contains only one linearly independent eigenvector,

$$\vec{v}_1 = \begin{bmatrix} p(0) \\ q(0) \end{bmatrix} \tag{4.163}$$

which means that 1 is an eigenvalue with algebraic multiplicity equal to 2, but geometric multiplicity equal to 1. We should then be able to find a generalized eigenvector of grade 2 associated to this eigenvalue. For that, we need to find a vector $\vec{v}_2$ such that $(\Psi(T) - I_2)^2 \vec{v}_2 = 0$, but $(\Psi(T) - I_2) \vec{v}_2 \neq 0$. Since $(\Psi(T) - I_2)^2 = 0I_2$, our task reduces to finding any vector that is not a multiple of $\vec{v}_1$. For simplicity, we can use

$$\vec{v}_2 = \begin{bmatrix} -p(0) \\ 0 \end{bmatrix} \tag{4.164}$$

it is easy to see that $(\Psi(T) - I_2) \vec{v}_2 = \bar{\Gamma}\vec{v}_1$. We have then obtained the Jordan canonical form of the monodromy matrix

$$\Psi(T) = QJQ^{-1} \tag{4.165}$$

where

$$Q := \left[\bar{\Gamma}\vec{v}_1|\vec{v}_2\right] \tag{4.166}$$

$$J := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{4.167}$$

To conclude this remark, we resort to Floquet theory to say that there exists a T-periodic matrix $P(t)$ and a constant matrix R such that

$$\Psi(t) = P(t)e^{Rt} \tag{4.168}$$

which implies that

$$\begin{aligned} \Psi(nT+t) &= P(nT+t)e^{R(nT+t)} = P(t)e^{Rt}e^{nRT} \\ &= \Psi(t)\Psi(T)^n \end{aligned} \tag{4.169}$$

The next result provides the variance of the first-order solution at multiples of the period.

Corollary 4.1. *Define*

$$\Upsilon = Q^{-1}\left(\int_0^T e^{-Ru}P^{-1}(u)\mathbf{1}\mathbf{1}^\intercal P^{-\intercal}(u)e^{-R^\intercal u}\varsigma^2(\lambda_0(u))\,du\right)Q^{-\intercal} \tag{4.170}$$

We then have that the variance of the solution $(\vec{v}_\varepsilon(t))_{t\geq 0}$ *of the system* (4.137) *satisfies*

$$Var\left[\vec{v}_\varepsilon(nT)\right] = G(0)Q\sum_{m=1}^{n}\left\{\begin{bmatrix}1 & m\\ 0 & 1\end{bmatrix}\Upsilon\begin{bmatrix}1 & 0\\ m & 1\end{bmatrix}\right\}Q^{\intercal}G(0) \tag{4.171}$$

Proof. As mentioned in Proposition 4.12, once we apply the appropriate initial condition to the solution (4.142), $\vec{v}_\varepsilon(0) = [0,0]^{\intercal}$, we find the martingale

$$\vec{v}_\varepsilon(t) = G(t)\Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{1}\varsigma(\lambda_0(s))\,dW_s \tag{4.172}$$

with covariance matrix given by Itô Isometry

$$\begin{aligned} Var\left[\vec{v}_\varepsilon(t)\right] &= \mathbb{E}\left[\vec{v}_\varepsilon(t)\vec{v}_\varepsilon(t)^{\intercal}\right]\\ &= G(t)P(t)\int_0^t e^{R(t-s)}P^{-1}(s)M(s)P^{-\intercal}(s)e^{R^{\intercal}(t-s)}\,ds\,P^{\intercal}(t)G(t) \end{aligned} \tag{4.173}$$

where

$$M(t) = \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\varsigma^2(\lambda_0(t)) \tag{4.174}$$

is a T-periodic matrix. We can then study how the variance behaves after multiples of the period have elapsed using the results obtained in Remark 4.8

$$\begin{aligned} Var\left[\vec{v}_\varepsilon(nT)\right] &= G(0)\int_0^{nT} e^{R(nT-s)}P^{-1}(s)M(s)P^{-\intercal}(s)e^{R^{\intercal}(nT-s)}\,ds\,G(0)\\ &= G(0)\sum_{k=0}^{n-1}\left[\int_{kT}^{(k+1)T} e^{R(nT-s)}P^{-1}(s)M(s)P^{-\intercal}(s)e^{R^{\intercal}(nT-s)}\,ds\right]G(0)\\ &= G(0)\sum_{m=1}^{n}\left[e^{mRT}\left(\int_0^T e^{-Ru}P^{-1}(u)M(u)P^{-\intercal}(u)e^{-R^{\intercal}u}\,du\right)e^{mR^{\intercal}T}\right]G(0)\\ &= G(0)Q\sum_{m=1}^{n}\left\{\begin{bmatrix}1 & m\\ 0 & 1\end{bmatrix}\Upsilon\begin{bmatrix}1 & 0\\ m & 1\end{bmatrix}\right\}Q^{\intercal}G(0) \end{aligned} \tag{4.175}$$

□

Corollary 4.1 shows that the variance of the first-order solution grows cubically with respect to the cycles of the Goodwin model, suggesting that the first-order approximation can only be accurate for short time horizons.

4.4 Example

To illustrate the results obtained in this chapter, we choose the functions μ, Φ and v as in (3.9), (3.25) and (4.5), with constants according to (3.24) and

$$\Phi(0) = -0.04 \qquad \Phi^{-1}(\alpha) = 0.80 \tag{4.176}$$

Unlike the numerical example provided in the Chapter 3, we decided to use a smaller value for $\overline{\lambda} = \Phi^{-1}(\alpha)$ for didactic reasons. The qualitative results would be the same if we had calibrated Φ with the previous value of $\overline{\lambda} = 0.96$, yet the figures would seem rather confusing and messy.

The bound on σ given by (4.6) turns out to be $0 \leq \sigma < 0.2550$. We have simulated several sample paths and integrated the solutions using XPPAUT with 4th order Runge-Kutta scheme for the deterministic part, and Euler scheme for the stochastic part. As an illustration on the general behaviour of the model (4.2) under different values of σ ranging from 0.05 to 0.25, we refer to Figure 4.2.

The stochastic orbits, which were analytically studied in the previous section, are well exemplified in Figure 4.2. There is room for improvement, however. A question one might have is how these solutions behave, on average, at the time they cross the line $\lambda = \widetilde{\rho}\omega$. We already know that solutions which start too close to the point $(\widetilde{\omega}, \widetilde{\lambda})$ must inevitable drift away from it. As well, we have that solutions cannot converge to the boundary of the domain D, and that they must loop around $(\widetilde{\omega}, \widetilde{\lambda})$ indefinitely ad infinitum. Resorting to numerical methods, we have simulated the system 2000 times

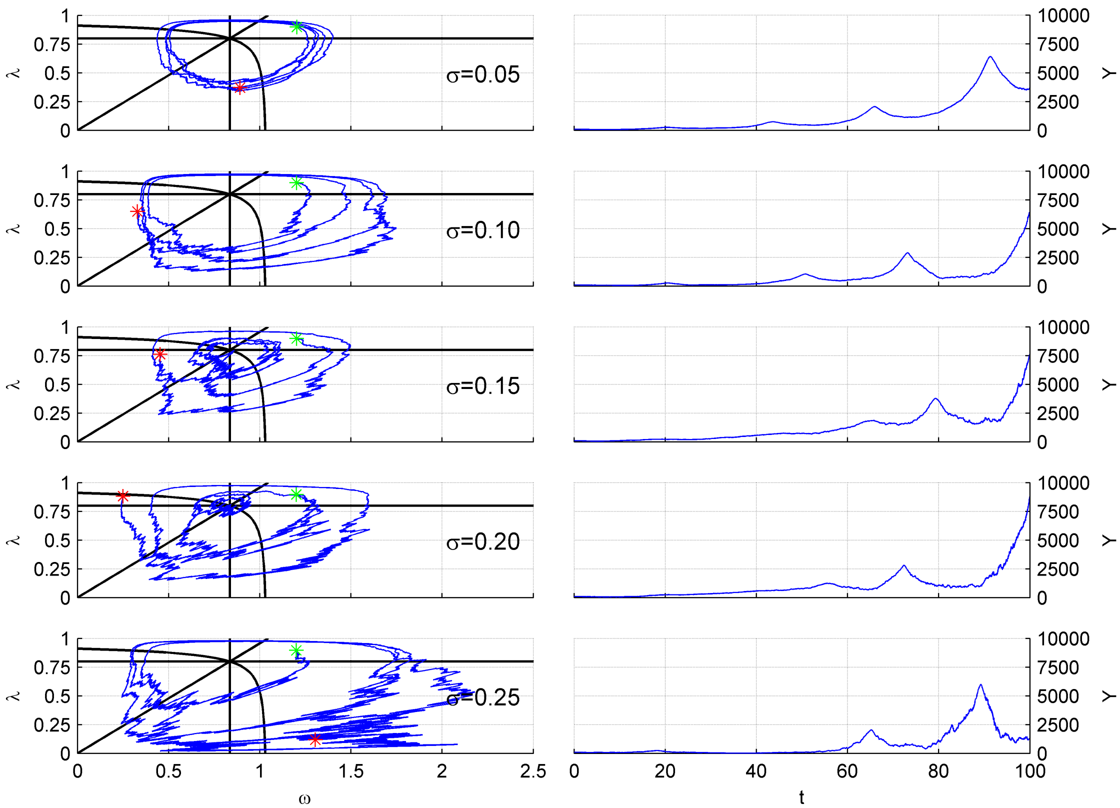

Figure 4.2: Examples of solutions of (4.2) for different values of volatility. On the left column, we have a phase diagram $\omega \times \lambda$, where the green star denotes the initial point, while the red start represents the last point. On the right column, we have the evolution of output Y over time t.

for 100 different starting points lying on the line $(\widetilde{\omega}, \widetilde{\lambda})$ and recorded the position at the time when this line is crossed the second time, that is, the positions after a full loop. Figure 4.3 contains such examination for an array of values of σ. The expected time it takes to complete a full-loop is also illustrated. As observed, there seems to be a stable fixed point in terms of the expected value of the solution after a full loop. If the starting point is picked too close to $(\widetilde{\omega}, \widetilde{\lambda})$, the expected crossing value after one loop is further away from it. On the other hand, if the one starts extremely far away from $(\widetilde{\omega}, \widetilde{\lambda})$, say with $\lambda(0)$ below 0.25, then the expected value after one loop is higher. In expected terms, this indicates that the solution after one loop should

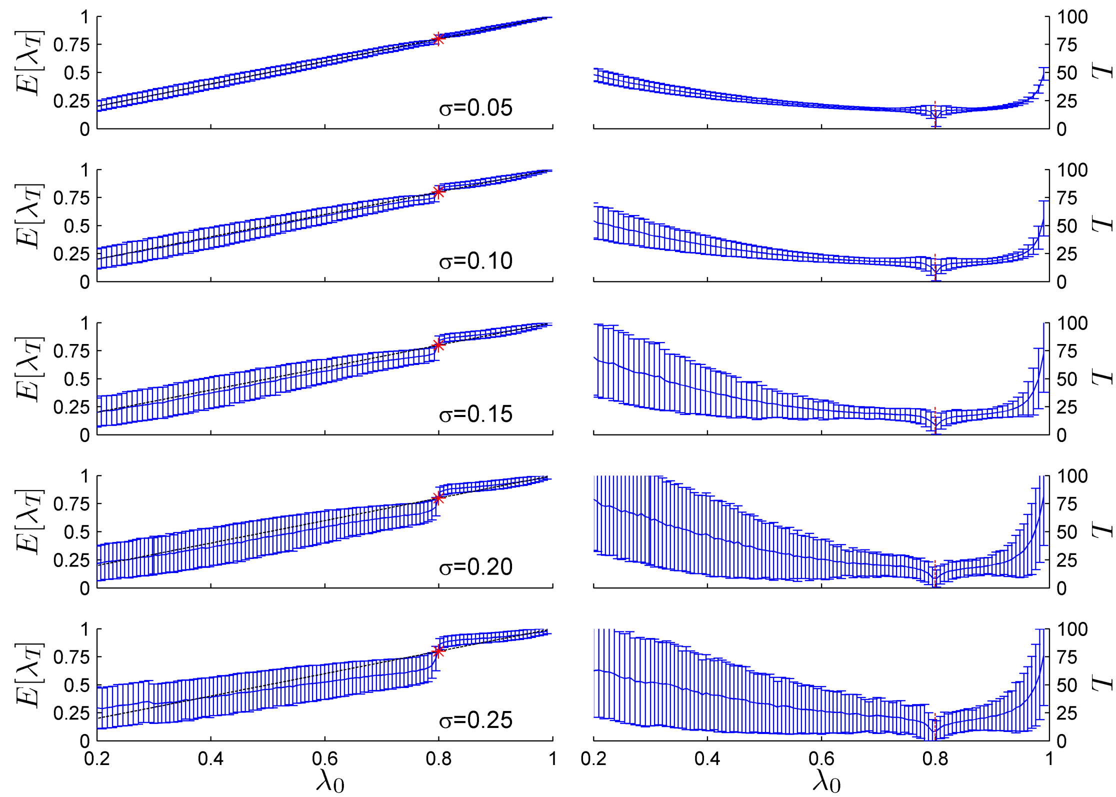

Figure 4.3: Expected values of employment after one full loop λ_T (left), and time elapsed T (right). Computation performed in MATLAB, with 2000 simulations for every value single one of the 100 initial values taken along the line $\lambda = \widetilde{\rho}\omega$.

converge to some value.

For smaller values of the volatility parameter, we can approximate the solution by (4.130) using (4.12). The quality of such approximation has been analyzed through numerical integration. For values of σ ranging from 0.001 to 0.025, we point to Figure 4.4.

4.5 Conclusion

This Chapter accomplishes two goals. First, it extends the Goodwin model (3.12) with a random productivity function. Secondly, and perhaps, more importantly, it

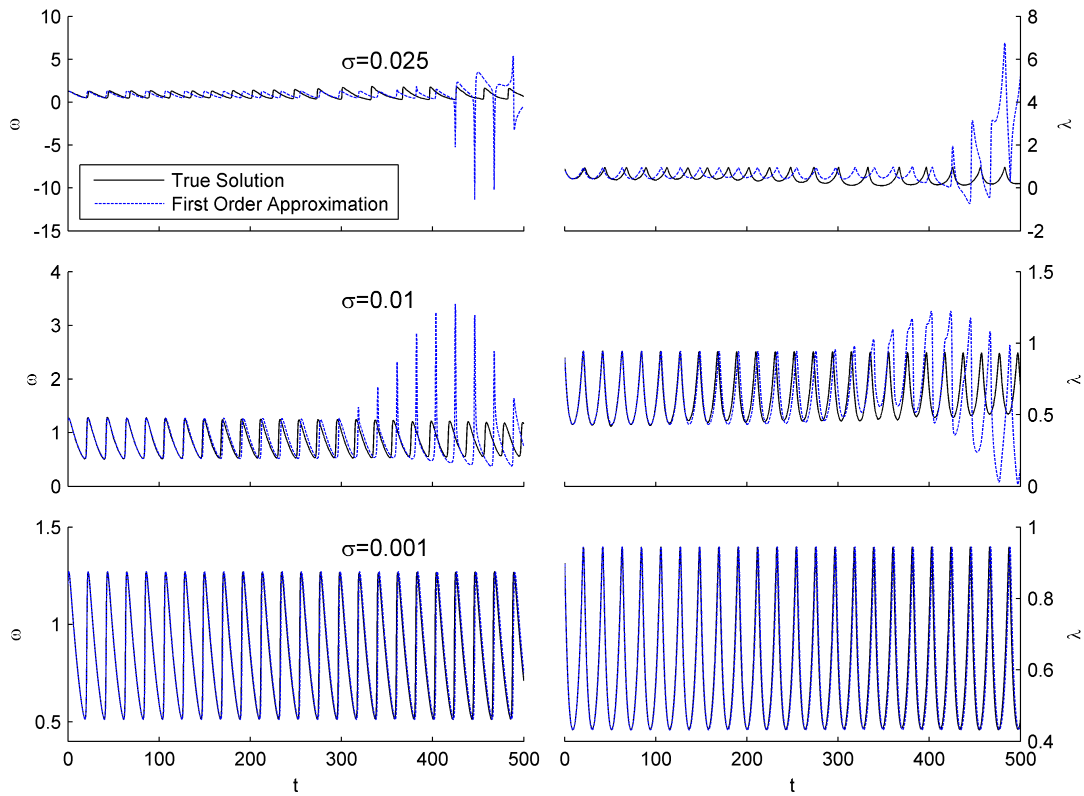

Figure 4.4: Examples of solutions of both (4.2) and (4.130) for different values of volatility. On the left column, we have the evolution of ω versus t, while on the right column we have $\lambda \times t$. The exact solution is drawn in a solid black line, whereas the approximate solution is represented by the dashed blue line.

contains a variety of tools that, we believe, will ultimately be employed when dealing with stochasticity in Minsky models. That being said, we do not exempt ourselves from responsibly proposing the macroeconomical insights that will define the extended model, hence the careful introduction of stochasticity in an intuitive, and plausible manner.

The stochastic extension proposed proves to be a valid one: not only it conserves the desired cyclical behaviour of its predecessor, it enriches it. By allowing the productivity to fluctuate randomly, where the noise decreases with employment, we have created a stochastic model that continuously extends the Goodwin model, yet also introduces some stability in expectation, as verified numerically. More importantly, we have produced several analytical properties, including: a probabilistic bound for the time it takes before the stochastic solution deviates away from the Goodwin model's solution, the almost surely finiteness of the period of stochastic orbits, and a closed-form solution to an approximation valid when the parameter σ is small.

Granted, the macroeconomical knowledge developed in this chapter might not be revolutionary, yet the tools and ideas utilized should be easily extendable to a family of more intriguing models. One immediate extension is to include the banking sector, and study a stochastic version of the Keen model [Kee95], where the stochasticity might arise from, for example, stochastic leverage ratio, credit worthiness, or risk appetite.

Chapter 5

Keen Model

Minsky's Financial Instability Hypothesis, described in numerous essays [Min82], links the expansion of credit with the inherent fragility of the financial system. Minsky provided a verbal prognosis of such hypothesis that follow a sequence of events. At a time when firms and banks are acting conservatively, perhaps due to the fresh memory of a recent crisis, defaults are rare, and moderate levels of debt are perfectly sustainable. Profit thirsty agents realize that they can, and ought to, borrow more to increase their revenue. Lenders and borrowers all around start feeling compelled to expand their balance sheet through credit to meet higher market expectations – euphoria ensues. Eventually, leverage reaches such a high level that any small downturn of the economy will be magnified beyond repair. To deleverage, some investors must sell assets, adding negative pressure to the prices. Suddenly, liquidity has dried out, assets are oversupplied, and euphoria becomes panic.

Despite his use of a persuasive verbal style aided by convincing diagrams and incisive exploration of data, Minsky refrained from presenting his ideas in a formal mathematical setting. This task was taken up by, among others, Keen [Kee95], where a system of differential equations is proposed as a simplified model incorporating the basic features of Minsky's hypothesis.

Being itself an extension of the Goodwin model [Goo67], discussed in Chapter 3, it benefits from the simplicity, while at the same time avoiding the much criticized structural instability. Instead of centers, we verify the existence of two key fixed points, which are locally stable under usual conditions. Notably, Keen's ideas have produced a much richer model, capable of exhibiting complex phenomena, resembling, for instance, what happened in the Great Moderation, followed by the destructive downward spiral initiated in 2008.

In this chapter, we introduce the Keen model, study its equilibria, determining their local stability, and then investigate global stability through numerical representation of the basin of attraction.

5.1 Mathematical formulation

The extension of the basic Goodwin model proposed by Keen [Kee95] consists of introducing a banking sector to finance new investments. By relaxing the assumption that capitalists invest the totality of their profits, and thus by introducing the variable D, the amount of debt in real terms, the net profit after paying wages and interest on debt is

$$(1 - \omega - rd)Y \tag{5.1}$$

where r is a constant real interest rate and $d = D/Y$ is the debt ratio in the economy. If capitalists reinvested all this net profit and nothing more, debt levels would remain constant over time. The key insight provided by Minsky [Min82] is that current cash-flows validate past liabilities and form the basis for future ones. In other words, high net profits lead to more borrowing whereas low net profits (possibly negative) lead to a deleveraging of the economy. Keen [Kee95] formalizes this insight by taking the change in capital stock to be

$$\dot{K} = \kappa(1 - \omega - rd)Y - \delta K \tag{5.2}$$

where the rate of new investment is a nonlinear increasing function κ of the net profit share $\pi = (1 - \omega - rd)$ and δ is a constant depreciation rate as before. Accordingly, total output evolves as

$$\frac{\dot{Y}}{Y} = \frac{\kappa(1 - \omega - rd)}{\nu} - \delta := \mu(1 - \omega - rd) \tag{5.3}$$

and the employment rate dynamics becomes

$$\frac{\dot{\lambda}}{\lambda} = \frac{\kappa(1 - \omega - rd)}{\nu} - \alpha - \beta - \delta, \tag{5.4}$$

whereas the time evolution for the wage share remains (3.8).

The new dynamic variable in this model is the amount of debt, which changes based on the difference between new investment and net profits. In other words, we have that

$$\dot{D} = \kappa(1 - \omega - rd)Y - (1 - \omega - rd)Y \tag{5.5}$$

whence it follows that

$$\frac{\dot{d}}{d} = \frac{\dot{D}}{D} - \frac{\dot{Y}}{Y} = \frac{\kappa(1 - \omega - rd) - (1 - \omega - rd)}{d} - \frac{\kappa(1 - \omega - rd)}{\nu} + \delta. \tag{5.6}$$

Combining (3.8), (5.4), (5.6) we arrive at the following three–dimensional system of autonomous differential equations:

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1 - \omega - rd)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d} &= d\left[r - \frac{\kappa(1 - \omega - rd)}{\nu} + \delta\right] + \kappa(1 - \omega - rd) - (1 - \omega)
\end{aligned} \tag{5.7}$$

For the analysis that follows, we assume henceforth that the rate of new investment in (5.2) is a continuously differentiable function κ satisfying

$$\kappa'(\pi) > 0 \text{ on } (-\infty, \infty) \tag{5.8}$$

$$0 \leq \lim_{\pi \to -\infty} \kappa(\pi) < \nu(\alpha + \beta + \delta) < \lim_{\pi \to +\infty} \kappa(\pi) < 1 \tag{5.9}$$

$$\lim_{\pi \to -\infty} \pi^2 \kappa'(\pi) < \infty \tag{5.10}$$

Recalling from Section 2.2.1 that consumption in the Keen model is given by $C = Y[1 - \kappa(\pi)]$, condition (5.9) ensures that the consumption ratio will belong to the $[0, 1]$ interval.

5.2 Equilibria in the Keen model

We see that

$$(\overline{\omega}_0, \overline{\lambda}_0, \overline{d}_0) = (0, 0, \overline{d}_0), \tag{5.11}$$

where $\overline{d}_0$ is any solution of the equation

$$d\left[r - \frac{\kappa(1 - rd)}{\nu} + \delta\right] + \kappa(1 - rd) - 1 = 0, \tag{5.12}$$

is an equilibrium point for (5.7). Equilibria of the form (5.11) are economically meaningless, and we expect them to be unstable in the same way that $(\overline{\omega}_0, \overline{\lambda}_0) = (0, 0)$ is saddle point in the original Goodwin model.

For a more meaningful equilibrium, observe that it follows from (5.9) that $\nu(\alpha + \beta + \delta)$ is in the image of κ so that we can define

$$\overline{\pi}_1 = \kappa^{-1}(\nu(\alpha + \beta + \delta)) \tag{5.13}$$

and verify by direct substitution that the point

$$\begin{aligned}\overline{\omega}_1 &= 1 - \overline{\pi}_1 - r\frac{\nu(\alpha+\beta+\delta) - \overline{\pi}_1}{\alpha+\beta} \\ \overline{\lambda}_1 &= \Phi^{-1}(\alpha) \\ \overline{d}_1 &= \frac{\nu(\alpha+\beta+\delta) - \overline{\pi}_1}{\alpha+\beta}\end{aligned} \tag{5.14}$$

satisfies the relation

$$1 - \overline{\omega}_1 - r\overline{d}_1 = \overline{\pi}_1 \tag{5.15}$$

and is an equilibrium for (5.7). This equilibrium corresponds to a finite level of debt and strictly positive employment rate and is therefore economically desirable, so in the next section we shall investigate conditions guaranteeing that it is locally stable. As with the Goodwin model, it is interesting to note that the growth rate of the economy at this equilibrium point is given by

$$\mu(\overline{\pi}_1) = \frac{\kappa(1 - \overline{\omega}_1 - r\overline{d}_1)}{\nu} - \delta = \alpha + \beta. \tag{5.16}$$

We can obtain yet another set of equilibrium points by setting $\omega = 0$ and

$$1 - rd = \overline{\pi}_1 = \kappa^{-1}(\nu(\alpha+\beta+\delta)) \tag{5.17}$$

so that $\dot{\omega} = \dot{\lambda} = 0$ in (5.7) regardless of the value of λ. However, to have $\dot{d} = 0$ as well we must have $d = \overline{d}_1$ as before. But this can only be satisfied simultaneously with (5.17) if the model parameters satisfy the following very specific condition

$$1 - r\frac{\nu(\alpha+\beta+\delta) - \kappa^{-1}(\nu(\alpha+\beta+\delta))}{\alpha+\beta} = \kappa^{-1}(\nu(\alpha+\beta+\delta)). \tag{5.18}$$

Provided (5.18) holds, we have that points on the line $(0, \lambda, \overline{d}_1)$ are equilibria for (5.7) for any value $0 < \lambda < 1$. We see that equilibria of this form are not only economically meaningless, but are also structurally unstable, since a small change in the model parameters leading to a violation of (5.18) makes them disappear.

Finally, if we rewrite the system with the change of variables $u = 1/d$, we obtain

$$\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-r/u)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{u} &= u\left[\frac{\kappa(1-\omega-r/u)}{\nu} - r - \delta\right] - u^2\left[\kappa(1-\omega-r/u) - (1-\omega)\right].
\end{aligned} \tag{5.19}$$

We now see that $(0, 0, 0)$ is an equilibrium of (5.19) corresponding to the point

$$(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty) \tag{5.20}$$

for the original system. This equilibrium for (5.19) corresponds to the economically undesirable but nevertheless important situation of a collapse in wages and employment when the economy as a whole becomes overwhelmed by debt, rendering of paramount importance to investigate its local stability. Observe that condition (5.9) guarantees that

$$\kappa(1-\omega-r/u) \to \kappa(-\infty) \tag{5.21}$$

as $\omega \to 0$ and $u \to 0^+$, so the vector field for (5.19) remains finite on trajectories approaching $(0, 0, 0)$ along positive values of u.

5.3 Local stability in the Keen model

Denoting $\pi = 1 - \omega - rd$, the Jacobian for (5.7) is

$$J(\omega,\lambda,d) = \begin{bmatrix} \Phi(\lambda)-\alpha & \omega\Phi'(\lambda) & 0 \\ -\frac{\lambda\kappa'(\pi)}{\nu} & \frac{\kappa(\pi)-\nu(\alpha+\beta+\delta)}{\nu} & -\frac{r\lambda\kappa'(\pi)}{\nu} \\ \frac{(d-\nu)\kappa'(\pi)+\nu}{\nu} & 0 & \frac{\nu(r+\delta)-\kappa(\pi)+r(d-\nu)\kappa'(\pi)}{\nu} \end{bmatrix} \tag{5.22}$$

At the equilibrium point $(0,0,\overline{d}_0)$ this reduces to the lower triangular matrix

$$J(0,0,\overline{d}_0) = \begin{bmatrix} \Phi(0)-\alpha & 0 & 0 \\ 0 & \frac{\kappa(\overline{\pi}_0)-\nu(\alpha+\beta+\delta)}{\nu} & 0 \\ \frac{(\overline{d}_0-\nu)\kappa'(\overline{\pi}_0)+\nu}{\nu} & 0 & \frac{\nu(r+\delta)-\kappa(\overline{\pi}_0)+r(\overline{d}_0-\nu)\kappa'(\overline{\pi}_0)}{\nu} \end{bmatrix} \tag{5.23}$$

where $\overline{\pi}_0 = 1 - r\overline{d}_0$. Its real eigenvalues are given by the diagonal entries, and it is hard to determine their sign a priori since $\overline{d}_0$ is given as the solution of equation (5.12). Although they can be readily determined once specific parameters are chosen, we observe that these equilibrium points are likely to be unstable, since a sufficiently large value of $\overline{\pi}_0$ makes the second diagonal term above positive, whereas a sufficiently small value of $\overline{\pi}_0$ (and correspondingly large value of $\overline{d}_0$) makes the third diagonal term above positive.

At the equilibrium $(\overline{\omega}_1,\overline{\lambda}_1,\overline{d}_1)$ the Jacobian takes the interesting form

$$J(\overline{\omega}_1,\overline{\lambda}_1,\overline{d}_1) = \begin{bmatrix} 0 & K_0 & 0 \\ -K_1 & 0 & -rK_1 \\ K_2 & 0 & rK_2-(\alpha+\beta) \end{bmatrix} \tag{5.24}$$

where

$$\begin{aligned} K_0 &= \overline{\omega}_1 \Phi'(\overline{\lambda}_1) > 0 \\ K_1 &= \frac{\overline{\lambda}_1 \kappa'(\overline{\pi}_1)}{\nu} > 0 \\ K_2 &= \frac{(\overline{d}_1 - \nu)\kappa'(\overline{\pi}_1) + \nu}{\nu} \end{aligned} \tag{5.25}$$

Therefore, the characteristic polynomial for the matrix in (5.24) is

$$p_3(y) = y^3 + [(\alpha + \beta) - rK_2]y^2 + K_0K_1 y + K_0K_1(\alpha + \beta). \tag{5.26}$$

According to the Routh-Hurwitz criterion, a necessary and sufficient condition for all the roots of a cubic polynomial of the form

$$p(y) = a_3 y^3 + a_2 y^2 + a_1 y + a_0 \tag{5.27}$$

to have negative real parts is

$$a_n > 0, \forall n \text{ and } a_2 a_1 > a_3 a_0. \tag{5.28}$$

Our characteristic polynomial already has three of its coefficients positive; therefore all we need is

$$(\alpha + \beta) > rK_2 \tag{5.29}$$

and

$$((\alpha + \beta) - rK_2)K_0K_1 > K_0K_1(\alpha + \beta) \tag{5.30}$$

Since we are already assuming that $\alpha > 0$ and $\beta > 0$, we see that the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ is stable if and only if $rK_2 < 0$, which is equivalent to

$$r\left[\frac{\kappa'(\overline{\pi}_1)}{\nu}\big(\overline{\pi}_1 - \nu\delta\big) - (\alpha + \beta)\right] > 0. \tag{5.31}$$

Because the real interest rate r can have any sign, condition (5.31) needs to be checked in each implementation of the model. Observe, however, that once the sign of r is chosen, the remaining terms are all independent of the magnitude of the interest rate. For the most common situation of $r > 0$, condition (5.31) imposes interesting constraints on the investment function κ. Condition (5.31) states that for $\overline{\pi}_1$ to correspond to a stable equilibrium we must have $\overline{\pi}_1 > \nu\delta$, suggesting a lower bound for equilibrium capitalists profits, while at the same time $\kappa'(\overline{\pi}_1)$ needs to be sufficiently large, leading to a rapid ramp-up of investment for net profits beyond $\overline{\pi}_1$.

As we mentioned in the previous section, equilibria of the form $(0, \lambda, \overline{d}_1)$ depend on a very specific choice of parameters satisfying (5.18), making them structurally unstable. Therefore we are not going to discuss them any further, except by verifying that they do not arise for the parameters used in the numerical example implemented later.

Finally, regarding the point $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty)$, observe that the Jacobian for the modified system (5.19) is

$$J(\omega, \lambda, u) = \begin{bmatrix} \Phi(\lambda) - \alpha & \omega\Phi'(\lambda) & 0 \\ -\frac{\lambda\kappa'(\pi)}{\nu} & \frac{\kappa(\pi)-\nu(\alpha+\beta+\delta)}{\nu} & -\frac{r\lambda\kappa'(\pi)}{u^2\nu} \\ \frac{(\nu u^2-u)\kappa'(\pi)-\nu u^2}{\nu} & 0 & \frac{\kappa(\pi)(1-2u)+r\kappa'(\pi)(1/u-1)+2u\nu(1-\omega)-\nu(r+\delta)}{\nu} \end{bmatrix}$$

where $\pi = 1 - \omega - r/u$. At the equilibrium $(\overline{\omega}_2, \overline{\lambda}_2, \overline{u}_2) = (0, 0, 0)$, conditions (5.9) and (5.10) ensure that this reduces to the diagonal matrix

$$J(0,0,0) = \begin{bmatrix} \Phi(0) - \alpha & 0 & 0 \\ 0 & \frac{\kappa(-\infty)-\nu(\alpha+\beta+\delta)}{\nu} & 0 \\ 0 & 0 & \frac{\kappa(-\infty)-\nu(r+\delta)}{\nu} \end{bmatrix}, \tag{5.32}$$

from which all real eigenvalues can be readily obtained. Observe that the first two eigenvalues are negative by virtue of conditions (3.16) and (5.8) on the functions Φ and κ, so this equilibrium is stable if and only if

$$\mu(-\infty) = \frac{\kappa(-\infty)}{\nu} - \delta < r. \tag{5.33}$$

Recalling expression (5.3), we see that this equilibrium is stable if and only if the real interest rate exceeds the growth rate of the economy at infinite levels of debt and zero wages.

It is interesting to note that assumptions (3.16) and (5.8) were made in order to guarantee the existence of the economically desirable equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$, but perversely contribute to the stability of the undesirable point $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty)$. Moreover, in view of (5.8), we see that a sufficient condition for (5.33) to hold is

$$\alpha + \beta < r. \tag{5.34}$$

Recalling (5.16) we conclude that a sufficient condition for $(\overline{\omega}_2, \overline{\lambda}_2, \overline{u}_2) = (0, 0, 0)$ to be a locally stable equilibrium for (5.19) is that the real interest rate r exceeds the growth rate of the economy at the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$, which resembles the condition derived by Tirole [Tir85] for the absence of rational bubbles in an overlapping generation model, corresponding to an "efficient" economy.

5.4 Example

Choosing the fundamental economic constants to be the same as in (3.24) with the addition of

$$r = 0.03, \tag{5.35}$$

taking the Phillips curve as in (3.25) and (3.26), and defining the investment function κ as

$$\kappa(x) = \kappa_0 + \kappa_1 \tan^{-1}(\kappa_2 x + \kappa_3) \tag{5.36}$$

where the constants $\kappa_0, \kappa_1, \kappa_2$ and κ_3 are chosen according to

$$\begin{aligned} \kappa(-\infty) &= 0, \qquad \kappa(+\infty) = 1, \\ \overline{\pi}_1 &= 0.16, \qquad \kappa'(\overline{\pi}_1) = 5 \end{aligned} \tag{5.37}$$

it follows that conditions (5.8)–(5.10) are satisfied.

Observe first that the real solutions for equation (5.12) in this case are

$$\overline{d}_0 = \begin{cases} -0.021 \\ 32.503 \end{cases} . \tag{5.38}$$

The eigenvalues for $J(0,0,\overline{d}_0)$ for these two points are

$$\begin{aligned} &(-0.2915, -0.0650, 0.2763) \\ &(0.0849, -0.0650, -0.0448), \end{aligned}$$

confirming that the equilibrium $(0,0,\overline{d}_0)$ is unstable in either case, as expected.

Moving to the economically meaningful equilibrium, we obtain the equilibrium values

$$(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1) = (0.8367, 0.9600, 0.1111). \tag{5.39}$$

with corresponding eigenvalues

$$(-0.0451, -0.0572 + 2.0855i, -0.0572 - 2.0855i) \tag{5.40}$$

all of which have negative real part. Alternatively, we find that

$$\frac{\kappa'(\overline{\pi}_1)}{\nu}\big(\overline{\pi}_1 - \nu\delta\big) - (\alpha + \beta) = 0.1717 > 0, \tag{5.41}$$

so that (5.31) is satisfied and this equilibrium is locally stable. When the initial conditions are chosen sufficiently close to the equilibrium values, we observe the convergent behaviour shown in the phase portrait for employment and wages in Figure 5.1. The oscillatory behaviour of all variables can be seen in Figure 5.2, where we also show the growing output Y as a function of time.

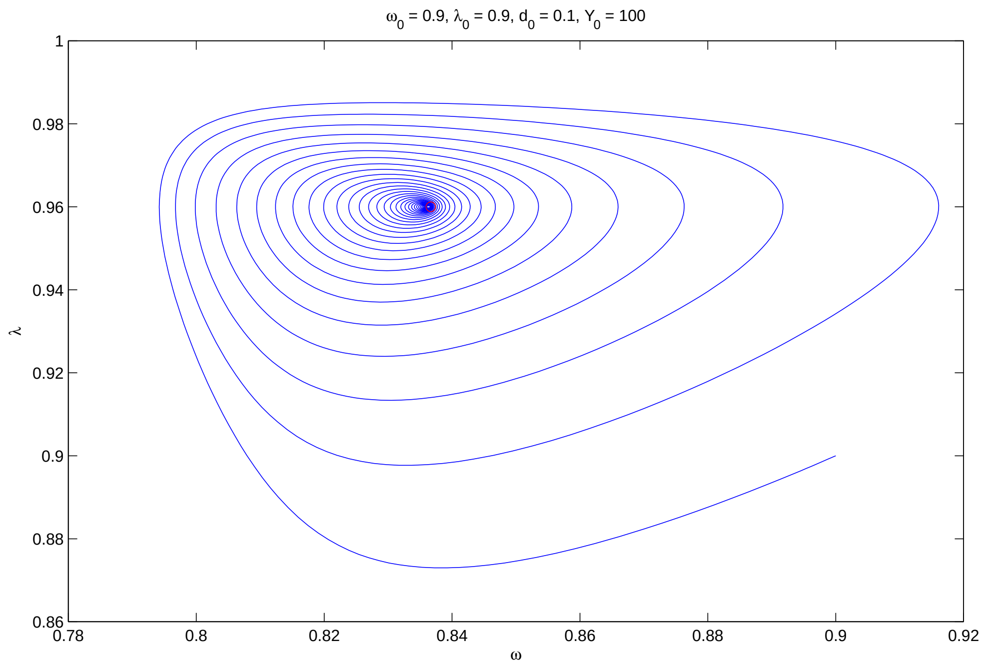

Figure 5.1: Phase portrait of employment and wages converging to a stable equilibrium with finite debt in the Keen model.

We notice that condition (5.18) is violated by our model parameters, so we do not need to consider the structurally unstable equilibria of the form $(0, \lambda, \overline{d}_1)$. Moving on to the equilibrium with infinite debt, we observe that

$$\frac{\kappa(-\infty)}{\nu} - \delta - r = -0.04 < 0, \tag{5.42}$$

so that (5.33) is satisfied and $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_2) = (0, 0, +\infty)$ corresponds to a stable equilibrium of (5.19). Therefore we expect to observe ever increasing debt levels when the initial conditions are sufficiently far from the equilibrium values $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$. This is

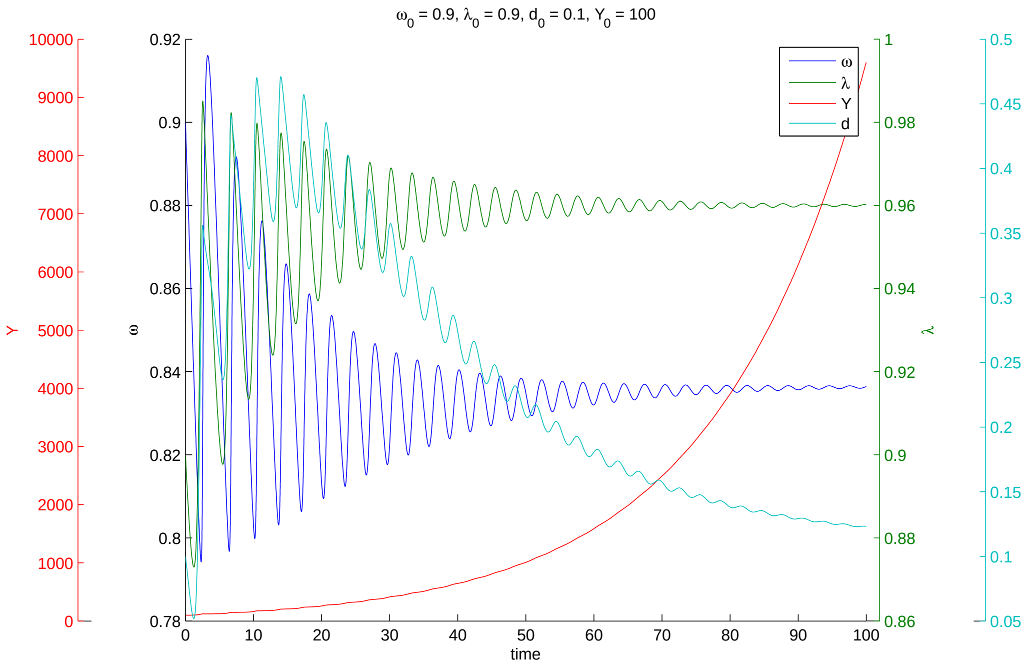

Figure 5.2: Employment, wages, debt and output as functions of time converging to a stable equilibrium with finite debt in the Keen model.

depicted in Figure 5.3, where we can see both wages and employment collapsing to zero while debt explodes to infinity. We also show the output Y which increases to very high levels propelled by the increasing debt before starting an inexorable descent.

While it is difficult to determine the basin of convergence for the equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_1)$ analytically, we plot in Figure 5.4 the set of initial conditions for which we observed convergence to this equilibrium numerically. As expected, the set of initial values for wages and employment leading to convergence becomes smaller as the initial value for debt increases.

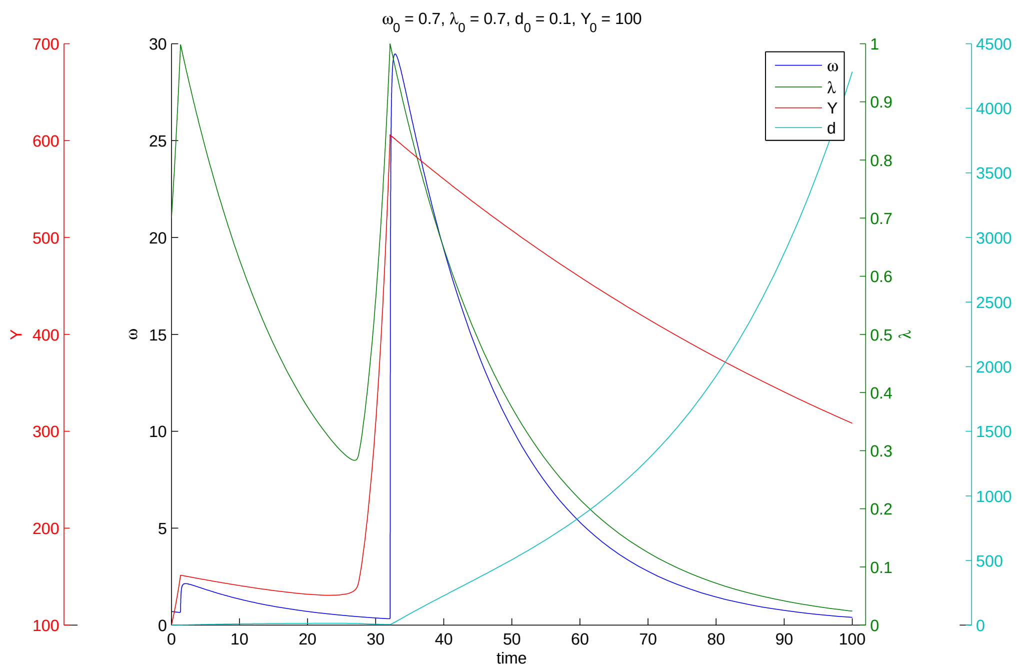

Figure 5.3: Employment, wages, debt and output as functions of time converging to a stable equilibrium with infinite debt in the Keen model.

5.5 Conclusion

Introducing debt to finance new investment leads to the three–dimensional Keen model exhibiting two distinct equilibria, a good one with finite debt and strictly positive employment and wage share, and a bad one with infinite debt and zero employment and wage share. We have determined that for typical model parameters, both can be locally stable.

As we have seen, this simple model is able to generate remarkably rich dynamics, but can still be generalized in a variety of ways. Staying in the realm of deterministic models, one possible extension already considered by Keen [Kee95, Kee09], and fully explored in Chapter 8, consists of introducing a government sector with corresponding spending and taxation, increasing the dimensionality of model and the complexity of its outcomes. Alternatively, in Chapter 7 we propose a second extension, where capital

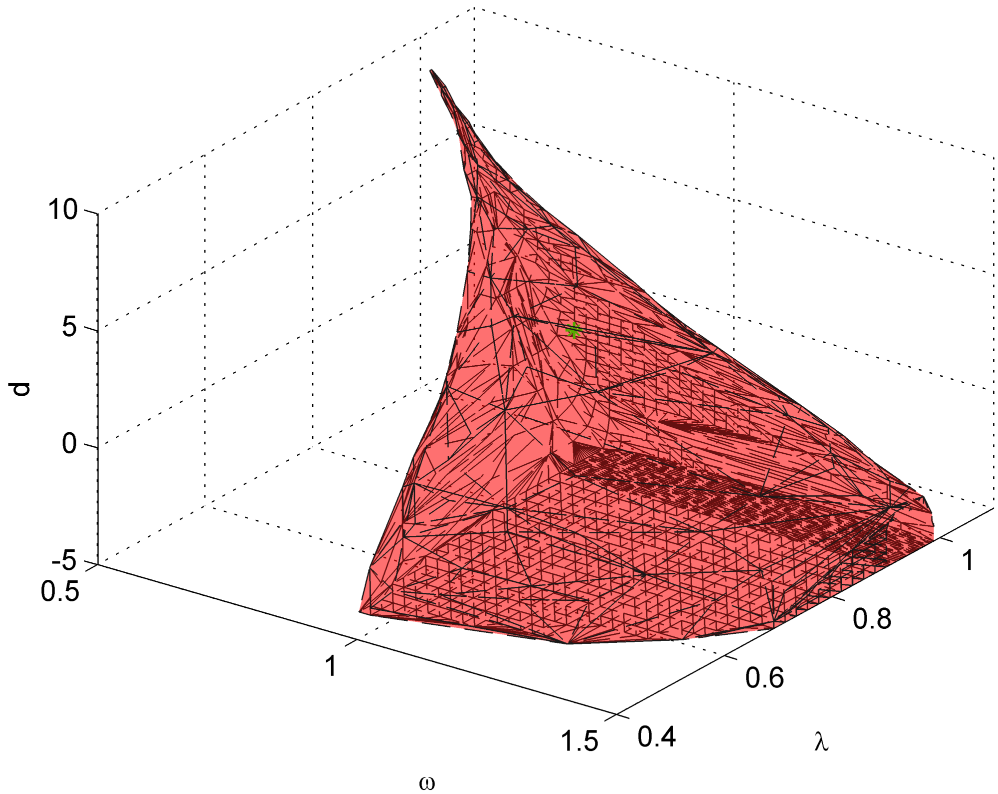

Figure 5.4: Basin of convergence for the Keen model for wages, employment and private debt.

investment projects are not immediately developed.

Chapter 6

Low Interest Rate Regimes

The purpose of this chapter is to further develop to analytical analysis of the Keen model when the real interest rate is assumed to be close to zero. To begin with, observe that if $r = 0$, the first two equations in (5.7) decouple, and we have the following system for the variables, which we will write in terms of (ω_0, λ_0) for wage share and employment, respectively

$$\begin{aligned} \dot{\omega}_0 &= \omega_0 \left[\Phi(\lambda_0) - \alpha\right] \\ \dot{\lambda}_0 &= \lambda_0 \left[\frac{\kappa(1-\omega_0)}{\nu} - (\alpha + \beta + \delta)\right] \end{aligned} \tag{6.1}$$

with the capitalists' debt, d_0, solving its own (non-autonomous, but one-dimensional) ODE

$$\dot{d}_0 = \kappa(1-\omega_0) - (1-\omega_0) - d_0 \left[\frac{\kappa(1-\omega_0)}{\nu} - \delta\right] \tag{6.2}$$

This system has only one fixed point,

$$\overline{\omega}_0 = 1 - \kappa^{-1}(\nu(\alpha+\beta+\delta)), \quad \overline{\lambda}_0 = \Phi^{-1}(\alpha), \quad \overline{d}_0 = \frac{\overline{\omega}_0 - (1 - \nu(\alpha+\beta+\delta))}{\alpha+\beta} \tag{6.3}$$

In result, $\omega_0(t), \lambda_0(t)$ must follow cycles given by the energy function

$$\begin{aligned} V(\omega_0(t), \lambda_0(t)) &= \int_{\overline{\omega}_0}^{\omega_0(t)} \frac{\kappa(1-\overline{\omega}_0) - \kappa(1-x)}{\nu x}\, dx + \int_{\overline{\lambda}_0}^{\lambda_0(t)} \frac{\Phi(y) - \Phi(\overline{\lambda}_0)}{y}\, dy \\ &= V(\omega_0(0), \lambda_0(0)), \end{aligned} \tag{6.4}$$

which can be verified from the fact that $\dot{V} = 0$ everywhere. Theorem 6.1 gives the period of such cycles, which depends only on the initial energy $V(\omega_0(0), \lambda_0(0))$.

The assumption of low interest rates will be translated into assuming that r equals some ε arbitrarily small. Suppose next that we can expand the solution to the Keen model (5.7) linearly as follows

$$\begin{aligned} \omega(t) &= \omega_0(t) + \varepsilon\omega_\varepsilon(t) + O(\varepsilon^2) \\ \lambda(t) &= \lambda_0(t) + \varepsilon\lambda_\varepsilon(t) + O(\varepsilon^2) \\ d(t) &= d_0(t) + \varepsilon d_\varepsilon(t) + O(\varepsilon^2) \end{aligned} \tag{6.5}$$

The continuously differentiable functions κ and Φ can be expanded as

$$\begin{aligned} \Phi(\lambda) &= \Phi(\lambda_0 + \varepsilon\lambda_\varepsilon) = \Phi(\lambda_0) + \varepsilon\lambda_\varepsilon\Phi'(\lambda_0) + O(\varepsilon^2) \\ \kappa(1-\omega-rd) &= \kappa(1-\omega_0 - \varepsilon(\omega_\varepsilon + d_0) + O(\varepsilon^2)) \\ &= \kappa(1-\omega_0) - \varepsilon(\omega_\varepsilon + d_0)\kappa'(1-\omega_0) + O(\varepsilon^2) \end{aligned} \tag{6.6}$$

Differentiating (6.5) with respect to time, we have

$$\begin{aligned} \dot{\omega}_0 + \varepsilon\dot{\omega}_\varepsilon + O(\varepsilon^2) &= \left(\omega_0 + \varepsilon\omega_\varepsilon + O(\varepsilon^2)\right)\left[\Phi(\lambda_0) + \varepsilon\lambda_\varepsilon\Phi'(\lambda_0) + O(\varepsilon^2) - \alpha\right] \\ &= \omega_0\left[\Phi(\lambda_0) - \alpha\right] + \varepsilon\left[\omega_\varepsilon\left[\Phi(\lambda_0) - \alpha\right] + \omega_0\lambda_\varepsilon\Phi'(\lambda_0)\right] + O(\varepsilon^2) \end{aligned} \tag{6.7}$$

$$\begin{aligned}
\dot{\lambda}_0 + \varepsilon\dot{\lambda}_\varepsilon + O(\varepsilon^2) &= \big(\lambda_0 + \varepsilon\lambda_\varepsilon + O(\varepsilon^2)\big)\left[\frac{\kappa(1-\omega_0)}{\nu} - \alpha - \beta - \delta\right.\\
&\qquad\left. - \varepsilon(\omega_\varepsilon + d_0)\frac{\kappa'(1-\omega_0)}{\nu} + O(\varepsilon^2)\right]\\
&= \lambda_0\left[\frac{\kappa(1-\omega_0)}{\nu} - (\alpha+\beta+\delta)\right] + O(\varepsilon^2)\\
&\quad + \varepsilon\left\{\lambda_\varepsilon\left[\frac{\kappa(1-\omega_0)}{\nu} - (\alpha+\beta+\delta)\right] - \lambda_0(\omega_\varepsilon + d_0)\frac{\kappa'(1-\omega_0)}{\nu}\right\}
\end{aligned} \tag{6.8}$$

$$\begin{aligned}
\dot{d}_0 + \varepsilon\dot{d}_\varepsilon + O(\varepsilon^2) &= \kappa(1-\omega_0) - \varepsilon(\omega_\varepsilon + d_0)\kappa'(1-\omega_0) + O(\varepsilon^2) - (1 - \omega_0 - \varepsilon(\omega_\varepsilon + d_0))\\
&\quad - (d_0 + \varepsilon d_\varepsilon + O(\varepsilon^2))\left[\frac{\kappa(1-\omega_0)}{\nu} - \delta - \varepsilon(\omega_\varepsilon + d_0)\frac{\kappa'(1-\omega_0)}{\nu} + O(\varepsilon^2)\right]\\
&= \kappa(1-\omega_0) - (1-\omega_0) - d_0\left[\frac{\kappa(1-\omega_0)}{\nu} - \delta\right] + O(\varepsilon^2)\\
&\quad + \varepsilon\left\{(\omega_\varepsilon + d_0)\kappa'(1-\omega_0)\left(\frac{d_0}{\nu} - 1\right) + \omega_\varepsilon + d_0 - d_\varepsilon\left[\frac{\kappa(1-\omega_0)}{\nu} - \delta\right]\right\}
\end{aligned} \tag{6.9}$$

The fundamental theorem of perturbation theory [SMJ98] allows us to say that the terms accompanying the powers of ε must be the same on both sides of the above equations. Accordingly, one finds that ω_0 and λ_0 do indeed solve (6.4), while the remaining variables solve

$$\dot{d}_0 = \kappa(1-\omega_0) - (1-\omega_0) - d_0\left[\frac{\kappa(1-\omega_0)}{\nu} - \delta\right] \tag{6.10}$$

$$\dot{\omega}_\varepsilon = \omega_\varepsilon\left[\Phi(\lambda_0) - \alpha\right] + \omega_0\lambda_\varepsilon\Phi'(\lambda_0) \tag{6.11}$$

$$\dot{\lambda}_\varepsilon = \lambda_\varepsilon\left[\frac{\kappa(1-\omega_0)}{\nu} - (\alpha+\beta+\delta)\right] - \lambda_0\frac{\kappa'(1-\omega_0)}{\nu}(\omega_\varepsilon + d_0) \tag{6.12}$$

$$\dot{d}_\varepsilon = (\omega_\varepsilon + d_0)\kappa'(1-\omega_0)\left(\frac{d_0}{\nu} - 1\right) + \omega_\varepsilon + d_0 - d_\varepsilon\left(\frac{\kappa(1-\omega_0)}{\nu} - \delta\right) \tag{6.13}$$

$$\tag{6.14}$$

6.1 Zero-order solution

So far, we know that wage share ω_0 and employment λ_0 must oscillate in a cycle, while d_0 is given by

$$d_0(t) = f_d(t)\left[d_0(0) + \int_0^t f_d(u)^{-1} h_d(u)\, du\right] \tag{6.15}$$

where

$$f_d(u) = exp\left[-\int_0^t \left(\frac{\kappa(1-\omega_0(s))}{\nu} - \delta\right) ds\right] = \frac{\lambda_0(0)}{\lambda_0(t)} e^{-(\alpha+\beta)t} \tag{6.16}$$

and

$$h_d(u) = \kappa(1-\omega_0(u)) + \omega_0(u) - 1 \tag{6.17}$$

The following analysis follows closely the procedure developed in [ST11a]. The homogeneous term is

$$d_H(t) = f_d(t) d_0(0) = \frac{d_0(0)\lambda_0(0)}{\lambda_0(t)} e^{-(\alpha+\beta)t} \xrightarrow[t\to+\infty]{} 0 \tag{6.18}$$

Denoting the period of the oscillations of ω_0 and λ_0 as T, one can easily see that $f_d(T)$ reduces to

$$f_d(T) = \exp\left(-\int_0^T \left(\frac{\kappa(1-\omega_0(s))}{\nu} - \delta\right) ds\right) = e^{-(\alpha+\beta)T} < 1 \tag{6.19}$$

while $f_d(nT)$ factors into

$$f_d(nT) = \frac{\lambda_0(0)}{\lambda_0(nT)} e^{-(\alpha+\beta)nT} = \left(e^{-(\alpha+\beta)T}\right)^n = f_d(T)^n \tag{6.20}$$

Hence, we can calculate

$$d_H(nT) = d_0(0) f_d(T)^n \tag{6.21}$$

It can also be shown that the full solution for d_0 at multiples of the period T

satisfies

$$d_0(nT) = f_d(T)^n d_0(0) + f_d(T)\frac{1 - f_d(T)^n}{1 - f_d(T)} \int_0^T f_d(u)^{-1} h_d(u)\, du \tag{6.22}$$

from which we readily obtain that

$$\lim_{n\to\infty} d_0(nT) = \frac{f_d(T)}{1 - f_d(T)} \int_0^T f_d(u)^{-1} h_d(u)\, du \tag{6.23}$$

since $f_d(T) < 1$. In fact, we can show that d_0 converges to a steady state solution. For that purpose, let $t \in (0,T)$, $n = 1, 2, \cdots$, and observe that the particular solution d_p can be expressed as

$$\begin{aligned}
d_p(t+nT) &= f_d(t+nT) \int_0^{t+nT} f_d(u)^{-1} h_d(u)\, du \\
&= f_d(t) f_d(T)^n \left[\left(\int_0^T + \int_T^{2T} + \cdots + \int_{(n-1)T}^{nT} + \int_{nT}^{nT+t} \right) f_d(u)^{-1} h_d(u)\, du \right] \\
&= f_d(t) f_d(T)^n \Bigg\{ \Bigg[\left(1 + f_d(T)^{-1} + \cdots + f_d(T)^{-(n-1)}\right) \int_0^T \\
&+ f_d(T)^{-n} \int_0^t \Bigg] f_d(u)^{-1} h_d(u)\, du \Bigg\} \\
&= f_d(t) \Bigg[\left(f_d(T) + f_d(T)^2 + \cdots f_d(T)^n\right) \int_0^T f_d(u)^{-1} h_d(u)\, du \\
&+ \int_0^t f_d(u)^{-1} h_d(u)\, du \Bigg] \\
&= f_d(t) \left[f_d(T) \frac{1 - f_d(T)^n}{1 - f_d(T)} \int_0^T f_d(u)^{-1} h(u)\, du + \int_0^t f_d(u)^{-1} h_d(u)\, du \right]
\end{aligned} \tag{6.24}$$

which converges, as $n \to +\infty$, to the steady state solution d_{ss} as follows

$$d_{ss}(t+nT) = f_d(t) \left[\frac{f_d(T)}{1 - f_d(T)} \int_0^T f_d(u)^{-1} h_d(u)\, du + \int_0^t f_d(u)^{-1} h_d(u)\, du \right] = RHS(t) \tag{6.25}$$

Notice that $RHS(t)$ is periodic with period T, since

$$\begin{aligned}
RHS(t+T) &= f_d(t+T)\left[\frac{f_d(T)}{1-f_d(T)}\int_0^T f_d(u)^{-1}h_d(u)\,du + \int_0^{T+t} f_d(u)^{-1}h_d(u)\,du\right] \\
&= f_d(t)\Bigg[\left(f_d(T)+\frac{f_d(T)^2}{1-f_d(T)}\right)\int_0^T f_d(u)^{-1}h_d(u)\,du \\
&+ f_d(T)\int_T^{T+t} f_d(u)^{-1}h_d(u)\,du\Bigg] \\
&= f_d(t)\left[\frac{f_d(T)}{1-f_d(T)}\int_0^T f_d(u)^{-1}h_d(u)\,du + f_d(T)\int_0^t f_d(u)^{-1}f_d(T)^{-1}h_d(u)\,du\right] \\
&= X(t)\left[\frac{f_d(T)}{1-f_d(T)}\int_0^T f_d(u)^{-1}h_d(u)\,du + \int_0^t f_d(u)^{-1}h_d(u)\,du\right] \\
&= RHS(t)
\end{aligned} \tag{6.26}$$

Recalling the fact that the homogeneous term decays to 0 as $t \to +\infty$ (exponentially fast, with rate $-(\alpha+\beta)$), we can conclude that $d_0(t)$ converges to $d_{ss}(t)$ as $t \to +\infty$.

6.2 First-order solution

The first order variables solve the following linear system

$$\begin{aligned}
\begin{bmatrix} \dot{\omega}_\varepsilon \\ \dot{\lambda}_\varepsilon \\ \dot{d}_\varepsilon \end{bmatrix} &= \begin{bmatrix} \Phi(\lambda_0)-\alpha & \omega_0\Phi'(\lambda_0) & 0 \\ -\lambda_0\frac{\kappa'(1-\omega_0)}{\nu} & \frac{\kappa(1-\omega_0)}{\nu}-(\alpha+\beta+\delta) & 0 \\ 1+\frac{\kappa'(1-\omega_0)}{\nu}(d_0-\nu) & 0 & -\left(\frac{\kappa(1-\omega_0)}{\nu}-\delta\right) \end{bmatrix} \begin{bmatrix} \omega_\varepsilon \\ \lambda_\varepsilon \\ d_\varepsilon \end{bmatrix} \\
&+ \begin{bmatrix} 0 \\ -d_0\lambda_0\frac{\kappa'(1-\omega_0)}{\nu} \\ d_0\frac{\kappa'(1-\omega_0)}{\nu}(d_0-\nu)+d_0 \end{bmatrix}
\end{aligned} \tag{6.27}$$

Likewise the perturbation model proposed for the stochastic Goodwin model in Section 4.3, we can express the solution to (6.2) in closed-form. This is the objective of the next proposition.

Proposition 6.1. *Let*

$$p(t) := \Phi(\lambda_0(t)) - \alpha \tag{6.28}$$

$$q(t) := \kappa(1 - \omega_0(t))/\nu - \alpha - \beta - \delta \tag{6.29}$$

$$G(t) := \begin{bmatrix} \omega_0(t) & 0 \\ 0 & \lambda_0(t) \end{bmatrix} \tag{6.30}$$

$$\vec{h}_\varepsilon(t) := [0, -d_0(t)\kappa'(1 - \omega_0(t))/\nu]^\intercal \tag{6.31}$$

where the pair $(\omega_0(t), \lambda_0(t))$ *solve the system* (6.1) *with initial condition* $(\omega_0(0), \lambda_0(0)) \neq (\overline{\omega}, \overline{\lambda})$*, while* $d_0(t)$ *is given by* (6.15)*. Denote* $\vec{v}_\varepsilon(t) = [\omega_\varepsilon(t), \lambda_\varepsilon(t)]^\intercal$*. The solution of the system* (6.27) *is then*

$$\begin{aligned} \vec{v}_\varepsilon(t) &= G(t)\Psi(t)\left[G^{-1}(0)\vec{v}_\varepsilon(0) + \int_0^t \Psi^{-1}(s)\vec{h}_\varepsilon(s)\,ds\right] \\ &= G(t)\Psi(t)\int_0^t \Psi^{-1}(s)\vec{h}_\varepsilon(s)\,ds \quad \textit{for } \vec{v}_\varepsilon(0) = [0,0]^\intercal \end{aligned} \tag{6.32}$$

where

$$\Psi(t) := \begin{bmatrix} \frac{p(t)}{p(0)} - q(0)\Gamma_x(t) & p(0)\Gamma_x(t) \\ q(0)\Gamma_y(t) & \frac{q(t)}{q(0)} - p(0)\Gamma_y(t) \end{bmatrix} \tag{6.33}$$

with

$$\begin{aligned} \Gamma_x(t) &:= p(t)\int_0^t \frac{\lambda_0(s)\Phi'(\lambda_0(s))}{p(s)^2}\,ds \\ \Gamma_y(t)) &:= q(t)\int_0^t \frac{\omega_0(s)\mu'(\omega_0(s))}{q(s)^2}\,ds \end{aligned} \tag{6.34}$$

Finally, $d_\varepsilon(t)$ is given by

$$d_\varepsilon(t) = \int_0^t \frac{\lambda_0(s)}{\lambda_0(t)} e^{-(\alpha+\beta)(t-s)} \left[\omega_\varepsilon(s) + d_0(s)\right] \left[1 + \frac{\kappa'(1-\omega_0(s))}{\nu}(d_0(s) - \nu)\right] ds \quad (6.35)$$

Proof. Making the change of variables $x = \omega_\varepsilon/\omega_0$ and $y = \lambda_\varepsilon/\lambda_0$, we obtain the following model for x and y

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & \lambda_0\Phi'(\lambda_0) \\ -\omega_0\frac{\kappa'(1-\omega_0)}{\nu} & 0 \end{bmatrix}}^{A(t)} \begin{bmatrix} x \\ y \end{bmatrix} + \overbrace{\begin{bmatrix} 0 \\ -d_0\frac{\kappa'(1-\omega_0)}{\nu} \end{bmatrix}}^{\vec{h}_\varepsilon(t)} \quad (6.36)$$

First, we will solve the homogeneous version

$$\begin{bmatrix} \dot{x}_H \\ \dot{y}_H \end{bmatrix} = \begin{bmatrix} 0 & \lambda_0\Phi'(\lambda_0) \\ -\omega_0\frac{\kappa'(1-\omega_0)}{\nu} & 0 \end{bmatrix} \begin{bmatrix} x_H \\ y_H \end{bmatrix} \quad (6.37)$$

With the exact same procedure developed in Proposition 4.12, we find that $\Psi(t)$ from (6.33) is the state-density matrix that solves (6.37), and thus $\vec{z}(t) = [x(t), y(t)]^\intercal$ satisfies

$$\vec{z}(t) = \Psi^{-1}(t)\left[\vec{z}(0) + \int_0^t \Psi(s)\vec{h}_\varepsilon(s)\,ds\right] \quad (6.38)$$

In other words,

$$\begin{aligned} \vec{v}_\varepsilon(t) &= G(t)\vec{z}(t) \\ &= G(t)\Psi(t)\left[G^{-1}(0)\vec{v}_\varepsilon(0) + \int_0^t \Psi^{-1}(s)\vec{h}_\varepsilon(s)\,ds\right] \end{aligned} \quad (6.39)$$

Taking into consideration that $\vec{v}_\varepsilon(0) = [0,0]^\intercal$, we obtain (6.32). The solution for $d_\varepsilon(t)$ follows immediately from the observation that

$$\frac{d}{dt}\left[\lambda_0(t)e^{(\alpha+\beta)t}d_\varepsilon(t)\right] = (\omega_\varepsilon(t) + d_0(t))\frac{\kappa'(1-\omega_0(t))}{\nu}(d_0(t) - \nu) \quad (6.40)$$

which can be solved a priori. □

Remark 6.1. Borrowing the results from Remark 4.8, we can express the solution at multiples of the period as

$$\begin{aligned}\begin{bmatrix}\omega_\varepsilon(nT)/\omega_0(nT)\\ \lambda_\varepsilon(nT)/\lambda_0(nT)\end{bmatrix} &= \Psi(T)^n\sum_{k=1}^{n}\left[\int_{(k-1)T}^{kT}\Psi^{-1}(s)\vec{h}_\varepsilon(s)\,ds\right]\\ &= \Psi^n(T)\sum_{k=1}^{n}\left[\Psi^{-1}\left((k-1)T\right)\int_0^T\Psi^{-1}(u)\vec{h}_\varepsilon(u+(k-1)T)\,du\right]\\ &= \sum_{k=1}^{n}\left[\Psi^{n-k+1}(T)\int_0^T\Psi^{-1}(u)\vec{h}_\varepsilon(u+(k-1)T)\,du\right]\end{aligned} \tag{6.41}$$

Observe that since

$$\int_0^T\Psi^{-1}(u)\vec{h}_\varepsilon(u+(k-1)T)\,du \to \int_0^T\Psi^{-1}(u)\vec{h}_{ss}(u)\,du \quad \text{as } k\to+\infty \tag{6.42}$$

where we have defined

$$\vec{h}_{ss}(t) := \begin{bmatrix}0\\ -d_{ss}(t)\frac{\kappa'(1-\omega_0(t))}{\nu}\end{bmatrix}, \tag{6.43}$$

together with the fact that

$$\Psi^{n-k+1}(T) = VJ^{n-k+1}V^{-1} = V\begin{bmatrix}1 & n-k+1\\ 0 & 1\end{bmatrix}V^{-1}, \tag{6.44}$$

we can conclude that both $\omega_\varepsilon(nT)$ and $\lambda_\varepsilon(nT)$ diverge as $n\to\infty$, indicating that the approximate solution is only valid for finite time horizons.

6.3 Period of a generalization of the Lotka-Volterra model

The work presented here was motivated by [Hsu83]. In this section, we make the contribution of deriving the period of a general non-linear Lotka-Volterra model[1] ([Lot25] and [Vol27]). Here, we consider general cross-dependence between x and y through the functions f and g which are only assumed to be increasing, with a and b in the interior of their respective images

$$\begin{aligned} \dot{x} &= x\,[a - f(y)] \\ \dot{y} &= y\,[g(x) - b] \end{aligned} \tag{6.45}$$

The next Theorem provides a closed-form expression for the period of (6.45).

Theorem 6.1. *If $f(x)$ satisfies*

$$\int_0^{f^{-1}(a)} \frac{a - f(y)}{y}\,dy = \int_{f^{-1}(a)}^{+\infty} \frac{f(y) - a}{y}\,dy = +\infty \tag{6.46}$$

then the period of the non-linear Lotka-Volterra model (6.45) *is given by*

$$T = \int_{\log(x_{min})}^{\log(x_{max})} \frac{1}{F_1^{-1}\left(G(z)\right)}\,dz + \int_{\log(x_{max})}^{\log(x_{min})} \frac{1}{F_2^{-1}\left(G(z)\right)}\,dz \tag{6.47}$$

where x_{min} and x_{max} are the two roots of

$$\int_{g^{-1}(b)}^{x} \frac{g(\eta) - b}{\eta}\,d\eta = V_0 = \int_{g^{-1}(b)}^{x(0)} \frac{g(\eta) - b}{\eta}\,d\eta + \int_{f^{-1}(a)}^{y(0)} \frac{f(\xi) - a}{\xi}\,d\xi \tag{6.48}$$

also

$$G(z) = V_0 + \int_{g^{-1}(b)}^{e^z} \frac{b - g(x)}{x}\,dx \tag{6.49}$$

[1] As opposed to the original Lotka-Volterra model, which is quadratic in x and y.

and $F_1(w)$, $F_2(w)$ are the restrictions of

$$F(w) = \int_{f^{-1}(a)}^{f^{-1}(a-w)} \frac{f(y)-a}{y}\,dy \tag{6.50}$$

to $\big[0, a - f(0)\big)$ and $(-\infty, 0]$, respectively.

Proof. Solutions of the dynamical system (6.45) are cycles centered at

$$\overline{x} = g^{-1}(b) \tag{6.51}$$

$$\overline{y} = f^{-1}(a) \tag{6.52}$$

We can separate variables and find an energy potential

$$V(x,y) = \int_{\overline{x}}^{x} \frac{g(\eta)-b}{\eta}\,d\eta + \int_{\overline{y}}^{y} \frac{f(\xi)-a}{\xi}\,d\xi \tag{6.53}$$

that must remain constant on the trajectories, that is, for a given starting point $(x(0), y(0))$, the solution $(x(t), y(t))$ must solve

$$V(x(t), y(t)) = V(x(0), y(0)) =: V_0 \tag{6.54}$$

The variable $x(t)$ oscillates between x_{min} and x_{max}, where $x_{min} < x_{max}$ are the roots of

$$\int_{\overline{x}}^{x} \frac{g(\eta)-b}{\eta}\,d\eta = V_0 \tag{6.55}$$

Likewise, $y(t)$ fluctuates between $y_{min} < y_{max}$, roots of

$$\int_{\overline{y}}^{y} \frac{f(\xi)-\alpha}{\xi}\,d\xi = V_0 \tag{6.56}$$

Observe that we can write

$$y = f^{-1}\left(a - \frac{\dot{x}}{x}\right) \tag{6.57}$$

and thus

$$\begin{aligned}
\ddot{x} &= \dot{x}\,\overbrace{[a - f(y)]}^{\frac{\dot{x}}{x}} - x f'(y) y \,[g(x) - b] \\
&= \frac{(\dot{x})^2}{x} - f' o f^{-1}\left(a - \frac{\dot{x}}{x}\right) f^{-1}\left(a - \frac{\dot{x}}{x}\right)[g(x) - b] \\
&= \frac{(\dot{x})^2}{x} - h\left(a - \frac{\dot{x}}{x}\right)[g(x) - b]
\end{aligned} \tag{6.58}$$

where $h(\zeta) = f' o f^{-1}(\zeta) f^{-1}(\zeta)$. Performing the change of variable $x = e^z$ give us $\dot{x} = e^z \dot{z}$, $\ddot{x} = e^z\left[(\dot{z})^2 + \ddot{z}\right]$, along with

$$\ddot{z} + h(a - \dot{z})\left[g\left(e^z\right) - b\right] = 0 \tag{6.59}$$

We can write this second-order ODE as a system of first order differential equations

$$\dot{z} = w \tag{6.60}$$

$$\dot{w} = \left[b - g\left(e^z\right)\right] h(a - w) \tag{6.61}$$

where we introduced the new variable $w = \dot{z} = \frac{\dot{x}}{x} = a - f(y)$.

By separation of variables, we have

$$\left[b - g\left(e^z\right)\right]\, dz = \frac{w}{h(a - w)}\, dw \tag{6.62}$$

Suppose that at $t = 0$, $x = x_{min}$ and $y = \overline{y}$. We want to find out how long it takes for x to reach x_{max} (at which point, y will return to $\overline{y}$). Suppose this happens at $t = T_1$. We know that for $0 \leq t \leq T_1$, $x_{min} \leq x \leq x_{max}$ and $0 < y \leq \overline{y}$. Furthermore, $\log(x_{min}) \leq z \leq \log(x_{max})$ and $0 \leq w < a - f(0)$. Hence,

$$F(w) := \int_0^w \frac{\xi}{h(a - \xi)}\, d\xi = \int_{\log(x_{min})}^{z} b - g\left(e^\eta\right)\, d\eta =: G(z) \tag{6.63}$$

Notice that

$$\begin{aligned} F(w) &= \int_0^w \frac{\xi}{f'\left(f^{-1}\left(a-\xi\right)\right) f^{-1}(a-\xi)}\, d\xi \qquad \text{let } y = f^{-1}(a-\xi)\text{, then } d\xi = -f'(y)dy \\ &= \int_{\overline{y}}^{f^{-1}(a-w)} \frac{f(y)-a}{y}\, dy \end{aligned} \tag{6.64}$$

exists for $w \in (a - Im(f))$. If f is bounded from above, then $a - \max\{f\} < w < a - f(0)$, otherwise, $-\infty < w < a - f(0)$. In any case, we will say $w_{inf} < w < a - f(0)$, with w_{inf} equal to either $a - \max\{f\}$ or $-\infty$. Given that a is in the interior of $Im(f)$, we obtain that

$$\int_0^{\overline{y}} \frac{a-f(y)}{y}\, dy = \left(\int_0^{\overline{y}/2} + \int_{\overline{y}/2}^{\overline{y}}\right) \frac{a-f(y)}{y}\, dy > \lim_{y\to 0^+} \left(a - f(\overline{y}/2)\right) \log\frac{\overline{y}}{2y} = +\infty \tag{6.65}$$

and

$$\int_{\overline{y}}^{+\infty} \frac{f(y)-a}{y}\, dy = \left(\int_{\overline{y}}^{y^*} + \int_{y^*}^{+\infty}\right) \frac{f(y)-a}{y}\, dy > \lim_{y\to +\infty} (y - y^*) \log\frac{y}{y^*} = +\infty \tag{6.66}$$

for some $y^* \in (\overline{y}, +\infty)$. It follows that $F(w)$ must satisfy the following properties

$$F(0) = 0 \tag{6.67}$$

$$F(w) \text{ is increasing for } 0 \le w < a - f(0) \tag{6.68}$$

$$F(w) \text{ is decreasing for } w_{inf} < w \le 0 \tag{6.69}$$

$$\lim_{w\to w_{inf}} F(w) = \lim_{w\to a-f(0)} F(w) = +\infty \tag{6.70}$$

Moreover, we can rewrite $G(z)$ as

$$G(z) = \overbrace{\int_{x_{min}}^{\overline{x}} \frac{b-g(x)}{x}\, dx}^{V_0} + \int_{\overline{x}}^{e^z} \frac{b-g(x)}{x}\, dx \tag{6.71}$$

Since

$$\frac{d}{dx}\left(\int_{\overline{x}}^{x}\frac{b-g(\eta)}{\eta}\,d\eta\right)=\frac{b-g(x)}{x}\begin{cases}>0 & \text{if } x<\overline{x}\\ <0 & \text{if } x>\overline{x},\end{cases} \tag{6.72}$$

we have that $G\left(\log(\overline{x})\right)=V_0$ is the maximum, while $G\left(\log(x_{min})\right)=G\left(\log(x_{max})\right)=0$ are the minima of $G(z)$ for $\log(x_{min})\le z\le\log(x_{max})$.

One can then define $F_1(w)$ to be the restriction of $F(w)$ to $\big[0,a-f(0)\big)$, from (6.63) we have

$$F_1^{-1}\left(G(z)\right)=w=\frac{dz}{dt} \tag{6.73}$$

which implies that

$$T_1=\int_{\log(x_{min})}^{\log(x_{max})}\frac{1}{F_1^{-1}\left(G(z)\right)}\,dz \tag{6.74}$$

For the rest of the cycle, suppose now that at $t=0$, $x=x_{max}$ and $y=\overline{y}$. We wish to find how long it takes for x to reach x_{min} (when also $y=\overline{y}$). Denote this time length T_2. We have that $z(0)=\log(x_{max})$, $z(T_2)=\log(x_{min})$, $w(0)=0$, $w(T_2)=0$ and $w(t)\le 0$ for $0\le t\le T_2$. From (6.62), we have

$$\begin{aligned}F(w)=\int_0^w\frac{\xi}{h(a-\xi)}\,d\xi=\int_{\log(x_{max})}^{z}b-g\left(e^{\eta}\right)\,d\eta=\overbrace{\int_{\overline{x}}^{x_{max}}\frac{g(x)-b}{x}\,dx}^{V_0}+\int_{\overline{x}}^{e^z}\frac{b-g(x)}{x}\,dx\\ =G(z)\end{aligned} \tag{6.75}$$

Similarly, we can define $F_2(w)$ to be the restriction of $F(w)$ to $(-\infty,0]$ and conclude that

$$T_2=\int_{\log(x_{max})}^{\log(x_{min})}\frac{1}{F_2^{-1}\left(G(z)\right)}\,dz \tag{6.76}$$

□

Remark 6.2. The zero-order model for ω_0 and λ_0 (6.1) can be written as our generalization of the Lotka-Volterra model if we define its variables as follows

$$\begin{aligned} x &= \lambda_0 \\ y &= \omega_0 \\ g(x) &= \Phi(x) \\ b &= \alpha \\ f(y) &= -\frac{\kappa(1-y)}{\nu} \\ a &= -(\alpha+\beta+\delta) \end{aligned} \tag{6.77}$$

By Theorem 6.1, λ_0 cycles between λ_{min} and λ_{max}, roots of

$$V_0 = \int_{\overline{\lambda}_0}^{x} \frac{\Phi(\eta)-\alpha}{\eta}\, d\eta \tag{6.78}$$

The function F is then

$$F(w) = \int_{\overline{\omega}_0}^{1-\kappa^{-1}(\nu(w+\alpha+\beta+\delta))} \frac{\nu(\alpha+\beta+\delta)-\kappa(1-y)}{\nu y}\, dy \tag{6.79}$$

where $\overline{\omega}_0 = 1-\kappa^{-1}(\nu(\alpha+\beta+\delta))$. As consequence, the functions F_1 and F_2 are the restrictions of this function F to $\big[0, \kappa(1)/\nu - (\alpha+\beta+\delta)\big)$ and $(-\infty, 0]$, respectively. The function G, in turn, can be written as

$$G(z) = V_0 - \int_{\overline{\lambda}_0}^{e^z} \frac{\Phi(x)-\alpha}{x}\, dx \tag{6.80}$$

where $\overline{\lambda}_0 = \Phi^{-1}(\alpha)$.

Around the equilibrium point $(\overline{\omega}_0, \overline{\lambda}_0)$, linearization shows that the period converges to

$$\frac{2\pi}{\sqrt{\overline{\omega}_0\overline{\lambda}_0\Phi'(\overline{\lambda}_0)\kappa'(1-\overline{\omega}_0)/\nu}} \tag{6.81}$$

6.4 Example

In this section, we take the parameters as (3.24), besides using the functions Φ and κ as defined in (3.25), and (5.36), calibrated according to (3.26) and (5.37).

First, to illustrate Remark 6.2, we refer to Figure 6.1, where the period of the zero-order model (6.1) for different starting values of (ω_0, λ_0) is illustrated. There seems to be a linear relationship between V_0 and T, with vertical intercept at 3.2. This value agrees with (6.81), which gives us a base period of 3.0039.

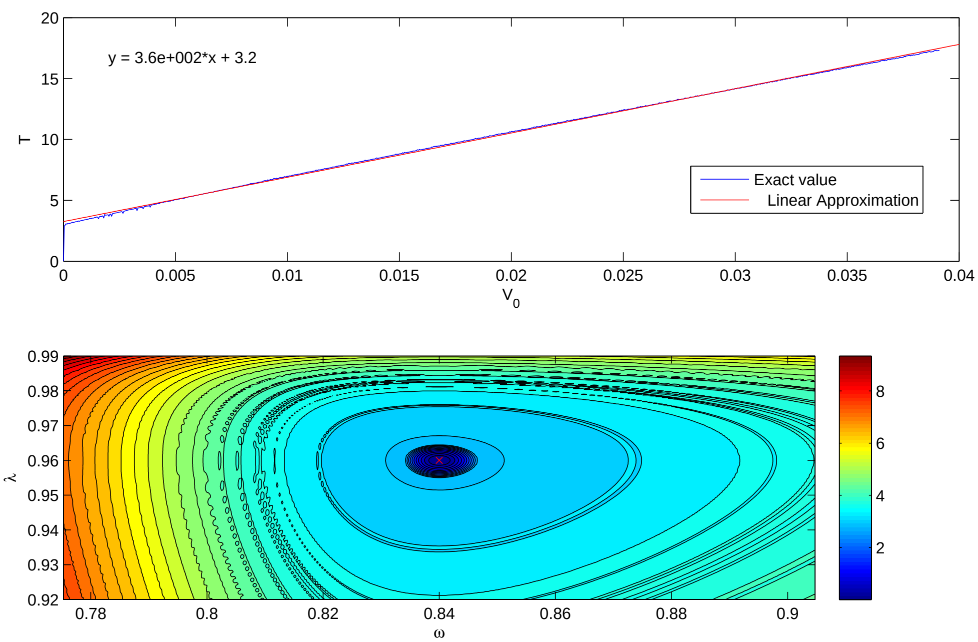

Figure 6.1: Period of the zero-order system.

Moreover, the limit cycle described in (6.25) can be observed in Figure 6.2, while the quality of the overall approximation developed in this chapter can be assessed through Figure 6.3.

As expected, due to the fact that the monodromy matrix (4.165) cannot be di-

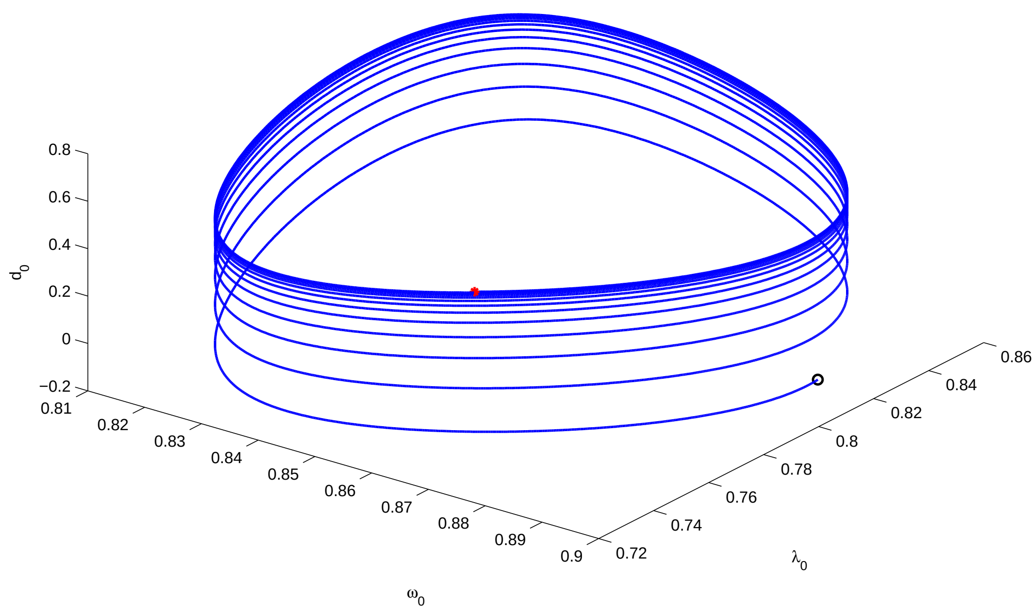

Figure 6.2: Example of the zero-order solution $(\omega_0, \lambda_0, d_0)$, depicting the limit cycle in d_0 as described in (6.25).

agonalized, the first-order solution grows unbounded with time, spoiling the approximation when t is large. In our numerical example, with a value of the interest rate of order 10^{-4}, the approximation seems to be almost indistinguishable from the true solution until at least $t = 100$, whereas once we raise the interest rate to 10^{-3}, at $t = 80$, we can already see the growing trend of the error taking place.

6.5 Conclusion

In this chapter, we have studied the Keen model under regimes of extremely low levels of the real rate of interest. Through perturbation methods, we are able to derive simpler dynamical systems that can be fully solved analytically. More specifically, we find that the zero order variables ω_0, λ_0 belong to a cycle specified by an energy functional, with period that can also be analytically determined. The zero order debt

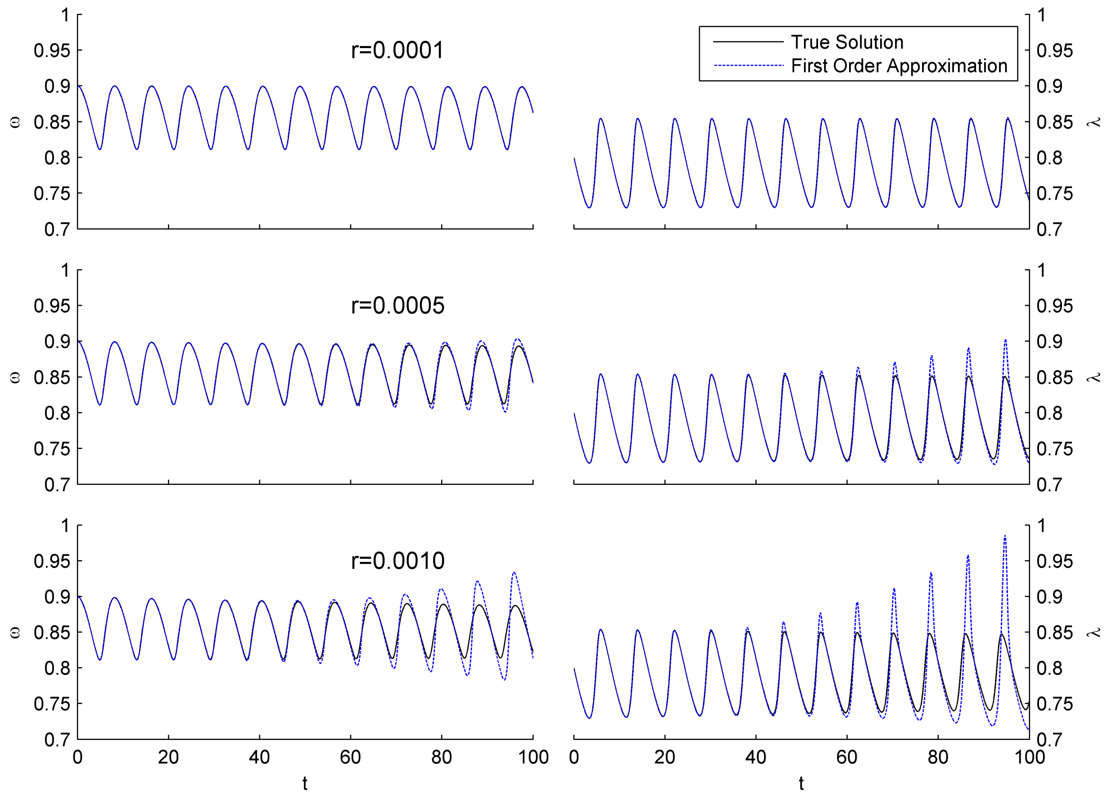

Figure 6.3: Examples of solutions of both (5.7), and (6.1) plus (6.32), for different values of the real interest rate. On the left column, we have the evolution of ω versus t, while on the right column we have $\lambda \times t$. The exact solution is drawn in a solid black line, whereas the approximate solution is represented by the dashed blue line.

d_0, on the other hand, converges a steady state periodic function of time, meaning that the zero order solution, as a whole, converges to a limit cycle.

Unfortunately, as the monodromy matrix associated to the problem (6.37) cannot be diagonalized, the first-order solution inevitable grows quadratically with time, compromising the accuracy of the approximation for large values of t. Still, as numerical examples show, when the interest rate is of order 10^{-4}, the error is virtually inexistent for at least a century.

Chapter 7

Distributed Time Delay

Behind the capital goods dynamical equation (5.2) lies the assumption that capital goods are added to the system as soon as the cash is invested. In reality, capital goods expansion cannot be instantly developed. Every time capitalists decide to invest in a new enterprise, the money spent at time t will only generate revenues at a later time $t + \tau$. In the context of the Keen model, this means that capital should not respond to new investment immediately. We can attempt to capture this effect by adding a delay in the the capital dynamics, representing the amount of time between when the profits are invested and when new capital is effectively added to the economy.

Previously, in the Keen model, capital would be driven by

$$\dot{K} = \kappa(\pi)Y - \delta K \tag{7.1}$$

Introducing a discrete delay can be achieve by replacing the previous equation by

$$\dot{K}(t) = \kappa\big(\pi(t-\tau)\big)Y(t-\tau) - \delta K(t) \tag{7.2}$$

When analyzing the dynamical equation (7.2), one needs to deploy the arsenal of delay differential equations, notably quite involving. We can alternatively emulate the time delay through a series of exponentially distributed times, as we will see next.

7.1 Alternative formulation

The technique described in this section is well documented in the Mathematical Biology literature (see, for example, [AD80]). Ultimately, the latent time for capital expansion will follow the Erlang distribution. The main idea behind this trick is that the sum of independent exponentially distributed random variables is Gamma distributed[1]. If we break the project implementation stage in n sub-stages, each exponentially distributed with mean $\frac{\tau}{n}$, then the total completion time will follow an Erlang distribution with shape parameter n and scale parameter $\frac{\tau}{n}$. In the limit case where $n \to \infty$, this converges to the delay differential equation described by (7.2).

We will introduce a new variable, Θ, to represent the amount of money invested in projects yet to be completed. In the simplest case, where $n = 1$, we have

$$\dot{\Theta} = \kappa(\pi)Y - \frac{1}{\tau}\Theta \tag{7.3}$$

Equation (7.3) contains the inflow of capital goods, $\kappa(\pi)$ and the outflow representing projects being completed at an exponential rate. The capital dynamics, in this case, is simply

$$\dot{K} = \frac{1}{\tau}\Theta - \delta K \tag{7.4}$$

Capital assets are now completed according to the inflow represented by Θ and depreciate at an exponential rate given by δ. The debt dynamics remain unchanged, as it is driven by the amount of money being currently invested.

In the general case, we have n intermediate investment stages.

[1] If $X \sim \Gamma\left(n, \frac{\tau}{n}\right)$ for $n \in \mathbb{N}$, then $X \sim \mathit{Erlang}\left(n, \frac{n}{\tau}\right)$. Also, $\mathbb{E}[X] = \tau$.

$$\begin{aligned}
\dot{\Theta}_1 &= \kappa(\pi)Y - \frac{n}{\tau}\Theta_1 \\
\dot{\Theta}_2 &= \frac{n}{\tau}\left(\Theta_1 - \Theta_2\right) \\
&\vdots \\
\dot{\Theta}_n &= \frac{n}{\tau}\left(\Theta_{n-1} - \Theta_n\right) \\
\dot{K} &= \frac{n}{\tau}\Theta_n - \delta K \\
\dot{D} &= \left(\kappa(\pi) - \pi\right)Y
\end{aligned} \tag{7.5}$$

Using several sub-stages, which can be understood either as a purely mathematical device or even as the several intermediate stages present in real capital projects, the projects take an Erlang distributed amount of time to be completed.

Defining $\theta_k = \frac{\Theta_k}{Y}, k = 1, \cdots, n$, we can fully specify the model using $\{\omega, \lambda, d, \theta_1, \theta_2, \cdots, \theta_n\}$. A bit of algebra leads to

$$\begin{aligned}
\dot{\omega} &= \omega(\Phi(\lambda) - \alpha) \\
\dot{\lambda} &= \lambda\left(\frac{n}{\tau\nu}\theta_n - (\alpha + \beta + \delta)\right) \\
\dot{d} &= \kappa(\pi) - \pi - d\left(\frac{n}{\tau\nu}\theta_n - \delta\right) \\
\dot{\theta}_1 &= \kappa(\pi) - \theta_1\left[\frac{n}{\tau}\left(1 + \frac{1}{\nu}\theta_n\right) - \delta\right] \\
\dot{\theta}_2 &= \frac{n}{\tau}\left(\theta_1 - \theta_2\right) - \theta_2\left(\frac{n}{\tau\nu}\theta_n - \delta\right) \\
&\vdots \\
\dot{\theta}_k &= \frac{n}{\tau}\left(\theta_{k-1} - \theta_k\right) - \theta_k\left(\frac{n}{\tau\nu}\theta_n - \delta\right) \\
&\vdots \\
\dot{\theta}_n &= \frac{n}{\tau}\left(\theta_{n-1} - \theta_n\right) - \theta_n\left(\frac{n}{\tau\nu}\theta_n - \delta\right)
\end{aligned} \tag{7.6}$$

The added terms on the $\dot{\theta}_k$ equations are due to the evolution of Y, the economy output (for more details, refer to the derivation of the Keen model in Chapter 5).

This model has an equilibrium, henceforward denoted as the "good" equilibrium, determined by

$$\begin{aligned}
\hat{\lambda}_1 &= \Phi^{-1}(\alpha) \\
\hat{\theta}_{n,1} &= \frac{\tau\nu}{n}(\alpha+\beta+\delta) \\
&\vdots \\
\hat{\theta}_{n-k,1} &= \hat{\theta}_{n,1}\left[\frac{\tau}{n}(\alpha+\beta+n/\tau)\right]^k \\
&\vdots \\
\hat{\theta}_{1,1} &= \hat{\theta}_{n,1}\left[\frac{\tau}{n}(\alpha+\beta+n/\tau)\right]^{n-1} \\
\hat{\pi}_1 &= \kappa^{-1}\left[\hat{\theta}_{1,1}(\alpha+\beta+n/\tau)\right] \\
\hat{d}_1 &= \frac{\kappa(\hat{\pi}_1)-\hat{\pi}_1}{\alpha+\beta} \\
\hat{\omega}_1 &= 1-\hat{\pi}_1-r\hat{d}_1
\end{aligned} \tag{7.7}$$

If we linearize the system (7.6) around this point, we arrive at the following Jacobian matrix:

$$
\begin{bmatrix}
0 & \hat{\omega}_1 \Phi'(\hat{\lambda}_1) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{n}{\tau\nu}\hat{\lambda}_1 \\
1-\kappa'(\hat{\pi}_1) & 0 & r\left(1-\kappa'(\hat{\pi}_1)\right)-(\alpha+\beta) & 0 & 0 & \dots & 0 & 0 & -\frac{n}{\tau\nu}\hat{d}_1 \\
-\kappa'(\hat{\pi}_1) & 0 & -r\kappa'(\hat{\pi}_1) & -\left(\alpha+\beta+\frac{n}{\tau}\right) & 0 & \dots & 0 & 0 & -\frac{n}{\tau\nu}\hat{\theta}_{1,1} \\
0 & 0 & 0 & \frac{n}{\tau} & -\left(\alpha+\beta+\frac{n}{\tau}\right) & \dots & 0 & 0 & -\frac{n}{\tau\nu}\hat{\theta}_{2,1} \\
\vdots & \vdots & \vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{n}{\tau} & -\left(\alpha+\beta+\frac{n}{\tau}\right) & -\frac{n}{\tau\nu}\hat{\theta}_{n-1,1} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{n}{\tau} & -\left(\frac{n}{\tau}+2\alpha+2\beta+\delta\right)
\end{bmatrix}
\tag{7.8}
$$

There is also a second equilibrium (from now referred to as the "bad" equilibrium), characterized by

$$\begin{aligned}
\hat{\lambda}_2 &= 0 \\
\hat{\omega}_2 &= 0 \\
\hat{\theta}_{1,2} &= 0 \\
&\vdots \\
\hat{\theta}_{k,2} &= 0 \\
&\vdots \\
\hat{\theta}_{n,2} &= 0 \\
\hat{d}_2 &\to +\infty
\end{aligned} \tag{7.9}$$

which, as one would expect, expresses the collapsed state of the economy.

7.2 Bifurcations

For the numerical experiments conducted in this chapter, we adopted the fundamental constants according to (3.24). The function Φ is the same as (3.25), with parameters given by (3.26). In addition, the function κ is the one defined in (5.36), but calibrated according to

$$\begin{aligned}
\kappa(-\infty) = 0, &\qquad \kappa(+\infty) = 1, \\
\overline{\pi}_1 = 0.16, &\qquad \kappa'(\overline{\pi}_1) = 500
\end{aligned} \tag{7.10}$$

Armed with the Jacobian matrix, we can investigate what pairs of τ and n produce a locally stable "good" equilibrium (that is, for what values of τ and n, matrix (7.8) contains only eigenvalues with negative real part). Figure 7.1 depicts the threshold at which stability is lost.

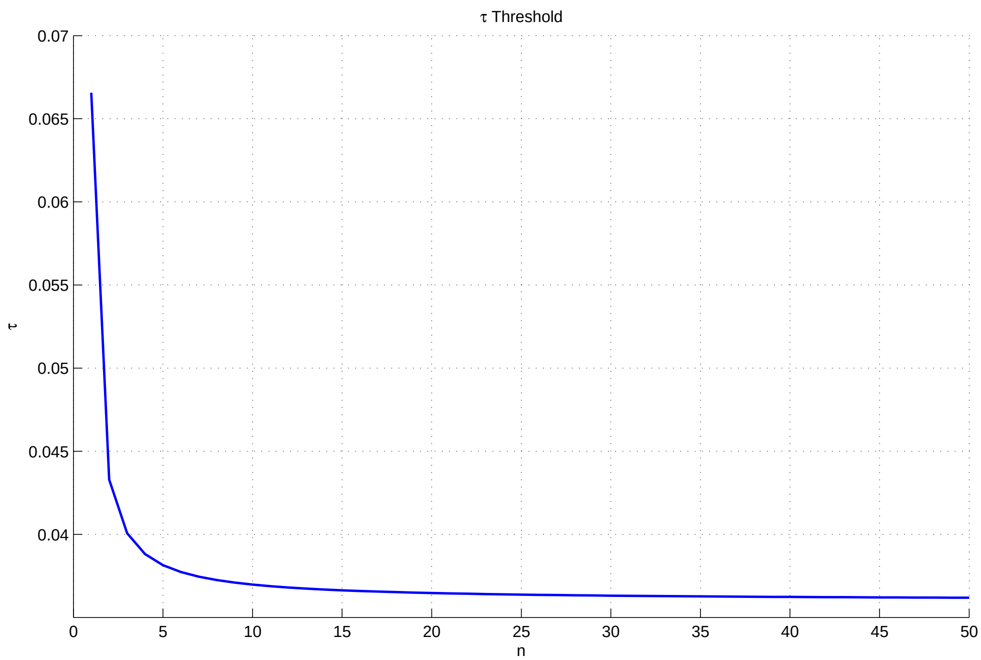

Figure 7.1: Threshold values of τ where local stability of the good equilibrium disappears (for the model with project development delay). For each n, the "good" equilibrium is stable for values of τ below this curve.

We can see that as the number of intermediate investment stages increases, the critical value for the average completion time decreases. There seems to be an asymptote around $t = 0.030$, about ten days. This value is arbitrarily determined in terms of the parameters chosen for this section.

Fixing the number of intermediate investment stages to $n = 10$, we can study bifurcations with respect to the parameter τ. Figure 7.2 shows bifurcation diagram containing the amplitude of ω for values of τ ranging from zero to one year(s). There seems to be a Hopf bifurcation for τ somewhere between 0.035 and 0.040. The period of the cycles is shown in Figure 7.3.

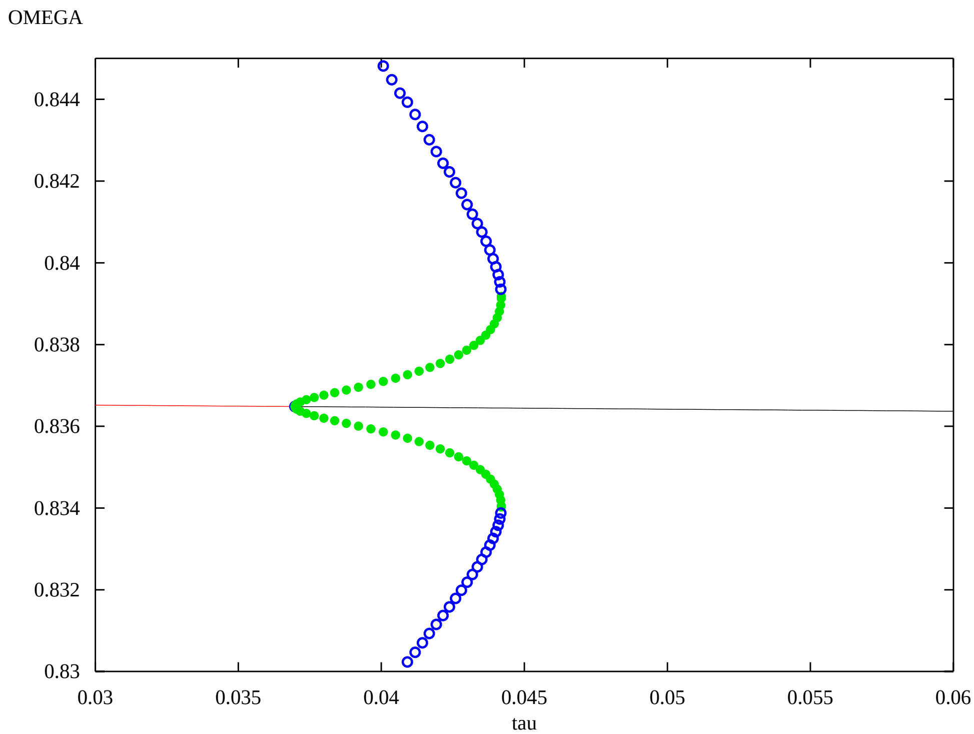

Figure 7.2: Bifurcations diagram for τ when $n = 10$ showing amplitude of ω versus τ. The solid (black) line denote local instability, while the solid (red) line represent local stability. Meanwhile, solid (green) circles denote stable limit cycles, while empty (blue) circles correspond to unstable limit cycles.

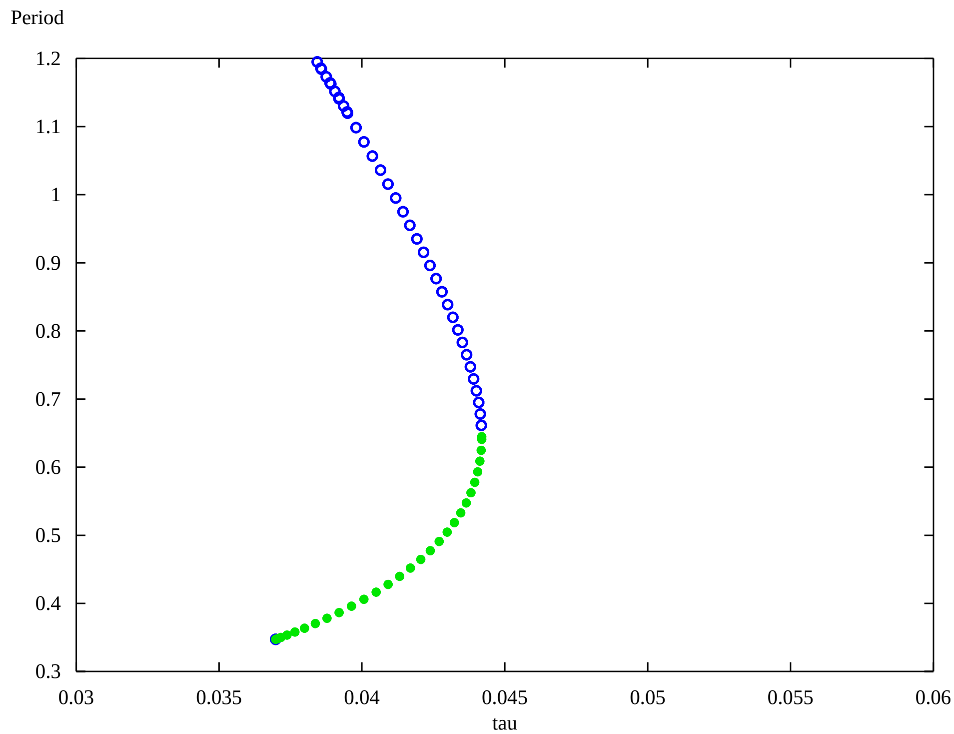

Figure 7.3: Bifurcations diagram for τ when $n = 10$ showing period of oscillations versus τ. Empty (blue) circles denote unstable limit cycles, while solid (green) circles correspond to stable limit cycles.

As expected, the "good" equilibrium, initially locally stable when τ is smaller than around 0.037, becomes locally unstable for large values of τ. This threshold value agrees with the results observed in Figure 7.1. Remarkably, the fixed point unfolds into a stable cycle at the supercritical Hopf bifurcation, indicating the existence of periodic solutions. These stable cycles with period shown in Figure 7.3 are only feasible for $\tau \in (0.03698, 0.04419)$. Moreover, unstable cycles with larger periods occur for higher amplitudes of ω.

A natural question is how these results would be affected if the number of multi-stages was not ten, but two, five, or thirty? As well, what would happen in the limiting case when the we use infinite intermediate stages of investment? To shed some light on these concerns, we repeated the previous analysis for different values of n. Figure 7.4 shows the bifurcation diagram, together with the 10% and 90% percentiles of the time delay for n in $\{2, 5, 10, 20, 50, 75\}$. The limiting case when $n = +\infty$, that is, when the time delay is discrete, was analyzed with brute force. By simulating the model with capital dynamics given by (7.2) for a long period of time, with τ varying from zero to one year, and recording the long-term behaviour of ω, we can observe the stable attractors, be it a fixed point or a limit cycle (see Figure 7.5).

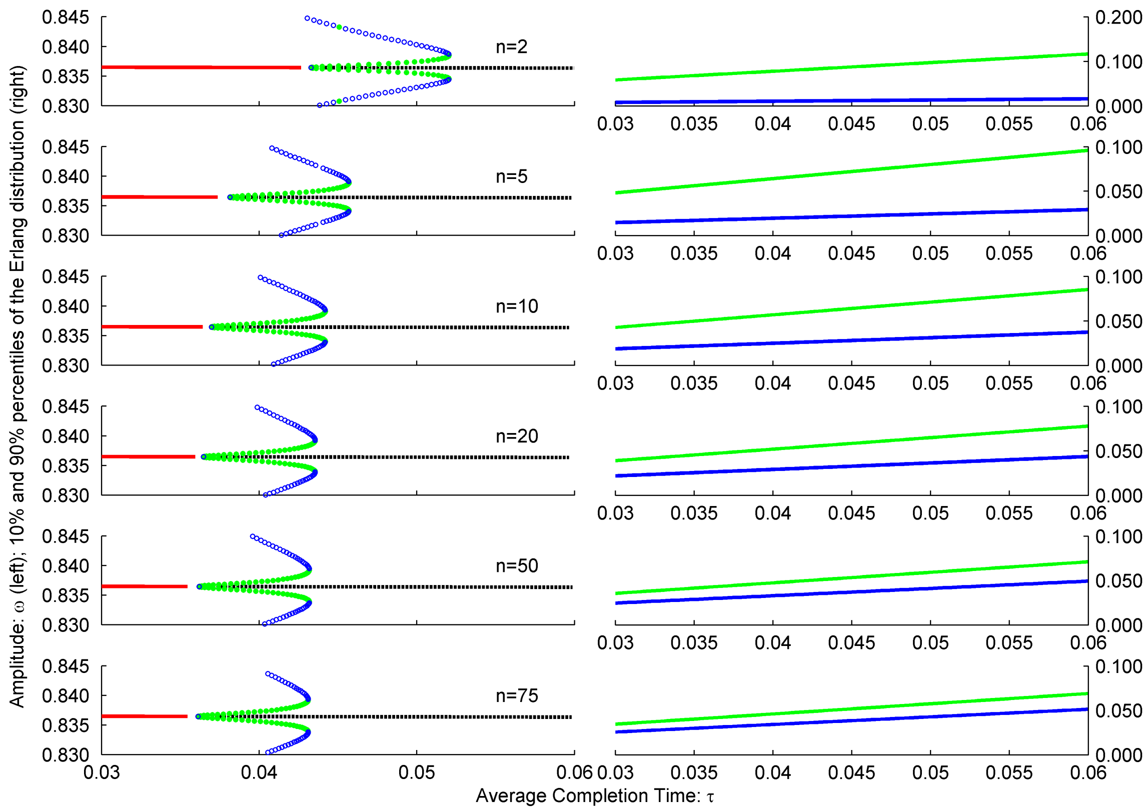

Figure 7.4: Bifurcation diagrams on the left. 10% and 90% percentiles of the delay distribution on the right.

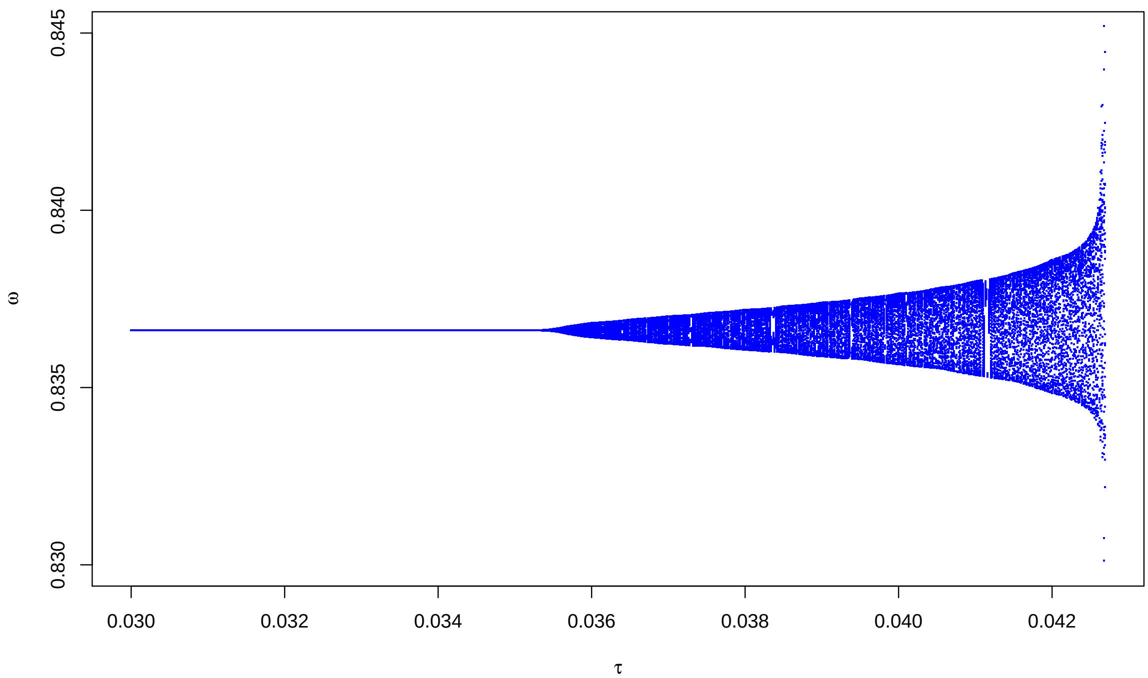

Figure 7.5: Brute Force bifurcation diagram for the model with discrete delay.

Altogether, we can study the dual effect of varying the number of investment stages and the average time for completion on the stability of the system in a two-parameter bifurcation diagram, as depicted in Figure 7.6. The threshold value τ_{min} corresponds to the point at which the good equilibrium, previously locally stable, becomes unstable. On the other hand, τ_{max} denotes the maximum value of τ for which there exists the stable limit cycle. The unstable limit cycle is not particularly interesting as it does not attract solutions. Besides, it seems to exist for values of τ that already generate the stable limit cycle. As shown in Figure 7.6, both threshold values τ_{min} and τ_{max} drop quickly as soon as n grows from zero, settling thereafter around 0.03534 and 0.04269, respectively.

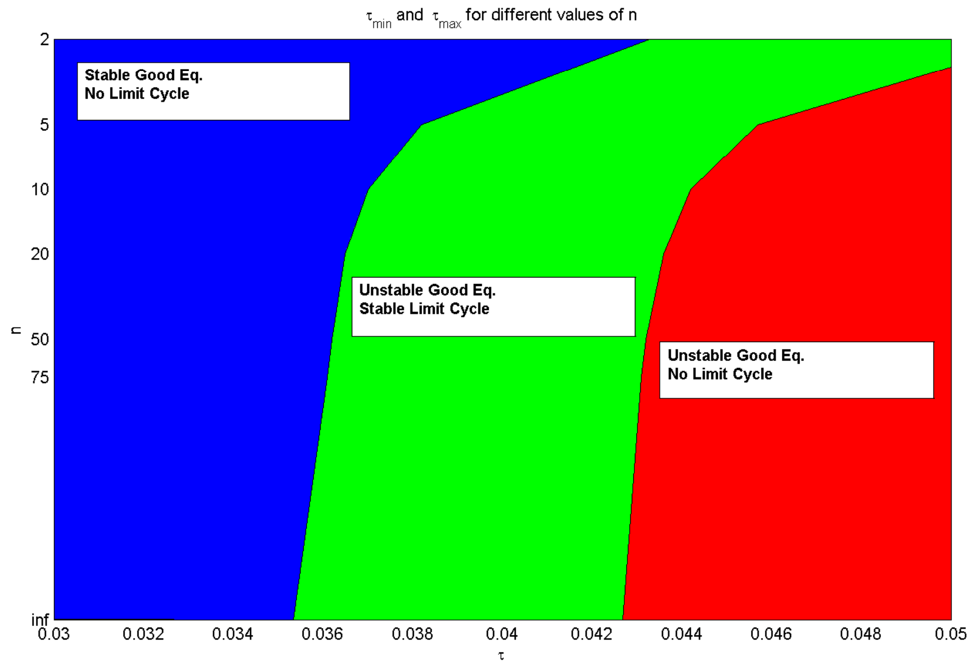

Figure 7.6: Two parameter (τ vs n) bifurcation diagram.

The transient behaviour of the system is studied next. First, we are interested in the case when the good equilibrium is still locally stable. Figure 7.7 was constructed with different values of n and τ. Apparently, higher model dimension and longer average completion times have a similar effect on stable solutions, though in different magnitudes: both seem to extend the transient convergence time, adding more oscillations to the result. Alternatively, Figure 7.8 exhibits the convergence to the limit cycle, as the values of n and τ were intentionally chosen inside the region where the good equilibrium is unstable, yet the stable limit cycle is present.

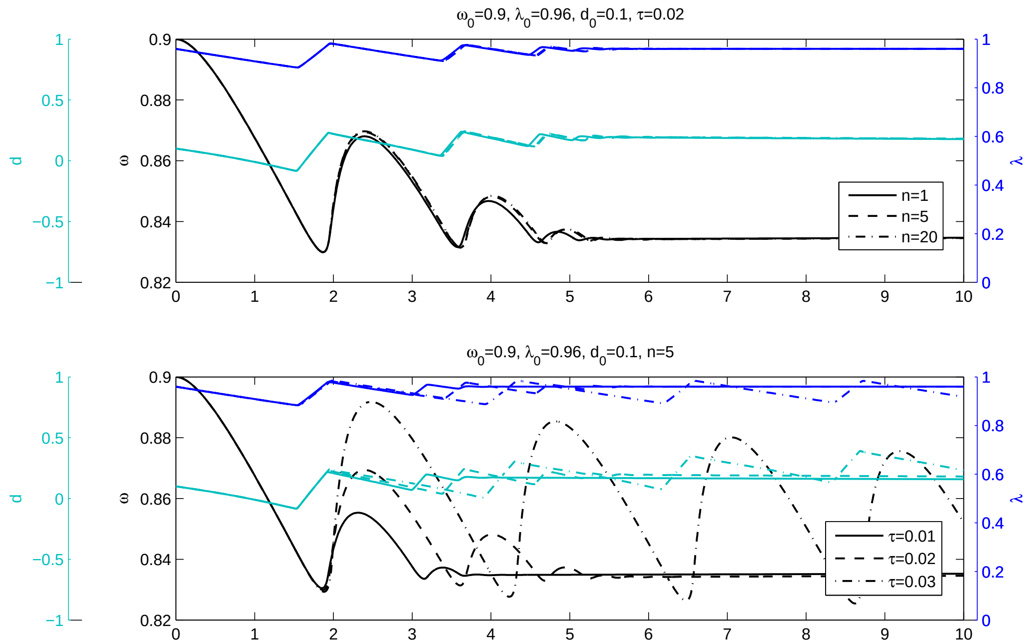

Figure 7.7: Solutions converging to the "good" equilibrium, for different values of n and τ.

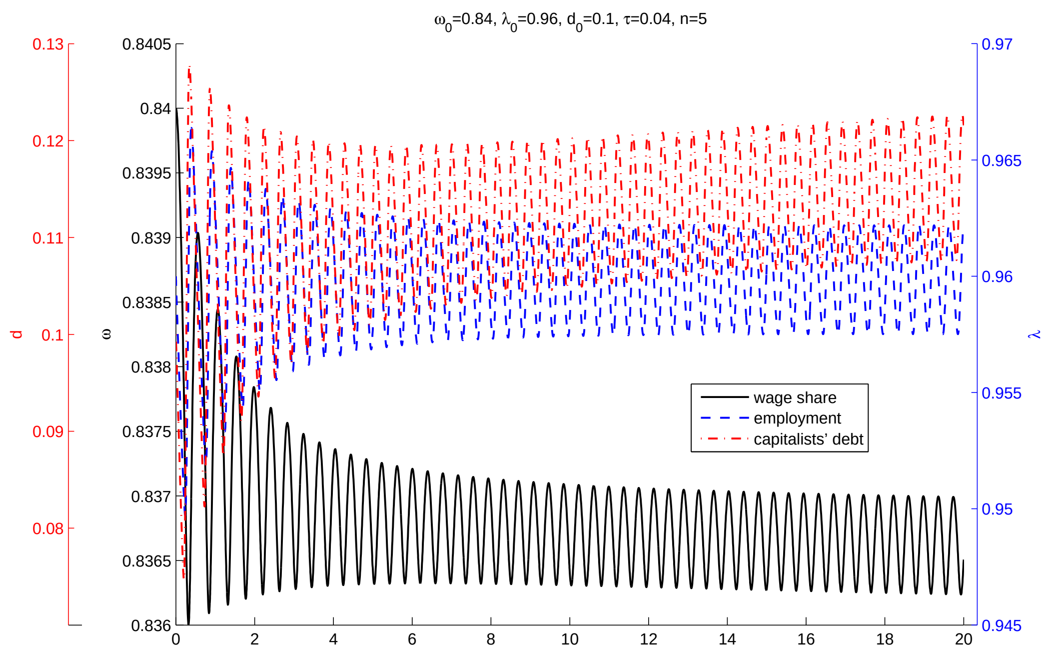

Figure 7.8: Solutions converging to the stable limit cycle.

7.3 Conclusion

In this chapter, we studied a modified model where projects are not immediately completed, but take an Erlang distributed time to be developed. The resulting model exhibits the same stability properties as the Keen model, namely two stable equilibria, when the mean time for completion, τ, is small enough.

Beyond a threshold, which depends on the number of intermediate stages one chooses to model, the good equilibrium becomes locally unstable, giving rise to a stable limit cycle. As this parameter increases, we verify a complete instability around the good equilibrium, where the limit cycle vanishes, leaving the bad equilibrium as the single stable attractor.

Put differently, when the gap between the time investment decisions are made and when projects start generating revenue becomes too large, the economy ceases to see positive effects from the investments, which inevitably leads to the collapsed state.

Chapter 8

Government Intervention

Ever since the financial crisis that brought the world's economies to their knees in 2008, Minsky's take on financial fragility has been a growing trend amongst economists. Accordingly, mathematical formulations of his Financial Instability Hypothesis have been gaining increasing popularity in the economics literature. Surprisingly, most of these models have not given enough substance to the government role, confining it to the task of regulating and/or issuing bonds to be purchased by more active players such as firms and households. For instance, as Santos [DS05] points out, Taylor and O'Connell in their early influential article [TO85] fall short of completely specifying the government policy, thus leaving room for "hidden" assumptions, that is, model ingredients that were not acknowledged by the authors, let alone analyzed. On the contrary, we model the government intervention explicitly, carefully analyzing its impact on the overall economy.

As Minsky himself had already pointed out throughout [Min82], the debt-deflation spiral can be interrupted with an appropriate intervention by the government, as government spending enter the Kalecki equation increasing firm profit. We model this behaviour by introducing government expenditure, subsidies and taxation into the Keen model in Section 8.1. We then perform local stability analysis of the many equilibria available in Sections 8.1.1 and 8.1.2. As before, we identify a "good" equilibrium, which proves to be stable under regular conditions, as expected. On the other hand, there are many undesirable equilibria, both with finite and infinite (negative) levels of profit. Fortunately, the finite undesirable equilibria are all either unstable or unachievable under usual assumptions. Moreover, we prove in Proposition 8.1 that all the equilibria with exploding negative profit are either unstable or unachievable provided the size of government subsidies is large enough around zero employment rate. In other words, even when the bad equilibrium is locally stable in the model without government, we are able to design the government intervention in a specific way that it destabilizes these unwanted fixed points associated to economic crises.

Moreover, we present in Section 8.2 our main result. As opposed to local stability analysis, persistence theory [ST11b] studies the behaviour of the solutions around a specific value for a single variable alone. Instead of answering questions about the convergence to a point in the space, we are now concerned about the convergence to a hyper-plane. Widely used in mathematical biology, where mathematicians are interested in the survival of a specific species over the long term, or whether a certain disease-control protocol will be able to eradicate the pathogen, we bring the same notion to macroeconomics. In our context, we are intrigued about the long-term "survival" of key economic variables, for instance profits, or employment. After proving preliminary results addressing positive exploding profits in Proposition 8.2, we show in Propositions 8.3 and 8.4 that under a collection of alternative reasonable conditions on government policy, we obtain uniform weak persistence for both the employment rate and the capitalists' profit. For a precise definition of persistence, we refer to Appendix C, though put simply, the main result proved in this chapter is that if the government is willing to be responsive enough in times of crises, both the employment rate and firm profit are guaranteed to not remain trapped at arbitrarily small values. Like any persistence result, these statements are global, the initial state of the economy is not addressed at all in the hypothesis. These represent a sharp improvement from the model without government intervention, where employment and profits were guaranteed to converge to zero and negative infinity, respectively, and stay there forever, if the initial conditions were sufficiently bad.

8.1 Introducing government

In accordance to Table 2.3, government intervention can be introduced in the model through expenditures G_e, subsidies GS and taxation T. We reserve the specification of government expenditure for the end of this section. For now, we define subsidies and taxes in the form

$$GS(t) = G_b(t) + G_s(t), \tag{8.1}$$

$$T(t) = T_b(t) + T_s(t), \tag{8.2}$$

where

$$\dot{G}_b = \Gamma_b(\lambda)Y, \qquad \dot{G}_s = \Gamma_s(\lambda)G_s, \tag{8.3}$$

$$\dot{T}_b = \Theta_b(\pi)Y, \qquad \dot{T}_s = \Theta_s(\pi)T_s. \tag{8.4}$$

We interpret G_b and T_b as base–level subsidies and taxation, whose dynamics depend primarily on the overall state of the economy as measured by the level of output Y. On the other hand, we interpret G_s and T_s as stimulative subsidies and taxation, supposed to react exponentially fast to changes in employment and firms profits with rates given by functions Γ_s and Θ_s. Initially, we only make the following general assumptions on the subsidies and taxation structural functions:

$$\Gamma_b'(\lambda) < 0 \text{ and } \Gamma_s'(\lambda) < 0 \text{ on } (0,1) \tag{8.5}$$

$$\Theta_b'(\pi) > 0 \text{ and } \Theta_s'(\pi) > 0 \text{ on } (-\infty,\infty) \tag{8.6}$$

$$\exists \theta_b(-\infty) = \lim_{\pi\to-\infty} \theta_b(\pi) \tag{8.7}$$

$$\exists \theta_s(-\infty) = \lim_{\pi\to-\infty} \theta_s(\pi) < \lim_{\pi\to-\infty} \frac{\kappa(\pi)}{\nu} - \delta = \frac{\kappa(-\infty)}{\nu} - \delta \tag{8.8}$$

$$\lim_{\pi\to-\infty} \pi^2\theta_s'(\pi) < \infty \tag{8.9}$$

Denoting capitalist and government debt respectively by D_k and D_g, and defining $g_b = G_b/Y$, $g_s = G_s/Y$, $g_e = G_e/Y$, $\tau_b = T_b/Y$, $\tau_s = T_s/Y$, $d_k = D_k/Y$, $d_g = D_g/Y$, it follows from Section 2.2.2 that the profit share of capitalists is now

$$\pi = 1 - \omega - rd_k + g_b + g_s - \tau_b - \tau_s \tag{8.10}$$

and government debt evolves according to

$$D_g = rD_g + G_b + G_s + G_e - T_b - T_s \tag{8.11}$$

To simplify the notation, let the rate of growth of the economy be denoted by

$$\mu(\pi) := \frac{\kappa(\pi)}{\nu} - \delta. \tag{8.12}$$

A bit of algebra leads to the following eight–dimensional system:

$$
\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\mu(\pi) - \alpha - \beta\right] \\
\dot{d}_k &= \kappa(\pi) - \pi - d_k\mu(\pi) \\
\dot{g}_b &= \Gamma_b(\lambda) - g_b\mu(\pi) \\
\dot{\tau}_b &= \Theta_b(\pi) - \tau_b\mu(\pi) \\
\dot{g}_s &= g_s\left[\Gamma_s(\lambda) - \mu(\pi)\right] \\
\dot{\tau}_s &= \tau_s\left[\Theta_s(\pi) - \mu(\pi)\right] \\
\dot{d}_g &= d_g\left(r - \mu(\pi)\right) + g_e + g_b + g_s - \tau_b - \tau_s
\end{aligned}
\tag{8.13}
$$

Notice that the capitalist profit share π in (8.10) does not depend on the government debt ratio d_g, which implies that the last equation in (8.13) can be solved separately from the rest of the system. Observe further that we can write

$$
\begin{aligned}
\dot{\pi} &= -\dot{\omega} - r\dot{d}_k + \dot{g}_s + \dot{g}_b - \dot{\tau}_s - \dot{\tau}_b \\
&= -\omega(\Phi(\lambda) - \alpha) - r(\kappa(\pi) - \pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_s(\pi) - \tau_s\Theta_s(\pi) \\
&+ (rd_k - g_s - g_b + \tau_s + \tau_b)\mu(\pi) \\
&= -\omega(\Phi(\lambda) - \alpha) - r(\kappa(\pi) - \pi) + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_s(\pi) - \tau_s\Theta_s(\pi),
\end{aligned}
$$

so that the model reduces to the following five–dimensional system:

$$
\begin{aligned}
\dot{\omega} =& \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} =& \lambda\left[\mu(\pi) - \alpha - \beta\right] \\
\dot{g}_s =& g_s\left[\Gamma_s(\lambda) - \mu(\pi)\right] \\
\dot{\tau}_s =& \tau_s\left[\Theta_s(\pi) - \mu(\pi)\right] \\
\dot{\pi} =& -\omega(\Phi(\lambda) - \alpha) - r(\kappa(\pi) - \pi) + (1 - \omega - \pi)\mu(\pi) \\
& + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi)
\end{aligned}
\tag{8.14}
$$

We shall base our analytic results on the reduced system (8.14), since this will be enough to characterize the equilibria in which the economy either prospers or collapses. Observe that when working with the reduced system (8.14), we cannot recover d_k, g_b and τ separately, but rather the combination

$$rd_k - g_b + \tau_b = 1 - \omega - \pi + g_s - \tau_s \tag{8.15}$$

For numerical simulations, however, we compute the trajectories for the full system (8.13), so that the evolution of each individual variable can be followed separately.

For completeness, we can express the dynamics of total debt $d = d_k + d_g$ as

$$\begin{aligned} \dot{d} = \dot{d}_k + \dot{d}_g &= \kappa(\pi) - \pi - d_k\mu(\pi) + rd_g - d_g\mu(\pi) + g_e + g_b + g_s - \tau_b - \tau_s \\ &= \kappa(\pi) - (1 - \omega - rd) + g_e \end{aligned} \tag{8.16}$$

The hyperplanes $g_s = 0$ and $\tau_s = 0$ are invariant manifolds, indicating that if the initial value for either g_s or τ_s is positive (or negative), the corresponding solution is also positive (or negative). In general, $\tau_s \geq 0$, as we understand that tax cuts will be captured by the decreasing function θ_s as opposed to negative taxes. As well, we will typically have $g_s > 0$, as we want to represent a government attempting to stimulate the economy with subsidies, although one could also have $g_s \leq 0$ in the case of austerity measures intended to reduce the government deficit (as a naive attempt to decrease the debt) when the economy performs badly.

We now return to the specification of government expenditures G_e. Observe that, since G_e does not affect the profit share in (8.10), its dynamics can be freely chosen without altering the solution of either the reduced system (8.14) or the full system (8.13). In fact, the only other variable affected by G_e is government debt, which is driven by (8.11).

For example, if we postulate the dynamics for expenditures in the form

$$\dot{G}_e = \Gamma(t, \omega, \lambda, \pi, g_s, \tau_s, G_e, Y), \tag{8.17}$$

we obtain

$$\dot{g}_e = \frac{\Gamma(\omega, \lambda, \pi, g_s, \tau_s, G_e, Y)}{Y} - g_e\mu(\pi) \tag{8.18}$$

In other words, as long as the dynamics for government expenditures does not depend explicitly on the level of government debt, equation (8.18) can be solved separately first and then used to solve the dynamics of d_g. Equivalently, we can model the government expenditure ratio directly as a function $g_e = g_e(t, \omega, \lambda, \pi, g_s, \tau_s)$.

8.1.1 Finite-valued equilibria

It is straightforward to see that the only possible finite-valued equilibria for the system (8.14) are given by the following six cases:

1. Define

$$\begin{aligned} \overline{\lambda}_1 &= \Phi^{-1}(\alpha) \\ \overline{\pi}_1 &= \mu^{-1}(\alpha + \beta) \end{aligned} \tag{8.19}$$

so that $\dot{\omega} = \dot{\lambda} = 0$. Discarding the structural coincidences $\Gamma_s(\overline{\lambda}_1) = \alpha + \beta$ or $\Theta_s(\overline{\pi}_1) = \alpha + \beta$, the only way to obtain $\dot{g}_s = \dot{\tau}_s = 0$ is to set $\overline{g}_{s1} = \overline{\tau}_{s1} = 0$. This leads us to

$$\overline{\omega}_1 = 1 - \overline{\pi}_1 - \frac{r(\nu(\alpha + \beta + \delta) - \overline{\pi}_1)}{\alpha + \beta} + \frac{\Gamma_b(\overline{\lambda}_1) - \Theta_b(\overline{\pi}_1)}{\alpha + \beta} \tag{8.20}$$

as the only way to obtain $\dot{\pi} = 0$. This defines what we call the "good equilibrium" for (8.14), that is, an equilibrium characterized by finite values for all variables and non-zero wage share. As we will see next, all remaining cases have the equilibrium wage share equal to zero.

2. Take $\overline{\omega}_2 = 0$ and $\overline{\pi}_2 = \overline{\pi}_1$ so that $\dot{\omega} = \dot{\lambda} = 0$. In this case, discarding the structural coincidence $\Theta_s(\overline{\pi}_1) = \alpha + \beta$, the only way to obtain $\dot{\tau}_s = 0$ is to set $\overline{\tau}_{s2} = 0$. For the remaining variables we define

$$\overline{\lambda}_2 = \Gamma_s^{-1}(\alpha + \beta) \tag{8.21}$$

so that $\dot{g}_s = 0$ and

$$\overline{g}_{s2} = \frac{\Theta_b(\overline{\pi}_1) - \Gamma_b(\overline{\lambda}_2)}{\alpha + \beta} + \frac{r(\nu\mu(\overline{\pi}_1) + \nu\delta - \overline{\pi}_1)}{\alpha + \beta} - (1 - \overline{\pi}_1) \tag{8.22}$$

so that $\dot{\pi} = 0$.

3. Take $\overline{\omega}_3 = \overline{\tau}_{s3} = 0$ and $\overline{\pi}_3 = \overline{\pi}_1$ and so that $\dot{\omega} = \dot{\lambda} = \dot{\tau}_s = 0$ as before. In addition take $\overline{g}_{s3} = 0$ so that $\dot{g}_s = 0$. To obtain $\dot{\pi} = 0$ define

$$\overline{\lambda}_3 = \Gamma_b^{-1}\left(r\left(\nu\mu(\overline{\pi}_1) + \nu\delta - \overline{\pi}_1\right) - \left(1 - \overline{\pi}_1\right)\left(\alpha + \beta\right) + \Theta_b\left(\overline{\pi}_1\right)\right). \tag{8.23}$$

4. Take $\overline{\omega}_4 = \overline{\lambda}_4 = \overline{g}_{s4} = \overline{\tau}_{s4} = 0$ so that $\dot{\omega} = \dot{\lambda} = \dot{g}_s = \dot{\tau}_s = 0$. To obtain $\dot{\pi} = 0$ define $\overline{\pi}_4$ as the solution of

$$-r(\nu\mu(\pi) + \nu\delta - \pi) + (1 - \pi)\mu(\pi) + \Gamma_b(0) - \Theta_b(\pi) = 0 \tag{8.24}$$

5. Take $\overline{\omega}_5 = \overline{\lambda}_5 = \overline{g}_{s5} = 0$ so that $\dot{\omega} = \dot{\lambda} = \dot{g}_s = 0$. To obtain $\dot{\tau}_s = 0$ define $\overline{\pi}_5$ as the solution of

$$\Theta_s(\pi) - \mu(\pi) = 0. \tag{8.25}$$

Finally, to obtain $\dot{\pi} = 0$ set

$$\overline{\tau}_{s5} = \frac{-r(\nu\mu(\overline{\pi}_5) + \nu\delta - \overline{\pi}_5) + (1 - \overline{\pi}_5)\Theta_s(\overline{\pi}_5) + \Gamma_b(0) - \Theta_b(\overline{\pi}_5)}{\Theta_s(\overline{\pi}_5)} \tag{8.26}$$

6. Take $\overline{\omega}_6 = \overline{\lambda}_6 = 0$ so that $\dot{\omega} = \dot{\lambda} = 0$. To obtain $\dot{g}_s = 0$, define

$$\overline{\pi}_6 = \mu^{-1}\left(\Gamma_s(0)\right). \tag{8.27}$$

Provided we discard again the structural coincidence $\Theta_s(\overline{\pi}_6) = \Gamma_s(0)$, this means that to obtain $\dot{\tau}_s = 0$ we must set $\overline{\tau}_{s6} = 0$. For the remaining variable we take

$$\overline{g}_{s6} = \frac{r(\nu\mu(\overline{\pi}_6) + \nu\delta - \overline{\pi}_6) - (1-\overline{\pi}_6)\Gamma_s(0) - \Gamma_b(0) + \Theta_b(\overline{\pi}_6)}{\Gamma_s(0)} \tag{8.28}$$

so that $\dot{\pi} = 0$.

To summarize, discarding equilibria whose existence depend on structurally unstable coincidences in the choice of parameter values, the finite-valued equilibria for system (8.14) are given by

$$(\omega, \lambda, g_s, \tau_s, \pi) = \begin{cases} (\overline{\omega}_1, \overline{\lambda}_1, 0, 0, \overline{\pi}_1) \\ (0, \overline{\lambda}_2, \overline{g}_{s2}, 0, \overline{\pi}_1) \\ (0, \overline{\lambda}_3, 0, 0, \overline{\pi}_1) \\ (0, 0, 0, 0, \overline{\pi}_4) \\ (0, 0, 0, \overline{\tau}_{s5}, \overline{\pi}_5) \\ (0, 0, \overline{g}_{s6}, 0, \overline{\pi}_6) \end{cases} \tag{8.29}$$

Once the system (8.14) converges to an equilibrium $(\overline{\omega}, \overline{\lambda}, \overline{g}_s, \overline{\tau}_s, \overline{\pi})$, the dependent variables g_b, τ_b, d_g must solve

$$\dot{g}_b = \Gamma_b(\overline{\lambda}) - g_b\mu(\overline{\pi}) \tag{8.30}$$

$$\dot{\tau}_b = \Theta_b(\overline{\pi}) - g_{T_1}\mu(\overline{\pi}) \tag{8.31}$$

$$\dot{d}_g = d_g[r - \mu(\overline{\pi})] + g_s + g_b + g_e - \tau_s - \tau_b \tag{8.32}$$

As a result, base government subsidies and taxation will converge exponentially fast to their corresponding equilibrium values

$$\begin{aligned}\overline{g}_b &= \frac{\Gamma_b(\overline{\lambda})}{\mu(\overline{\pi})}\\ \overline{\tau}_b &= \frac{\Theta_b(\overline{\pi})}{\mu(\overline{\pi})}\end{aligned} \tag{8.33}$$

Similarly, if the government expenditure ratio reaches an equilibrium value $\overline{g}_e$ compatible with the equilibrium values for the remaining variables, then the government debt ratio converges to

$$\overline{d}_g = \begin{cases} \dfrac{\overline{g}_s + \overline{g}_e + \overline{g}_b - \overline{\tau}_s - \overline{\tau}_b}{\mu(\overline{\pi}) - r} & \text{if } r < \mu(\overline{\pi}) \\ +\infty & \text{if } r > \mu(\overline{\pi}), \text{ or } r = \mu(\overline{\pi}) \text{ and } \overline{g}_s + \overline{g}_e + \overline{g}_b - \overline{\tau}_s - \overline{\tau}_b > 0 \\ 0 & \text{if } r = \mu(\overline{\pi}) \text{ and } \overline{g}_s + \overline{g}_e + \overline{g}_b - \overline{\tau}_S - \overline{\tau}_b < 0 \end{cases} \tag{8.34}$$

Local Stability

We begin our local stability analysis by determining the Jacobian matrix for the system (8.14):

$$\begin{bmatrix} \Phi(\lambda) - \alpha & \omega\Phi'(\lambda) & 0 & 0 & 0 \\ 0 & \mu(\pi) - \alpha - \beta & 0 & 0 & \lambda\mu'(\pi) \\ 0 & g_s\Gamma_s'(\lambda) & \Gamma_s(\lambda) - \mu(\pi) & 0 & -g_s\mu'(\pi) \\ 0 & 0 & 0 & \Theta_s(\pi) - \mu(\pi) & -\tau_s\mu'(\pi) \\ \alpha - \Phi(\lambda) - \mu(\pi) & \begin{array}{c} -\omega\Phi'(\lambda) + \Gamma_b'(\lambda) \\ +g_s\Gamma_s'(\lambda) \end{array} & \Gamma_s(\lambda) & -\Theta_s(\pi) & \begin{array}{c} r - \mu(\pi) \\ +\mu'(\pi)(1 - \omega - \pi - r\nu) \\ -\left(\Theta_b'(\pi) + \tau_s\Theta_s'(\pi)\right) \end{array} \end{bmatrix} \tag{8.35}$$

Returning to the equilibria defined in (8.29), we have the following six cases:

1. Defining the constant

$$K = r + \mu'(\overline{\pi}_1)\,(1 - \overline{\pi}_1 - r\nu) - (\alpha + \beta) - \Theta_b'(\overline{\pi}_1) \tag{8.36}$$

the characteristic polynomial for the Jacobian matrix (8.35) at the good equilibrium $(\overline{\omega}_1, \overline{\lambda}_1, 0, 0, \overline{\pi}_1)$ can be written as

$$\begin{aligned} \overline{p}_1(y) =& \Big[- y^3 + y^2(K - \overline{\omega}_1\mu'(\overline{\pi}_1)) + y\overline{\lambda}_1\mu'(\overline{\pi}_1)\left(\Gamma_b'(\overline{\lambda}_1) - \overline{\omega}_1\Phi'(\overline{\lambda}_1)\right) \\ & - (\alpha+\beta)\overline{\lambda}_1\mu'(\overline{\pi}_1)\overline{\omega}_1\Phi'(\overline{\lambda}_1)\Big] \\ & \times \Big(\Gamma_s(\overline{\lambda}_1) - (\alpha+\beta) - y\Big)\Big(\Theta_s(\overline{\pi}_1) - (\alpha+\beta) - y\Big) \end{aligned} \tag{8.37}$$

This equilibrium will be locally stable if and only if the polynomial (8.37) has only roots with negative real part. We can identify two of the real roots to be $\Gamma_s(\overline{\lambda}_1) - (\alpha + \beta)$ and $\Theta_s(\overline{\pi}_1) - (\alpha + \beta)$. The Routh-Hurwitz criterion gives us the remaining necessary and sufficient conditions for stability:

$$\Gamma_s(\overline{\lambda}_1) < \alpha + \beta \tag{8.38}$$

$$\Theta_s(\overline{\pi}_1) < \alpha + \beta \tag{8.39}$$

$$\overline{\omega}_1 > 0 \tag{8.40}$$

$$\frac{\overline{\omega}_1\mu'(\overline{\pi}_1) - K}{\alpha + \beta} > \frac{\overline{\omega}_1\Phi'(\overline{\lambda}_1)}{\overline{\omega}_1\Phi'(\overline{\lambda}_1) - \Gamma_b'(\overline{\lambda}_1)} \tag{8.41}$$

2. The characteristic polynomial at the equilibrium $(0, \overline{\lambda}_2, \overline{g}_{s2}, 0, \overline{\pi}_1)$ is

$$\begin{aligned} \overline{p}_2(y) =& \left(\Phi(\overline{\lambda}_2) - \alpha - y\right)\Big(\Theta_s\left(\overline{\pi}_1\right) - (\alpha+\beta) - y\Big)\Big\{ - y^3 + Ky^2 \\ & + \mu'(\overline{\pi}_1)\left[\overline{\lambda}_2\left(\Gamma_b'(\overline{\lambda}_2) + \overline{g}_{s2}\Gamma_s'(\overline{\lambda}_2)\right) - \overline{g}_{s2}\left(\alpha+\beta\right)\right] y + (\alpha+\beta)\,\overline{\lambda}_2\overline{g}_{s2}\mu'(\overline{\pi}_1)\Gamma_s'(\overline{\lambda}_2)\Big\} \end{aligned} \tag{8.42}$$

It follows that this equilibrium is locally stable if and only if the following conditions are satisfied:

$$\Phi(\overline{\lambda}_2) < \alpha \tag{8.43}$$

$$\Theta_s(\overline{\pi}_1) < \alpha + \beta \tag{8.44}$$

$$K < 0 \tag{8.45}$$

$$\overline{g}_{s2}\left[(\alpha+\beta) - \overline{\lambda}_2\Gamma_s'(\overline{\lambda}_2)\right] - \overline{\lambda}_2\Gamma_b'(\overline{\lambda}_2) > (\alpha+\beta)\overline{\lambda}_2\overline{g}_{s2}\Gamma_s'(\overline{\lambda}_2)/K \tag{8.46}$$

It is noteworthy that this equilibrium will only be attainable if $0 < \overline{\lambda}_2 = \Gamma_s^{-1}(\alpha+\beta) < 1$, for which it is necessary and sufficient to have $\Gamma_s(0) > \alpha+\beta > \Gamma_s(1)$.

On a different note, if we assume that the good equilibrium is stable, then not only we have $\Theta_s(\overline{\pi}_1) < \alpha+\beta$ but also $\Gamma_s(\overline{\lambda}_1) < \alpha+\beta = \Gamma_s(\overline{\lambda}_2)$, which shows us that $\overline{\lambda}_1 > \overline{\lambda}_2$ since Γ_s is a decreasing function. Since Φ is an increasing function, we have that $\alpha = \Phi(\overline{\lambda}_1) > \Phi(\overline{\lambda}_2)$, so the first two conditions (8.43) and (8.44) for stability of this equilibrium are satisfied.

3. The characteristic polynomial at the equilibrium $(0, \overline{\lambda}_3, 0, 0, \overline{\pi}_1)$ is

$$\begin{aligned}\overline{p}_3(y) = &\Big(\Phi(\overline{\lambda}_3) - \alpha - y\Big)\Big(\Theta_s(\overline{\pi}_1) - (\alpha+\beta) - y\Big)\Big(\Gamma_s(\overline{\lambda}_3) - (\alpha+\beta) - y\Big)\\ &\times \Big(y^2 - Ky - \overline{\lambda}_3\mu'(\overline{\pi}_1)\Gamma_b'(\lambda_3)\Big)\end{aligned} \tag{8.47}$$

Accordingly, local stability is guaranteed if and only if the following conditions are satisfied:

$$\Phi(\overline{\lambda}_3) < \alpha \tag{8.48}$$

$$\Theta_s(\overline{\pi}_1) < \alpha + \beta \tag{8.49}$$

$$\Gamma_s(\overline{\lambda}_3) < \alpha + \beta \tag{8.50}$$

$$K < 0 \tag{8.51}$$

Recalling that the employment level for this equilibrium is

$$\overline{\lambda}_3 = \Gamma_b^{-1}\left(\Gamma_b(\overline{\lambda}_1) - (\alpha+\beta)\overline{\omega}_1\right) > \overline{\lambda}_1,$$

we have that

$$\Phi(\overline{\lambda}_3) > \Phi(\overline{\lambda}_1) = \alpha,$$

which shows that this equilibrium is locally unstable whenever the good equilibrium is stable.

4. The Jacobian matrix at the equilibrium $(0, 0, 0, 0, \overline{\pi}_4)$ is diagonal, hence we can identify the eigenvalues at the diagonal and conclude that local stability is equivalent to the following conditions:

$$\mu(\overline{\pi}_4) < \alpha + \beta \tag{8.52}$$

$$\Gamma_s(0) < \mu(\overline{\pi}_4) \tag{8.53}$$

$$\Theta_s(\overline{\pi}_4) < \mu(\overline{\pi}_4) \tag{8.54}$$

$$r + \mu'(\overline{\pi}_4)(1 - \overline{\pi}_4 - r\nu) < \mu(\overline{\pi}_4) + \Theta_b'(\overline{\pi}_4) \tag{8.55}$$

These inequalities can only be satisfied simultaneously if $\Gamma_s(0) < \alpha + \beta$.

5. The characteristic polynomial at the equilibrium $(0, 0, 0, \overline{\tau}_{s5}, \overline{\pi}_5)$ is

$$\begin{aligned}\overline{p}_5(y) =& \Big[y^2 - y\left(r + \mu'(\overline{\pi}_5)(1 - \overline{\pi}_5 - r\nu) - \Theta_s(\overline{\pi}_5) - \Theta_b'(\overline{\pi}_5) - \overline{g}_{T_5}\Theta_s'(\overline{\pi}_5)\right) \\ & - \overline{g}_{T_5}\Theta_s(\overline{\pi}_5)\mu'(\overline{\pi}_5)\Big] \\ & \times \Big(\Phi(0) - \alpha - y\Big)\Big(\Theta_s(\overline{\pi}_5) - (\alpha + \beta) - y\Big)\Big(\Gamma_s(0) - \Theta_s(\overline{\pi}_5) - y\Big),\end{aligned} \tag{8.56}$$

from which we can derive the necessary and sufficient conditions for local stability:

$$\Theta_s(\overline{\pi}_5) < \alpha + \beta \tag{8.57}$$

$$\Gamma_s(0) < \Theta_s(\overline{\pi}_5) \tag{8.58}$$

$$r + \mu'(\overline{\pi}_5)(1 - \overline{\pi}_5 - r\nu) < \Theta_s(\overline{\pi}_5) + \Theta_b'(\overline{\pi}_5) + \overline{\tau}_{s5}\Theta_s'(\overline{\pi}_5) \tag{8.59}$$

$$\overline{\tau}_{s5}\Theta_s(\overline{\pi}_5) < 0 \tag{8.60}$$

Since $\tau_s(0) \geq 0$, this equilibrium can only be attained if $\overline{\tau}_{s5} > 0$. In that case, we need $\Theta_s(\overline{\pi}_5) < 0$, for it to be locally stable, which would then force $\Gamma_s(0)$ to be negative. Since this is not economically meaningful, we can conclude that this equilibrium will always be locally unstable to all effects and purposes.

6. The characteristic polynomial at the equilibrium $(0, 0, \overline{g}_{s6}, 0, \overline{\pi}_6)$ is

$$\begin{aligned}\overline{p}_6(y) = & \Big[y^2 - y\left(r + \mu'(\overline{\pi}_6)(1 - \overline{\pi}_6 - r\nu) - \Gamma_s(0) - \Theta_b'(\overline{\pi}_6)\right)y + \Gamma_s(0)\overline{g}_{S_6}\mu'(\overline{\pi}_6)\Big] \\ & \times \Big(\Phi(0) - \alpha - y\Big)\Big(\Gamma_s(0) - (\alpha + \beta) - y\Big)\Big(\Theta_s(\overline{\pi}_6) - \Gamma_s(0) - y\Big)\end{aligned} \tag{8.61}$$

Therefore, this equilibrium is locally stable if and only if the following conditions are satisfied:

$$\Gamma_s(0) < \alpha + \beta \tag{8.62}$$

$$\Theta_s(\overline{\pi}_6) < \Gamma_s(0) \tag{8.63}$$

$$r + \mu'(\overline{\pi}_6)(1 - \overline{\pi}_6 - r\nu) < \Gamma_s(0) + \Theta_b'(\overline{\pi}_6) \tag{8.64}$$

$$\overline{g}_{s6}\Gamma_s(0) > 0 \tag{8.65}$$

8.1.2 Infinite-valued equilibria

Our original motivation to introduce a government sector was to prevent the economy from reaching the bad equilibrium (5.20) in the Keen model without government. Because this equilibrium is characterized by infinitely negative profits caused by explosive private debt, we focus on the cases where $\pi \to -\infty$.

Making the change of variable $u = e^{\pi}$, we obtain the system

$$\begin{aligned}
\dot{\omega} =& \omega[\Phi(\lambda) - \alpha] \\
\dot{\lambda} =& \lambda[\mu(\log u) - \alpha - \beta] \\
\dot{g}_s =& g_s[\Gamma_s(\lambda) - \mu(\log u)] \\
\dot{\tau}_s =& \tau_s[\Theta_s(\log u) - \mu(\log u)] \\
\dot{u} =& u\Big[- \omega\big(\Phi(\lambda) - \alpha\big) - r\big(\nu\mu(\log u) + \nu\delta - \log u\big) + (1 - \omega - \log u)\mu(\log u) \\
& + \Gamma_b(\lambda) - \Theta_b(\log u) + g_s\Gamma_s(\lambda) - \tau_s\Theta_s(\log u)\Big]
\end{aligned} \tag{8.66}$$

The Jacobian matrix for this system is

$$\begin{bmatrix} \Phi(\lambda)-\alpha & \omega\Phi'(\alpha) & 0 & 0 & 0 \\ 0 & \mu(\log u)-\alpha-\beta & 0 & 0 & \lambda\mu'(\log u)/u \\ 0 & g_s\Gamma_s'(\lambda) & \Gamma_s(\lambda)-\mu(\log u) & 0 & -g_s\mu'(\log u)/u \\ 0 & 0 & 0 & \Theta_s(\log u)-\mu(\log u) & \tau_s[\Theta_s'(\log u)-\mu'(\log u)]/u \\ J_{5,1}(\lambda,u) & J_{5,2}(\lambda,g_s,u) & u\Gamma_s(\lambda) & -u\Theta_s(\log u) & J_{5,5}(\omega,\lambda,g_s,\tau_s,u) \end{bmatrix},$$

where

$$\begin{aligned} J_{5,1}(u,\lambda) &= -u(\Phi(\lambda)-\alpha+\mu(\log u)) \\ J_{5,2}(u,\lambda,g_s) &= u(g_s\Gamma_s'(\lambda)+\Gamma_b'(\lambda)-\omega\Phi'(\lambda)) \\ J_{5,5}(\omega,\lambda,g_s,\tau_s,u) &= -\log u[\mu(\log u)-r+\mu'(\log u)]+\Gamma_b(\lambda)-\Theta_b(\log u) \\ &\quad + r\,[1-\nu\mu(\log u)-\nu\delta-\nu\mu'(\log u)]+g_s\Gamma_s(\lambda)-\tau_s\Theta_s(\log u) \\ &\quad -\omega\,[\Phi(\lambda)-\alpha+\mu(\log u)+\mu'(\log u)]+\mu'(\log u) \\ &\quad -\Theta_b'(\log u)-\tau_s\Theta_s'(\log u) \end{aligned}$$

We can see that $(\omega,\lambda,g_{S_2},g_{T_2},u)=(0,0,0,0,0)$ is an equilibrium point for (8.66), since all terms inside square brackets in the right-hand side of (8.66) approach constants as $u\to 0^+$, with the exception of $\log u$, for which we have that $u\log u\to 0$. Assuming further that

$$\Gamma_b,\Gamma_s\in C^1[0,1] \tag{8.67}$$

we then have that the Jacobian at this equilibrium becomes

$$\begin{bmatrix} \Phi(0)-\alpha & 0 & 0 & 0 & 0 \\ 0 & \mu(-\infty)-\alpha-\beta & 0 & 0 & * \\ 0 & 0 & \Gamma_s(0)-\mu(-\infty) & 0 & * \\ 0 & 0 & 0 & \Theta_s(-\infty)-\mu(-\infty) & * \\ 0 & 0 & 0 & 0 & \infty\cdot(\mu(-\infty)-r) \end{bmatrix}.$$

where $*$ denotes any value, and we have replaced $\lim_{u\to 0^+}[-\log(u)]$ by ∞. Recall that we assumed in (5.9), (8.7), and (8.8) that the functions κ, θ_b and θ_s have horizontal asymptotes satisfying $\theta_s(-\infty) < \mu(-\infty)$. Local stability is then guaranteed if, in addition to the standard requirements (3.17), (5.9), (5.33), and the new condition (8.8), we impose that

$$\Gamma_s(0) < \mu(-\infty) \tag{8.68}$$

That is, the bad equilibrium $(\omega, \lambda, g_s, \tau_s, u) = (0, 0, 0, 0, 0)$ fails to be locally stable whenever condition (8.68) is violated, which is easy to achieve in practice, since $\mu(-\infty)$ is in general very small. This constitutes our first positive result regarding government intervention.

Unfortunately, this is not the only plausible equilibrium for the extended system (8.14) corresponding to the bad equilibrium (5.20) in the Keen model without government. Namely, allowing $\Gamma_s(0) \geq \mu(-\infty)$ in (8.66) gives rise to the possibility that $g_s \to \pm\infty$, depending on the initial condition $g_s(0)$. To investigate these other possibilities we make a second change of variables $v = 1/g_s$ and $x = g_s/\pi$, which leads to the modified system

$$\begin{aligned}
\dot{\omega} =& \omega\left[\Phi(\lambda)-\alpha\right] \\
\dot{\lambda} =& \lambda\left[\mu\left(\frac{1}{vx}\right)-\alpha-\beta\right] \\
\dot{v} =& v\left[\mu\left(\frac{1}{vx}\right)-\Gamma_s(\lambda)\right] \\
\dot{\tau}_s =& \tau_s\left[\Theta_s\left(\frac{1}{vx}\right)-\mu\left(\frac{1}{vx}\right)\right] \\
\dot{x} =& x\Bigg[\Gamma_s(\lambda)(1-x)-r+vx\Bigg(\omega(\Phi(\lambda)-\alpha)+r\nu\mu\left(\frac{1}{vx}\right)+r\nu\delta \\
& -(1-\omega)\mu\left(\frac{1}{vx}\right)+\Theta_b\left(\frac{1}{vx}\right)+\tau_s\Theta_s\left(\frac{1}{vx}\right)-\Gamma_b(\lambda)\Bigg)\Bigg].
\end{aligned} \tag{8.69}$$

We then see that $(\omega,\lambda,v,\tau_s,x)=(0,0,0^{\pm},0,0^{\mp})$ are equilibria for (8.69) since all terms in the square brackets on the right-hand side of (8.69) approach constant values as $v\to 0^{\pm}$ and $x\to 0^{\mp}$. Recalling (5.10) and (8.9), the associated Jacobian matrix for these fixed points is

$$J=\begin{bmatrix} \Phi(0)-\alpha & 0 & 0 & 0 & 0 \\ 0 & \mu(-\infty)-\alpha-\beta & 0 & 0 & 0 \\ 0 & 0 & \mu(-\infty)-\Gamma_s(0) & 0 & 0 \\ 0 & 0 & 0 & \Theta_s(-\infty)-\mu(-\infty) & 0 \\ 0 & * & * & 0 & \Gamma_s(0)-r \end{bmatrix} \tag{8.70}$$

Therefore, local stability for these equilibria is guaranteed by (3.17), (5.9), (8.8) and the new condition

$$\mu(-\infty)<\Gamma_s(0)<r. \tag{8.71}$$

If we assume that $\Gamma_s(0)\neq 0$, two other possible equilibria for (8.69) are given by

$$(\omega,\lambda,v,\tau_s,x)=\left(0,0,0^{\pm},0,\frac{\Gamma_s(0)-r}{\Gamma_s(0)}\right),$$

which are achievable provided either (i) $g_s(0) > 0$ and $\Gamma_s(0) < r$ (so that $v \to 0^+$ and $\pi \to -\infty$), or (ii) $g_s(0) < 0$ and $\Gamma_s(0) > r$ (so that $v \to 0^-$ and $\pi \to -\infty$). The associated Jacobian matrix for these fixed points is

$$J = \begin{bmatrix} \Phi(0)-\alpha & 0 & 0 & 0 & 0 \\ 0 & \mu(-\infty)-\alpha-\beta & 0 & 0 & 0 \\ 0 & 0 & \mu(-\infty)-\Gamma_s(0) & 0 & 0 \\ 0 & 0 & 0 & \Theta_s(-\infty)-\mu(-\infty) & 0 \\ 0 & * & * & 0 & r-\Gamma_s(0) \end{bmatrix} \tag{8.72}$$

Therefore, local stability for these equilibria is guaranteed by (3.17), (5.9), (5.33), (8.8) and $\Gamma_s(0) > r$.

The case $\Gamma_s(0) = 0$ allows for the possible equilibrium $(\omega, \lambda, v, \tau_s, x) = (0, 0, 0^{\pm}, 0, 0)$, depending on the sign of initial condition $g_s(0)$, with associated Jacobian matrix

$$J = \begin{bmatrix} \Phi(0)-\alpha & 0 & 0 & 0 & 0 \\ 0 & \mu(-\infty)-\alpha-\beta & 0 & 0 & 0 \\ 0 & 0 & \mu(-\infty)-\Gamma_s(0) & 0 & 0 \\ 0 & 0 & 0 & \Theta_s(-\infty)-\mu(-\infty) & 0 \\ 0 & * & * & 0 & 0 \end{bmatrix}, \tag{8.73}$$

so that its local stability can never be guaranteed.

We summarize the different results for infinite–valued equilibria in the next proposition.

Proposition 8.1. *If, in addition to the standing assumptions* (3.16)–(3.18), (5.8)–(5.10), (8.5)–(8.9), *we have that* (8.67) *holds, then the following are the infinite-valued equilibria of* (8.14) *corresponding to the bad equilibrium* (5.20) *for a Keen model* *without government:*

$$(\omega, \lambda, g_s, \tau_s, \pi) = (0, 0, 0, 0, -\infty) \tag{8.74}$$

$$(\omega, \lambda, g_s, \tau_s, \pi) = (0, 0, +\infty, 0, -\infty) \tag{8.75}$$

$$(\omega, \lambda, g_s, \tau_s, \pi) = (0, 0, -\infty, 0, -\infty) \tag{8.76}$$

Assuming additionally that (5.33) *is satisfied, the stability of these equilibria depend on stimulus government subsidies* g_s *as follows:*

(a) When $g_s(0) > 0$ *(stimulus):*

- *(i) if* $\Gamma_s(0) < \mu(-\infty)$*, then equilibrium* (8.74) *is locally stable, equilibrium* (8.75) *is achievable but unstable, and equilibrium* (8.76) *is unachievable.*
- *(ii) if* $\mu(-\infty) < \Gamma_s(0) < r$*, then equilibrium* (8.74) *is unstable, equilibrium* (8.75) *is achievable and locally stable, and equilibrium* (8.76) *is unachievable.*
- *(iii) if* $r < \Gamma_s(0)$*, then equilibrium* (8.74) *is unstable, equilibrium* (8.75) *is achievable and unstable, and equilibrium* (8.76) *is unachievable.*

(b) When $g_s(0) < 0$ *(austerity):*

- *(i) if* $\Gamma_s(0) < \mu(-\infty)$*, then equilibrium* (8.74) *is locally stable, equilibrium* (8.75) *is unachievable, and equilibrium* (8.76) *is achievable but unstable.*
- *(ii) if* $\mu(-\infty) < \Gamma_s(0)$*, then equilibrium* (8.74) *is unstable, equilibrium* (8.75) *is unachievable, and equilibrium* (8.76) *is achievable and locally stable.*

In other words, under a stimulus regime ($g_s(0) > 0$), any achievable equilibria with $\pi \to -\infty$ becomes unstable provided $\Gamma_s(0) > r$. On the other hand, under an austerity regime ($g_s(0) < 0$), there is no value of $\Gamma_s(0)$ that eliminates the possibility of local stability from all achievable equilibria with $\pi \to -\infty$.

8.2 Persistence results

As it is typical in persistence analysis, although we are primarily interest in preventing the crisis situation characterized by the bad equilibrium, our first result will clear the concern of exploding positive profits.

Proposition 8.2. *If* $\tau_s(0) \geq 0$*, and conditions* (5.8),(5.9) *are satisfied, then the system described by* (8.14) *is* $e^{-\pi}$*-UWP.*

Proof. Showing this consists of demonstrating that

$$\limsup e^{-\pi} > \varepsilon$$

for some $\varepsilon > 0$, which is equivalent to saying that

$$\liminf \pi < m$$

for some $m \in \mathbb{R}$. We are going to show this by contradiction, so assume that $\liminf \pi > m$ for any m, as large (and positive) as we want. We can then find a t_0 such that $\pi(t) > m$ for all $t \geq t_0$.

First, we can then bound employment from below since for $t \geq t_0$ we have

$$\dot{\lambda}/\lambda = \mu(\pi) - \alpha - \beta \geq \mu(m) - \alpha - \beta$$

which is positive for m large enough. That means that $\lambda(t) > \lambda(t_0)e^{(\mu(m)-\alpha-\beta)(t-t_0)}$ for all $t > t_0$.

Consequently, there exists $t_1 > t_0$ for which $\Phi(\lambda(t_1)) > \alpha$ and thus

$$\dot{\omega}/\omega = \Phi(\lambda) - \alpha$$

will be positive. We then have that

$$\omega(t) \geq \omega(t_1) \exp\left[\Phi(\lambda(t_1)) - \alpha\right]$$

Next, for $t \geq t_1$, the government subsidies dynamics satisfy

$$\dot{g}_s/g_s = \Gamma_s(\lambda) - \mu(\pi) \leq \Gamma_s(\lambda(t_1)) - \mu(m)$$

which can be made negative for m large enough. Consequently,

$$|g_s(t)| \leq |g_s(t_1)| \exp\left[(\Gamma_s(\lambda(t_1)) - \mu(m))(t - t_1)\right]$$

for all $t > t_0$.

Finally, one can choose m big enough such that $\kappa(m) \geq 0$, $\theta_b(m) \geq 0$, $\theta_s(m) \geq 0$, and $\mu(m) > r$ – possible because of (5.9) – allowing us to find the following bound for $\dot{\pi}$, valid for all $t > t_1$:

$$\begin{aligned}\dot{\pi} &= -\omega[\Phi(\lambda) - \alpha] - r(\kappa(\pi) - \pi) + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\lambda) - \Theta_b(\pi) + g_s\Gamma_s(\lambda) - \tau_s\Theta_s(\pi) \\ &\leq \pi\left[r - \mu(m)\right] - \omega(t_1)e^{[\Phi(\lambda(t_1))-\alpha](t-t_1)}\left[\Phi(\lambda(t_1)) - \alpha\right] + C_m,\end{aligned} \tag{8.77}$$

where $C_m = \Gamma_b(\lambda(t_0)) + (g_s(t_0))^+\,\Gamma_s(\lambda(t_0))$ is a positive constant. Consequently, Gronwall's inequality gives the following bound, valid for any $t > t_1$

$$\begin{aligned}\pi(t) \leq \pi(t_1)e^{-(\mu(m)-r)(t-t_1)} &+ \frac{C_m}{\mu(m) - r}\left(1 - e^{-[\mu(m)-r](t-t_1)}\right) \\ - \frac{\omega(t_1)\left[\Phi(\lambda(t_1)) - \alpha\right]}{[\Phi(\lambda(t_1)) - \alpha] + [\mu(m) - r]}&\left(e^{[\Phi(\lambda(t_1))-\alpha](t-t_1)} - e^{-[\mu(m)-r](t-t_1)}\right)\end{aligned} \tag{8.78}$$

From (3.18), we can choose t_1 appropriately such that $\Phi(\lambda(t_1)) - \alpha \geq \mu(+\infty) - r$ and thus the RHS of (8.78) converges to $-\infty$ as t increases, which provides us with a contradiction.

□

Our core results are presented in the next two propositions. We first show that government intervention can achieve uniformly weak persistence of the functional e^{π} even when the bad equilibrium for the model without government is locally stable.

Proposition 8.3. *Suppose that the structural conditions* (3.16)–(3.18)*,* (5.8)–(5.10) *and* (8.5)–(8.9) *are satisfied, along with the local stability condition* (5.33) *for the bad equilibrium of the Keen model* (5.7) *without government. Assume further that* $g_s(0) > 0$*. Then the model with government (8.14) is* e^{π}*-UWP if either of the following conditions is satisfied:*

(1) $\Gamma_s(0) > r$*, or*

(2) $\lambda\Gamma_b(\lambda)$ *is bounded below as* $\lambda \to 0$*.*

Proof. We prove it by contradiction. If $\limsup_{t\to\infty} \pi(t) \leq -m$ for any given large $m > 0$, there exists $t_0 \geq 0$ such that $\pi(t) \leq -m$ for $t > t_0$. From the equation for $\dot{\lambda}$, it follows that

$$\lambda(t) \leq \lambda(t_0)e^{(t-t_0)(\mu(-m)-\alpha-\beta)},$$

for $t > t_0$. Choosing $m > 0$ large enough so that $\mu(-m) < \alpha + \beta$ (recall condition (5.9)), we get that for any small $\varepsilon > 0$, there exists $t_1 > t_0$ such that $\lambda(t) < \varepsilon$ for $t > t_1$. From the equation for $\dot{\omega}$, this readily implies that

$$\omega(t) < \omega(t_1)e^{(t-t_1)(\Phi(\varepsilon)-\alpha)},$$

for $t > t_1$. Again, we may choose $\varepsilon > 0$ sufficiently small that $\Phi(\varepsilon) < \alpha$ (recall conditions (3.16) and (3.17)). Hence, there exists $t_2 > t_1 > t_0$ such that $\omega(t) < \varepsilon$ for $t > t_2$. Finally, condition (8.8) guarantees that we can choose m large enough such that

$$\Theta_s(\pi) - \mu(\pi) < 0, \quad \forall \pi \leq -m.$$

It then follows from the equation for $\dot{\tau}_s$ that there exists $t_3 > t_2 > t_1 > t_0$ such that $\tau_s(t) < \varepsilon$ for $t > t_3$. In other words, we can bound ω, λ and τ_s by ε for t large enough.

At this point, we need to consider the hypothesis $\Gamma_s(0) > r$ and $\lambda\Gamma_b(\lambda)$ bounded separately. Assume first that $\Gamma_s(0) > r$. Since Γ is a decreasing function, we can immediately see from the equation for $\dot{g}_s$ that

$$\frac{\dot{g}_s}{g_s} = \Gamma_s(\lambda) - \mu(\pi) > \Gamma_s(\varepsilon) - \mu(\pi),$$

for $t > t_1$. Moreover, since $\Gamma_s(0) > r > \mu(-\infty)$ (see condition (5.33)), we can choose ε small enough and/or m big enough such that $\Gamma_s(\varepsilon) > \mu(-m)$. Accordingly, for any $t > s > t_1$, we have that

$$g_s(t) > g_s(s)e^{(t-s)[\Gamma_s(\varepsilon)-\mu(-m)]}.$$

Using the equation for $\dot{\pi}$ we have:

$$\begin{aligned}
\dot{\pi} =& -\omega[\Phi(\lambda)-\alpha] - r[\nu\mu(\pi)+\nu\delta-\pi] + (1-\omega-\pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi) \\
=& -\omega[\Phi(\lambda)-\alpha] - r\kappa(\pi) + \pi(r-\mu(\pi)) + (1-\omega)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi) \\
>& -r\max\{|\kappa(-\infty)|, |\kappa(-m)|\} + \pi(r-\mu(-\infty)) - \max\{|\mu(-\infty)|, |\mu(-m)|\} + \Gamma_b(\varepsilon) \\
& + \Gamma_s(\varepsilon)g_s(t_3)e^{(t-t_3)[\Gamma_s(\varepsilon)-\mu(-m)]} - \Theta_b(-m) - \varepsilon\max\{|\Theta_s(-\infty)|, |\Theta_s(-m)|\} \\
=& \, C + A\pi + De^{Et}
\end{aligned} \tag{8.79}$$

where C is finite and does not depend on t, $D = \Gamma_s(\varepsilon)g_s(t_3)e^{-t_3(\Gamma_s(\varepsilon)-\mu(-m))} > 0$, $A = r - \mu(-\infty) > 0$, and $E = \Gamma_s(\varepsilon) - \mu(-m) > 0$. Consequently, for $t > t_3$, we have that $\pi(t) > y(t)$, where $y(t)$ is the solution of

$$\dot{y} = C + Ay + De^{Et}, \quad y(t_3) = \pi(t_3) \tag{8.80}$$

that is,

$$y(t) = \pi(t_3)e^{A(t-t_3)} + \frac{C}{A}\left(e^{A(t-t_3)} - 1\right) + \frac{D}{E-A}e^{Et_3}\left(e^{E(t-t_3)} - e^{A(t-t_3)}\right). \qquad (8.81)$$

At last, since $\Gamma_s(0) > r$, we can choose ε sufficiently small and m sufficiently large such that

$$E - A = \Gamma_s(\varepsilon) - r + \mu(-\infty) - \mu(-m) > 0,$$

which leads us to conclude that e^{Et} dominates the solution $y(t)$ when $t \to \infty$, that is,

$$\lim_{t\to\infty} y(t) = \frac{D}{E-A}e^{Et} = +\infty.$$

Yet, since $\pi(t) > y(t)$ for $t > t_3$, we must have also $\pi(t) \xrightarrow{t\to\infty} +\infty$, which contradicts the fact that $\pi(t) \leq -m$ for $t > t_0$.

Alternatively, assume now that $\lambda\Gamma_b(\lambda)$ is bounded from below as $\lambda \to 0$. We can still bound ω, λ and τ_s by ε for t large enough as before. Moreover, since $\lambda\Gamma_b(\lambda) > L$ for some positive L as $\lambda \to 0$, we now have that $\Gamma_b(\lambda) > \Gamma_b(\lambda)\lambda/\varepsilon > L/\varepsilon$. From the equation for $\dot{\pi}$ we then have

$$\begin{aligned}
\dot{\pi} =& -\omega[\Phi(\lambda) - \alpha] - r[\nu\mu(\pi) + \nu\delta - \pi] + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi) \\
=& -\omega[\Phi(\lambda) - \alpha] - r\kappa(\pi) + \pi(r - \mu(\pi)) + (1 - \omega)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) - \tau_s\Theta_s(\pi) \\
>& - r\max\{|\kappa(-\infty)|, |\kappa(-m)|\} + \pi(r - \mu(-\infty)) - \max\{|\mu(-\infty)|, |\mu(-m)|\} + L/\varepsilon \\
& - \Theta_b(-m) - \varepsilon\max\{|\Theta_s(-\infty)|, |\Theta_s(-m)|\} \\
=& \tilde{C}(\varepsilon) + \tilde{A}y
\end{aligned} \qquad (8.82)$$

where $\tilde{C}$ can be made arbitrarily large by choosing ε sufficiently small, while $\tilde{A} = r - \mu(-\infty) > 0$. Therefore, for $t > t_3$, we have that $\pi(t) \geq y(t)$, where $y(t)$ is now the solution of

$$\dot{y}(t) = \tilde{C} + \tilde{A}y, \quad y(t_3) = \pi(t_3)$$

that is,

$$y(t) = \frac{\left(\tilde{C}(\varepsilon) + \tilde{A}y(t_3)\right) e^{\tilde{A}(t-t_3)} - \tilde{C}}{\tilde{A}}.$$

We can then choose ε small enough such that $\tilde{C}(\varepsilon)+\tilde{A}y(0) > 0$ and hence $\lim_{t\to\infty} y(t) = +\infty$. But this implies that $\pi(t) \xrightarrow{t\to\infty} +\infty$, which again contradicts the fact that $\pi(t) \leq -m$ for $t > t_0$. □

Although profits play a key role in the model, from the point of view of economic policy, arguably the most important variable in (8.14) is the rate of employment. Our next and final result shows that under slightly stronger conditions we can still obtain uniformly weak persistence with respect to the functional λ itself. Before stating it, define the function

$$h(x) = -r[\nu\mu(x) + \nu\delta - x] + (1-x)\mu(x) + \Gamma_b(0) - \Theta_b(x), \tag{8.83}$$

and observe that it has the the properties:

(i) $h(\overline{\pi}_1) = \overline{\omega}_1(\alpha+\beta) + \Gamma_b(0) - \Gamma_b(\overline{\lambda}_1) > 0$,

(ii) $\lim\limits_{x\to\pm\infty} h(x) = -\infty$, and

(iii) $\max[h(\pi)] < +\infty$.

Proposition 8.4. *Suppose that the structural conditions* (3.16)–(3.18)*,* (5.8)–(5.10) *and* (8.5)–(8.9) *are satisfied, along with the local stability condition* (5.33) *for the bad equilibrium of the Keen model* (5.7) *without government. Assume further that* $g_s(0) > 0$*. Then the system* (8.14) *is* λ*-UWP if either of the following four conditions is satisfied:*

(1) $\tau_s(0) = 0$ *and* $\Gamma_s(0) > \max\{r, \alpha+\beta\}$*, or*

(2) $\tau_s(0) = 0$ *and* $\lambda\Gamma_b(\lambda)$ *is bounded below as* $\lambda \to 0$*, or*

(3) $\tau_s(0) = 0$*,* $r < \Gamma_s(0) \leq \alpha + \beta$*, and* $h(x) > 0$ *whenever* $\mu(x) \in [\Gamma_s(0), \alpha + \beta]$*, or*

(4) $\Gamma_s(0) > \max\{r, \alpha + \beta\}$*,* $\Theta_s(-\infty) < 0$*,* $\Theta_s(\overline{\pi}_1) < \alpha + \beta$*, and* Θ_s *is convex.*

Proof. We prove the result by contradiction again. If $\limsup_{t\to\infty} \lambda(t) \leq \varepsilon$ for any $\varepsilon > 0$, then there exists $t_0 > 0$ such that $\lambda(t) \leq \varepsilon$ for $t > t_0$. Since we can always choose ε small enough so that $\Phi(\varepsilon) - \alpha < 0$, it follows from the equation for $\dot{\omega}$ as before that there exists $t_1 > t_0$ such that $\omega(t) < \varepsilon$ for all $t > t_1$.

For items (1) and (2), observe that it follows from UWP of e^π obtained in Proposition 8.3 that we can find a large $m_1 > 0$ such that $\limsup_{t\to\infty} \pi(t) > -m_1$. In addition, we have that $\liminf_{t\to\infty} \pi < \mu^{-1}(\alpha + \beta) = m_2$, since otherwise λ cannot converge to zero and there is nothing left to prove. Let $m = \max\{m_1, m_2\}$.

If $\Gamma_s(0) > \max\{r, \alpha + \beta\}$, we see from the equation for $\dot{\lambda}$ that

$$\exp\left[\int_{t_1}^{t} \mu(\pi_s) ds\right] < \frac{\varepsilon}{\lambda(t_1)} e^{(\alpha+\beta)(t-t_0)} \quad \forall t > t_1,$$

which implies that

$$g_s(t) > \frac{\lambda(t_1) g_s(t_1)}{\varepsilon} \exp\left[\left(\Gamma_s(\varepsilon) - (\alpha + \beta)\right)(t - t_1)\right] \quad \forall t > t_1$$

In other words, given any large $L > 0$, provided we choose ε sufficiently small so that $\Gamma_s(\varepsilon) > \alpha + \beta$, there exists $t_2 > t_1$ such that $g_s(t) > L$ for $t > t_2$. Alternatively, if $\lambda\Gamma_b(\lambda)$ is bounded below as $\lambda \to 0$, given any large $L > 0$, we can choose ε sufficiently small so that $\Gamma_b(\lambda) > L$ for $\lambda < \varepsilon$ (since $\Gamma_b(\lambda) > L_0/\lambda > L_0/\varepsilon_1$ for some $L_0 > 0$).

In either case, we can find $\varepsilon > 0$ small enough and/or $t_2 > t_1$ such that

$$-\omega[\Phi(\lambda) - \alpha] - r[\nu\mu(\pi) + \nu\delta - \pi] + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\lambda) + g_s\Gamma_s(\lambda) - \Theta_b(\pi) > \varepsilon \quad (8.84)$$

for all $\omega \in [0, \varepsilon]$, $\lambda \in [0, \varepsilon]$, $\pi \in [-m, m]$ and $t > t_2$. Since $\limsup \pi > -m$ and

$\liminf \pi < m$, we can find $t_3 > t_2$ such that $\pi(t_3) \in (-m, m)$, from which it follows from (8.84) and the equation for $\dot{\pi}$ that $\dot{\pi}(t_3) > 0$. Furthermore, $\dot{\pi}(t) > 0$ for all $t > t_3$ with $\pi(t) \leq m$. Hence, there exists $t_4 > t_3$ such that $\pi(t_4) = m$ and $\pi(t) > m$ for all $t > t_4$. But this contradicts the fact $\liminf \pi < m$, and UWP of λ follows.

For item (3), we can again find a sufficiently small ε and a sufficiently large $t_0 > 0$ such that $\omega(t) < \varepsilon$ and $\lambda(t) < \varepsilon$ for all $t > t_0$, and

$$-\omega[\Phi(\lambda) - \alpha] - r[\nu\mu(\pi) + \nu\delta - \pi] + (1 - \omega - \pi)\mu(\pi) + \Gamma_b(\varepsilon) + g_s\Gamma_s(\varepsilon) - \Theta_b(\pi) > \varepsilon$$

for all $\omega \in [0, \varepsilon]$, $\lambda \in [0, \varepsilon]$ and π in the interval such that $\Gamma_s(0) \leq \mu(\pi) \leq \alpha + \beta$. We use the fact that $\Gamma_s(0) > r$, which implies $e^{\pi} - UWP$, to obtain that π does enter the interval $[-m, m]$, for some large $m \geq \mu^{-1}(\alpha + \beta))$, at some instant $t_1 > t_0$. But since $\dot{\pi}(t) > \varepsilon$ whenever $\pi(t)$ lies in the interval such that $\Gamma_s(0) \leq \mu(\pi) \leq \alpha + \beta)$, this in turn implies that $-m < \pi < \mu^{-1}(\Gamma_s(0))$ for all $t > t_1$, because otherwise $\pi > \mu^{-1}(\alpha + \beta)$ for all large t and $\lambda(t)$ could not go to zero. However, $\mu(\pi) < \Gamma_s(0)$ for all large t implies that $g_s(t)$ can be made arbitrarily large and we have that (8.84) holds, which again leads to a contradiction.

For item (4), let $\tau_s(0) > 0$, since otherwise this reduces to item (1) and there is nothing to prove. We start by defining $v = \frac{\tau_s}{g_s}$ and observing that

$$\frac{\dot{v}}{v} = \Theta_s(\pi) - \Gamma_s(\lambda).$$

We can write $\dot{\pi}$ in terms of v and h (defined in (8.83)) as

$$\dot{\pi} = -\omega[\Phi(\lambda) - \alpha] - \omega\mu(\pi) + h(\pi) + \Gamma_b(\lambda) - \Gamma_b(0) + g_s\left[\Gamma_s(\lambda) - v\Theta_s(\pi)\right] \quad (8.85)$$

Let us now choose ε small enough such that $\Phi(\varepsilon) < \alpha$, $\Gamma_s(\varepsilon) > \alpha + \beta$ and

$$\Gamma_s(\varepsilon)\frac{\Gamma_s(\varepsilon) - 2\varepsilon}{\Gamma_s(0) + 2\varepsilon} > \Theta_s(\overline{\pi}_1), \quad (8.86)$$

which is possible because by hypothesis $\Theta_s(\overline{\pi}_1) < \alpha + \beta < \Gamma_s(0)$.

There must then exists some $t_0 > 0$ such that $\lambda(t) \leq \varepsilon$ and $\omega(t) \leq \varepsilon$ for all $t > t_0$. From UWP of e^{π}, we can find $m > 0$ large enough such that $\limsup \pi > -m$ and $\liminf \pi < m$. Let us choose m large enough such that $-m < \Theta_s^{-1}(0)$ and

$$\frac{\Gamma_s(\varepsilon) - 2\varepsilon}{\Gamma_s(0) + 2\varepsilon}\Theta_s(m) > \Gamma_s(0).$$

Using the equations for $\dot{\lambda}$ and $\dot{g}_s$, it is straightforward to see that

$$\varepsilon g_s(t) > g_s(t_0)\lambda(t_0)e^{[\Gamma_s(\varepsilon)-(\alpha+\beta)](t-t_0)} \quad \forall t > t_0, \tag{8.87}$$

which grows exponentially since $\Gamma_s(\varepsilon) > \alpha + \beta$. Accordingly, we can find $t_1 > t_0$ such that:

(i) $\varepsilon g_s(t) > \varepsilon\left[\alpha - \Phi(0) - \mu(-\infty)\right] + \max_{\pi\in\mathbb{R}}[h(\pi)]$ and

(ii) $\varepsilon g_s(t) > \varepsilon\mu(m) + \max_{\pi\in[-m,m]}|h(\pi)| + \Gamma_b(0) - \Gamma_b(\varepsilon)$ and

(iii) $\varepsilon g_s(t) > \frac{\Gamma_s(0)^2}{4\Theta_s'(-m)}$ and

(iv) $\varepsilon g_s(t) > \frac{\Theta_s(m)[\Theta_s(m)-\Gamma_s(\varepsilon)]}{\Theta_s'(-m)}$

for all $t > t_1$. As a result, $\dot{\pi}$ can be globally bounded from above by

$$\begin{aligned}\dot{\pi} &< \varepsilon\left[\alpha - \Phi(0) - \mu(-\infty)\right] + \max[h(\pi)] + g_s[\Gamma_s(0) - v\Theta_s(\pi)] &(8.88)\\ &< g_s\left[\varepsilon + \Gamma_s(0) - v\Theta_s(\pi)\right] &(8.89)\end{aligned}$$

for all $t > t_1$. In addition, we have that $\dot{\pi}$ can be locally bounded from below by

$$\dot{\pi} > -\varepsilon\mu(m) - \max_{\pi\in[-m,m]} |h(\pi)| + \Gamma_b(\varepsilon) - \Gamma_b(0) + g_s[\Gamma_s(\varepsilon) - v\Theta_s(\pi)] \tag{8.90}$$

$$> g_s\left[-\varepsilon + \Gamma_s(\varepsilon) - v\Theta_s(\pi)\right] \tag{8.91}$$

for all $t > t_1$ such that $\pi(t) \in [-m, m]$. We can therefore conclude that, for $t > t_1$ and $\pi(t) \in [-m, m]$, if $v\Theta_s(\pi) \geq \Gamma_s(0) + 2\varepsilon$, then $\dot{\pi} \leq -\varepsilon g_s$ and if $v\Theta_s(\pi) \leq \Gamma_s(\varepsilon) - 2\varepsilon$, then $\dot{\pi} \geq \varepsilon g_s$.

Moreover, we can globally bound $\dot{v}$ from both sides as

$$\Theta_s(\pi) - \Gamma_s(0) < \frac{\dot{v}}{v} < \Theta_s(\pi) - \Gamma_s(\varepsilon), \tag{8.92}$$

so that $\pi < \Theta^{-1}\left(\Gamma_s(\varepsilon)\right)$ implies $\dot{v} < 0$, whereas $\pi > \Theta^{-1}\left(\Gamma_s(0)\right)$ implies $\dot{v} > 0$.

Observe further that $\liminf \pi \geq \Theta_s^{-1}(0)$, because when $\pi \in [-m, \Theta_s^{-1}(0)]$ the lower bound for $\dot{\pi}$ becomes strictly positive for $t > t_1$, forcing π to grow higher than $\Theta_s^{-1}(0)$. We can therefore assume, without loss of generality, that $0 \leq \Theta_s(\overline{\pi}_1) \leq \Theta_s(m)$, since otherwise we would be done ($\overline{\pi}_1 = \mu^{-1}(\alpha + \beta)$ would be smaller than the lower bound of the $\liminf \pi$ and λ could not go to zero).

We shall now define the following regions, contained in $[-m, m] \times \mathbb{R}^+$

- $V := \left\{(\pi, v) \in [\Theta_s^{-1}(0), m] \times \left[\frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(m)}, +\infty\right) : \Gamma_s(\varepsilon) - 2\varepsilon \leq v\Theta_s(\pi) \leq \Gamma_s(0) + 2\varepsilon\right\}$;

- $S := \left\{(\pi, v) \in [\Theta_s^{-1}(0), m] \times \left[\frac{\Gamma_s(\varepsilon)-2\varepsilon}{\Gamma_s(0)}, \frac{\Gamma_s(0)+2\varepsilon}{\Gamma_2(\varepsilon)}\right] : \Gamma_2(\varepsilon) - 2\varepsilon \leq v\Theta_s(\pi) \leq \Gamma_s(0) + 2\varepsilon\right\}$;

- $P := \left(\Theta_s^{-1}\left(\Gamma_s(0)\right), m\right] \times \left(0, \frac{\Gamma_2(0)+2\varepsilon}{\Theta_s(m)}\right)$

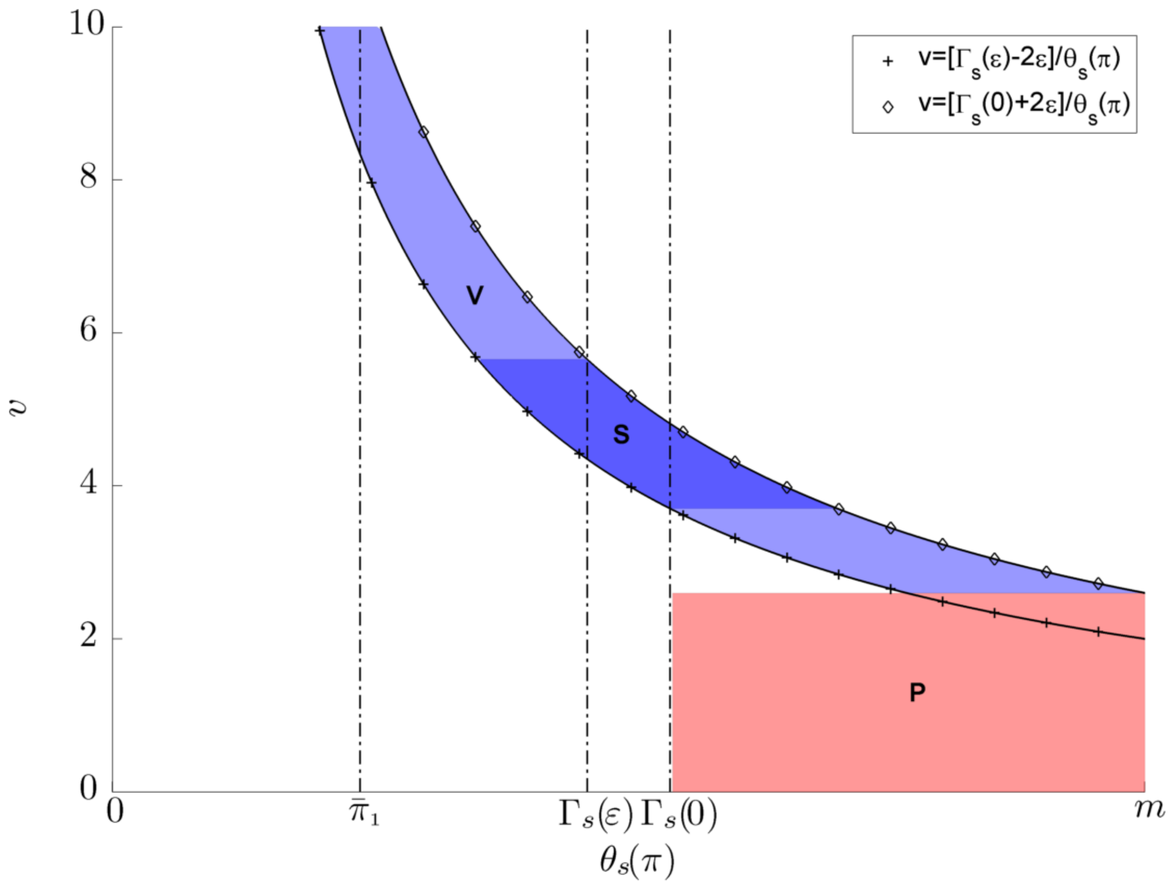

Figure 8.1: The part of the plane $(\Theta_s(\pi) \times v)$ where one can see the invariant regions V and S. Every solution that enters V eventually makes it to S and never leaves it. The region P is not invariant. Yet, solutions that enter it must eventually leave it and enter the basin of attraction of S, either directly, or after spending some time on $(\pi, v) \in [m, \infty) \times \mathbb{R}^+$.

With the bounds on $\dot{\pi}$ and $\dot{v}$ obtained above, one can observe the following (valid for $t > t_1$):

(i) The flow through $v = \frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(\pi)}$ goes inwards the region V. To see this, define the outward normal vector

$$\vec{n}_u := \left\{ \begin{array}{c} [\Gamma_s(0) + 2\varepsilon]\,\Theta_s'(\pi) \\ \Theta_s^2(\pi) \end{array} \right\}$$

and notice that the flow going through the curve obeys

$$
\begin{aligned}
\vec{n}_u \cdot \left\{ \begin{array}{c} \dot{\pi} \\ \dot{v} \end{array} \right\} &= [\Gamma_s(0) + 2\varepsilon]\, \Theta_s'(\pi)\dot{\pi} + \Theta_s^2(\pi) v \left[\Theta_s(\pi) - \Gamma_s(\lambda)\right] \\
&= [\Gamma_s(0) + 2\varepsilon]\, \Theta_s'(\pi)\dot{\pi} + \Theta_s^2(\pi) \frac{\Gamma_s(0) + 2\varepsilon}{\Theta_s(\pi)} \left[\Theta_s(\pi) - \Gamma_s(\lambda)\right] \\
&= [\Gamma_s(0) + 2\varepsilon] \left\{\Theta_s'(\pi)\dot{\pi} + \Theta_s(\pi) \left[\Theta_s(\pi) - \Gamma_s(\lambda)\right]\right\} \\
&\leq [\Gamma_s(0) + 2\varepsilon] \left\{-\varepsilon\Theta_s'(\pi) g_s + \Theta_s(\pi) \left[\Theta_s(\pi) - \Gamma_s(\varepsilon)\right]\right\} \\
&< [\Gamma_s(0) + 2\varepsilon] \left\{-\Theta_s'(\pi) \frac{\Theta_s(m)\left[\Theta_s(m) - \Gamma_s(\varepsilon)\right]}{\Theta_s'(-m)} + \Theta_s(m) \left[\Theta_s(m) - \Gamma_s(\varepsilon)\right]\right\} \\
&< 0
\end{aligned}
\tag{8.93}
$$

(ii) the flow through $v = \frac{\Gamma_s(\varepsilon) - 2\varepsilon}{\Theta_s(\pi)}$ also goes inwards the region V. To see this, define the outward normal vector

$$
\vec{n}_l := - \left\{ \begin{array}{c} [\Gamma_s(\varepsilon) - 2\varepsilon]\, \Theta_s'(\pi) \\ \Theta_s^2(\pi) \end{array} \right\}
$$

which yields

$$
\begin{aligned}
\vec{n}_l \cdot \left\{ \begin{array}{c} \dot{\pi} \\ \dot{v} \end{array} \right\} &= -\left[\Gamma_s(\varepsilon) - 2\varepsilon\right] \left\{\Theta_s'(\pi)\dot{\pi} + \Theta_s(\pi) \left[\Theta_s(\pi) - \Gamma_s(\lambda)\right]\right\} \\
&\leq -\left[\Gamma_s(\varepsilon) - 2\varepsilon\right] \left\{\varepsilon\Theta_s'(\pi) g_s + \Theta_s(\pi) \left[\Theta_s(\pi) - \Gamma_s(0)\right]\right\} \\
&< -\left[\Gamma_s(\varepsilon) - 2\varepsilon\right] \left\{\Theta_s'(\pi) \frac{\Gamma_s^2(0)}{4\Theta_s'(-m)} - \frac{\Gamma_s^2(0)}{4}\right\} < 0
\end{aligned}
\tag{8.94}
$$

where we have bounded $\Theta_s(\pi)\left[\Theta_s(\pi) - \Gamma_s(0)\right]$ by realizing that it is a quadratic polynomial like $y = x[x - \Gamma_s(0)]$ on $x \in [0, \Theta_s(m)]$, with minimum $y = -\Gamma_s^2(0)/4$.

(iii) the flow through the top side of P goes up. This is simply due to the fact that if $(\pi, v) \in P$, then $\pi > \Theta_s^{-1}(\Gamma_s(0))$, which implies that $\dot{v} > 0$.

(iv) the flow through the left side of P goes inside P. To see this, notice that for $\pi = \Theta_s^{-1}(\Gamma_s(0))$ and $v < \frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(m)}$, we have that $v\Theta_s(\pi) < \frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(m)}\Gamma_s(0) < \Gamma_s(\varepsilon) - 2\varepsilon$, hence $\dot{\pi} > 0$.

(v) once $(\pi, v) \in V$, there exists some $t_2 > t_1$ for which $(\pi, v) \in S$. One can be convinced of this from the fact that if $(\pi, v) \in V \setminus S$, then it must be either that $\pi < \Theta_s^{-1}(\Gamma_s(\varepsilon))$, in which case $\dot{v} < 0$, or that $\pi > \Theta_s^{-1}(\Gamma_s(0))$, and hence $\dot{v} > 0$. Either case, $\dot{v}$ drives the solution towards S.

Finally, the last argument goes as follows. Once π enters $[-m, m]$ (at time, say, $\hat{t}$), there exists some $t_2 > t_1$ for which $\pi(t) > \bar{\pi}_1$ for all $t > t_2$. To see this, observe that if $(\pi(\hat{t}), v(\hat{t}))$ is above the curve $v = \frac{\Theta_s(0)+2\varepsilon}{\Theta_s(\pi)}$, then it must eventually enter the region V, which then drives it to S at some future moment. If, however, $(\pi(\hat{t}), v(\hat{t}))$ starts below the curve $v = \frac{\Gamma_s(\varepsilon)-2\varepsilon}{\Theta_s(\pi)}$, then it might move to V, or P. If (π, v) enters V, we are done, as we know that it will eventually enter S and stay away from $\bar{\pi}_1$. If, however, it enters either P, we are done as well, since from that region the solution can either:

(i) leave P through its top side, entering the region of attraction of S, or

(ii) leave P through its right side, so π becomes bigger than m, while v continues growing. From there, the solution must return to $[-m, m]$ at some later time, at which it might return to P, or enter the region of attraction of V, eventually leading it to S.

In other words, every solution must eventually converge to the region S, where $\pi > \bar{\pi}_1$, which contradicts the facts that $\lambda \to 0$. Notice that it is crucial to this proof to have an unbounded region V, so we can guarantee that solutions entering $[-m, m]$ from the right, with v bigger than $\frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(\pi)}$ will eventually enter the band and find their way to the region S. Hence, the importance of having $\Theta_s(-\infty) < 0$. If this was not the case, we would not be able to eliminate cyclic solutions starting from the region P, exiting to $(m, +\infty) \times \mathbb{R}^+$, returning to $[-m, m]$ above the band, completely avoiding ($v > \frac{\Gamma_s(0)+2\varepsilon}{\Theta_s(\pi)}$ for $\pi \in [-m, m]$), escaping to $(-\infty, m) \times \mathbb{R}^+$, returning to $[-m, m]$ under the band and then return to P, which would not contradict the fact that $\lambda \to 0$. $\square$

8.3 Examples

In this section, we compare the behaviour of the solutions to the Keen model without government (5.7) to the model with government (8.14) studied in this chapter. We fixed the basic parameters according to (3.24), and chose the functions Φ and κ as in (3.25) and (5.36), with parameters according to the following constraints

$$\overline{\lambda}_1 = 0.96, \quad \Phi(0) = -0.04, \tag{8.95}$$

$$\overline{\pi}_1 = 0.16, \quad \kappa'(\bar{\pi}_1) = 5, \tag{8.96}$$

$$\kappa(-\infty) = 0, \quad \kappa(+\infty) = 1 \tag{8.97}$$

It is easily verified that the structural conditions (3.16)–(3.18), (5.8)–(5.10), (5.31), (5.33) are satisfied for these functions, meaning that, in the absence of government intervention, both the good and the bad equilibria are locally stable. The economy must converge to either of them depending on how close to the good equilibrium the solution starts.

For the model with government (8.13), we use the functions

$$\Gamma_b(\lambda) = \gamma_0(1-\lambda) \tag{8.98}$$

$$\Gamma_s(\lambda) = \gamma_1 - \gamma_2\lambda^{\gamma_3} \tag{8.99}$$

$$\Theta_b(\pi) = \theta_0 + \theta_1 e^{\theta_2\pi} \tag{8.100}$$

$$\Theta_s(\pi) = \theta_3 + \theta_4 e^{\theta_5\pi} \tag{8.101}$$

$$g_e(\pi,\lambda) = (1-\kappa(\pi))(1-\lambda)^{\gamma_4} \tag{8.102}$$

calibrated to satisfy the following

$$g_e(\overline{\pi}_1, \overline{\lambda}_1) = 0.20 \tag{8.103}$$

$$\overline{g}_{b1} = 0.004 \tag{8.104}$$

$$\overline{\tau}_{b1} = 0.08 \tag{8.105}$$

$$\Gamma_s(0) = \begin{cases} 0.02 & \text{for a timid government,} \\ 0.20 & \text{for a responsive government} \end{cases} \tag{8.106}$$

$$\Gamma_s(\overline{\lambda}_1) = \frac{1}{2}\min\{\alpha+\beta, \Gamma_s(0)\} \tag{8.107}$$

$$\Gamma_s'(\overline{\lambda}_1) = -0.5 \tag{8.108}$$

$$\lim_{\pi\to-\infty}\Theta_b(\pi) = -0.20 \tag{8.109}$$

$$\Theta_b(0) = 0 \tag{8.110}$$

$$\Theta_s(\bar{\pi}_1) = \frac{1}{2}(\alpha+\beta) \tag{8.111}$$

$$\lim_{\pi\to-\infty}\Theta_s(x) = -0.20 \tag{8.112}$$

$$\Theta_s(0) = 0 \tag{8.113}$$

The values of $\overline{g}_{b1}$ and $g_e(\overline{\pi}_1, \overline{\lambda}_1)$ were chosen according to the historical average of government subsidies and expenditure in the United States, as seen on Figures 8.2–

8.4. We chose the value of $\overline{\tau}_{b1}$ slightly higher than the historical average of government taxation as we believe that the dataset available illustrates a period of extremely low taxation.

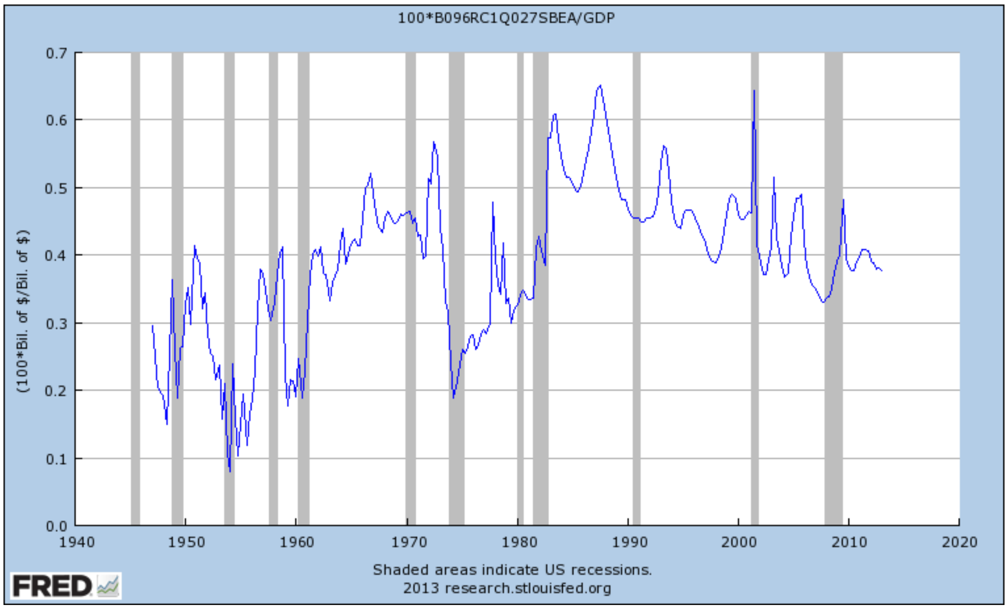

Figure 8.2: US Government subsidies over GDP.

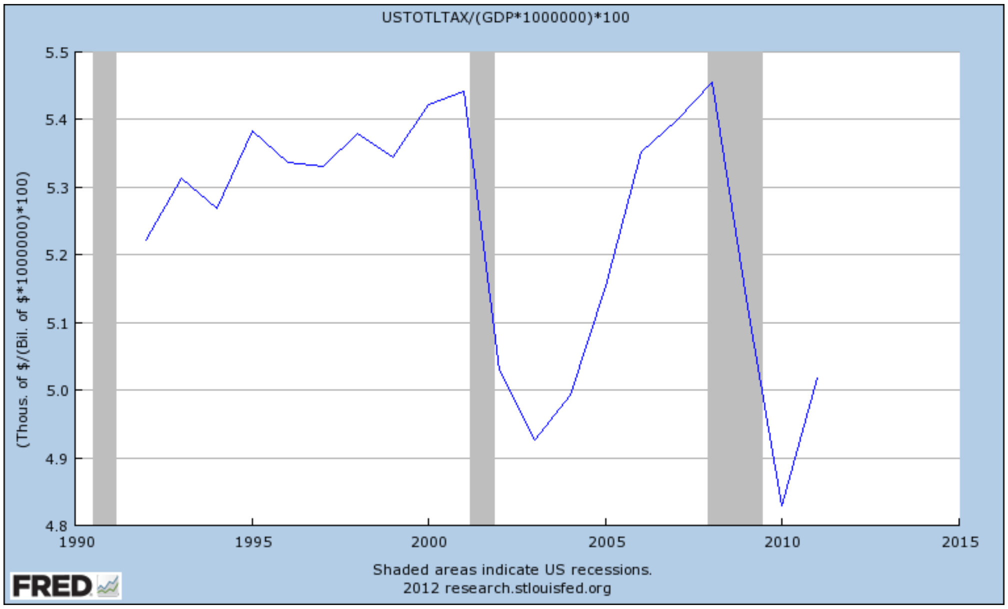

Figure 8.3: US Government taxes over GDP.

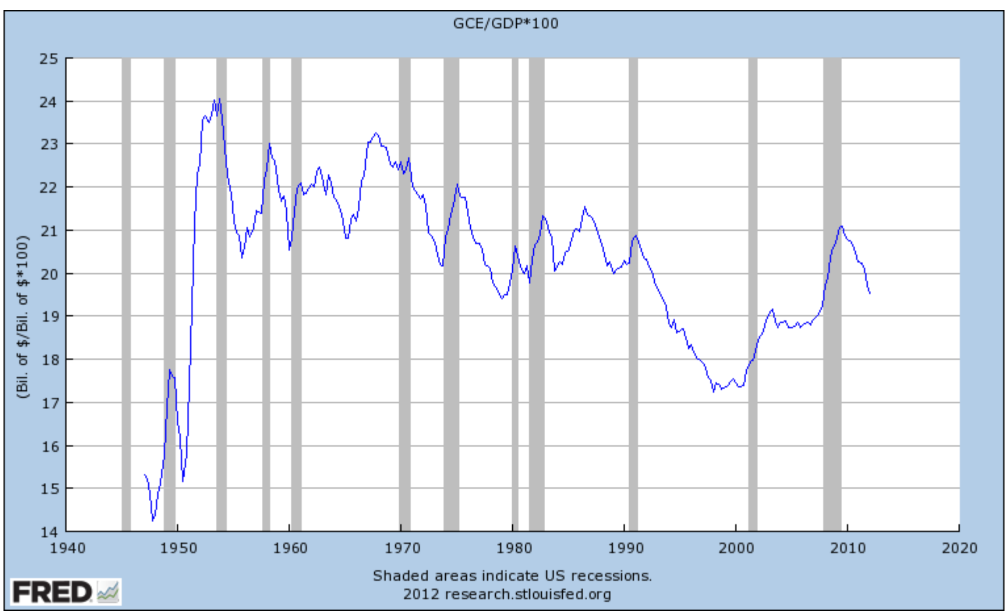

Figure 8.4: US Government expenditure over GDP.

We can again easily verify that the structural conditions (8.5)–(8.9) are satisfied.

On top of that, conditions (8.38)–(8.41) hold as well, so that the good equilibrium

$$(\omega, \lambda, g_s, \tau_s, \pi) = (\overline{\omega}_1, \overline{\lambda}_1, 0, 0, \overline{\pi}_1) = (0.76067, 0.96, 0, 0, 0.16)$$

is locally stable. Moreover, we can verify that the conditions for stability of the other finite-valued equilibria in (8.29) are easily violated for our choice of parameters, so that none of them are locally stable.

As we have seen in Proposition 8.1, the stability of the infinite-valued equilibria in the presence of government intervention depends crucially on the parameter $\Gamma_s(0) = \gamma_1$ corresponding to the maximum value of the stimulus subsidy function above.

It then follows from item (a) of Proposition 8.1 that in a stimulus regime, namely for initial conditions with $g_s(0) > 0$, equilibrium (8.74) is unstable in either case, whereas equilibrium (8.75) is stable in the case of a timid government but unstable in the case of a responsive government. On the other hand, it follows from item (b) that in an austerity regime, that is for initial conditions with $g_s(0) < 0$, equilibrium (8.76) is locally stable in either case.

Moving to the persistence results in Section 8.2, observe that condition (1) of Proposition 8.3 is satisfied in the case of a responsive government, but that neither conditions in this proposition are satisfied in the case of a timid government. As a result, provided $g_s(0) > 0$, the responsive government above ensures uniformly weakly persistence with respect to e^{π}, but the timid government does not.

Similarly, we can verify that condition (4) of Proposition 8.4 is satisfied by our responsive government even when $\tau_s(0) > 0$, but none of the conditions in this proposition are satisfied by the timid government. Consequently, provided $g_s(0) > 0$, the responsive government above ensures uniformly weakly persistence with respect to λ, whereas the timid government does not.

We illustrate these results in the next six figures. Choosing benign initial conditions, that is to say, high wage share (90% of GDP), high employment rate (90%), and low private debt (10% of GDP), we see in Figure 8.5 that the economy eventually converges to the corresponding good equilibrium with or without government intervention, even in the case of a timid government.

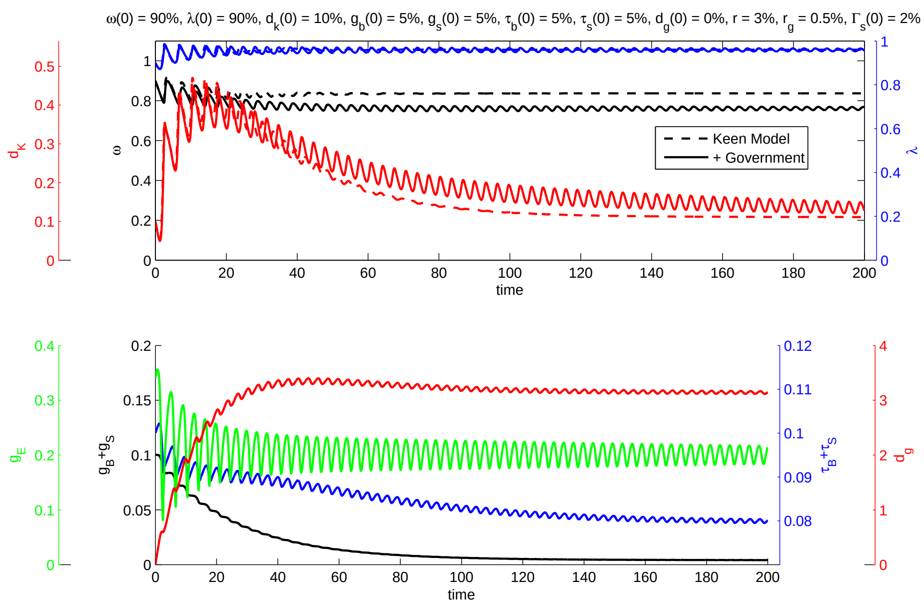

Figure 8.5: Solution to the Keen model with and without a timid government for good initial conditions.

As we move to worse initial conditions, that is lower wage share (75% of GDP), lower employment rate (80%), and higher private debt (50% of GDP), we see in Figure 8.6 that the "free economy" represented by the model without government eventually collapses to the bad equilibrium of zero wage share, zero employment and infinite private debt, whereas the model with a timid government is more robust and eventually converges to the good equilibrium.

A timid government, however, is not capable of saving the economy from a crash if the initial conditions are too extreme, for example a low wage share (75% of GDP), low employment rate (75%) and extremely high level of private debt (500% of GDP),

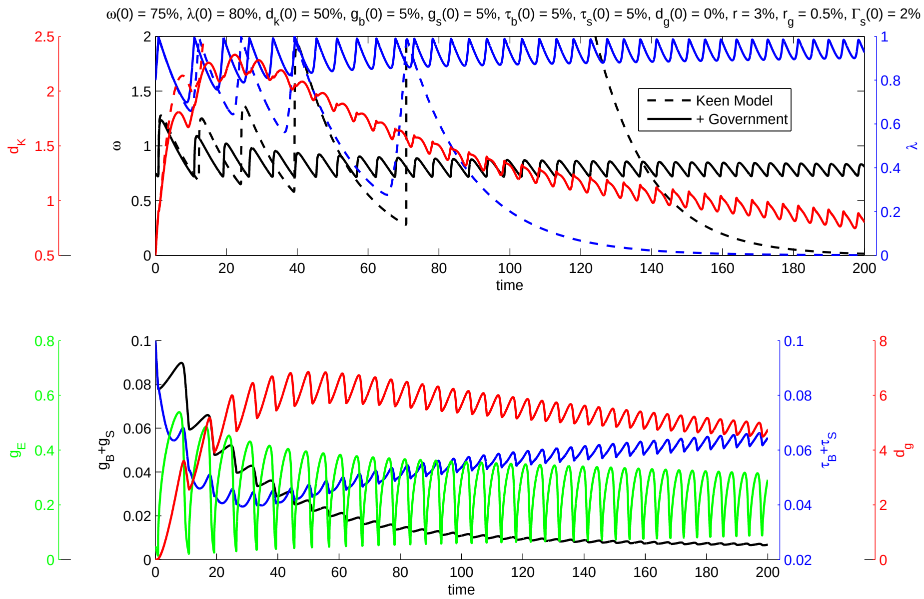

Figure 8.6: Solution to the Keen model with and without a timid government with bad initial conditions.

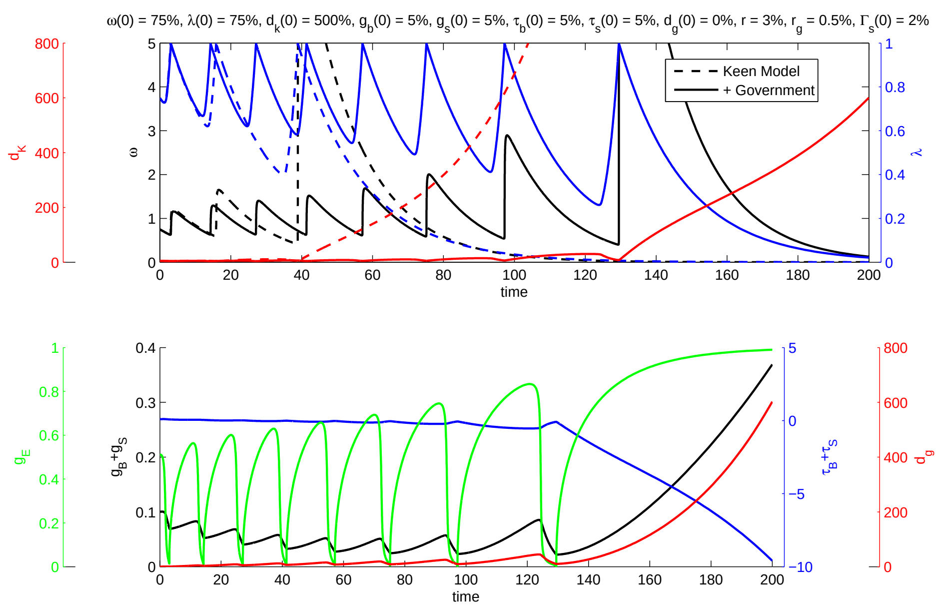

Figure 8.7: Solution to the Keen model with and without a timid government with extremely bad initial conditions.

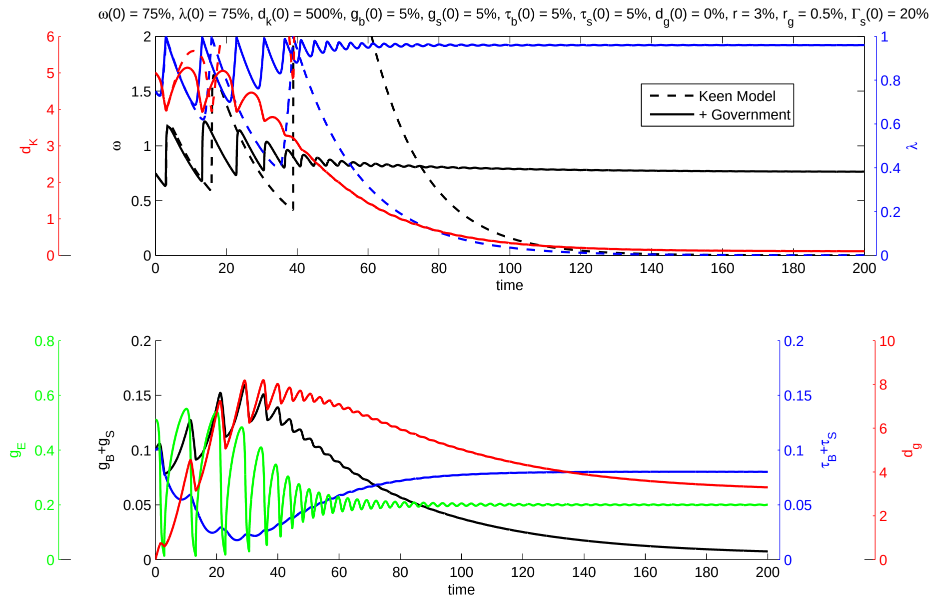

Figure 8.8: Solution to the Keen model with and without a responsive government with extremely bad initial conditions.

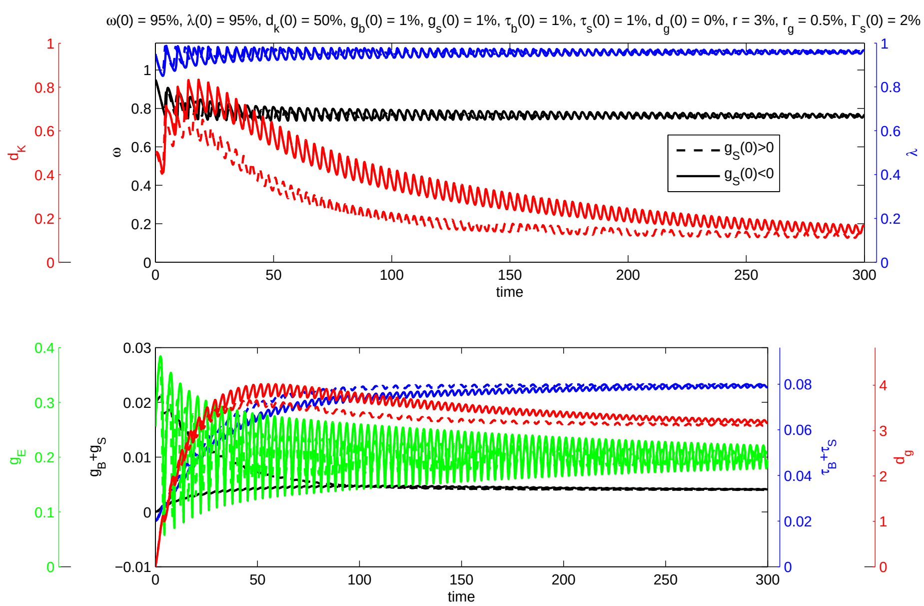

Figure 8.9: Solution to the Keen model, starting close the good equilibrium point, with positive (stimulus) and negative (austerity) government subsidies.

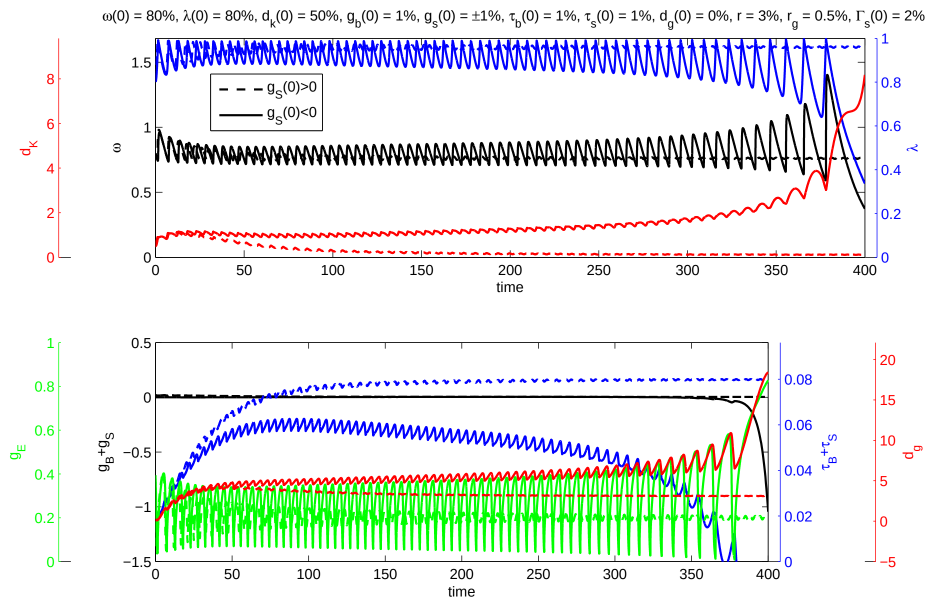

Figure 8.10: Solution to the Keen model, starting far from the good equilibrium point, with positive (stimulus) and negative (austerity) government subsidies.

as shown in Figure 8.7. On the other hand, a responsive government effectively brings the economy from the severe crisis induced by these extremely bad initial conditions, as shown in Figure 8.8.

Additionally, the effects of austerity measures are exemplified in Figures 8.9 and 8.10. For a healthy initial state, we see that the transient period suffers from the negative spending, compared to a positive stimulus, without any long term consequences. Once we push the initial state further away from the good equilibrium, we can immediately verify the disastrous consequences of austerity: the government focuses so much on reducing public debt that it throws the economy into a path of eventual collapse.

8.4 Concluding remarks

We proposed a macroeconomic model in which government intervention is a powerful tool to prevent economic meltdowns where unemployment soars and profits plunge. The model without government is essentially the one proposed by Keen in [Kee95], further analyzed in [GCL12]. There, firms make investment decisions based on their profit level: hight profits lead to heavier investments which accelerate the economy at the cost of increased private debt levels.

The extended model is not entirely novel, as Keen had already proposed an extension where government was present in [Kee95]. Unfortunately, there was no distinction between direct subsidies to firms and government expenditure in goods and services. This contrast is crucial, while the former affects the profit share in (8.10), the latter adds only to the output. This was eventually rectified in [Kee98], where Keen restricted government spending to subsidies alone, which meant that government debt no longer represented all government transactions. Rather, we explicitly model both government subsidies in (8.3) and expenditures in (8.17).

We successfully show that any of the undesirable equilibria characterized by zero employment, and zero wage share, on top of infinitely negative profit share, can be made either unstable or unachievable with sufficiently high government subsidies whenever close to full unemployment. More importantly, we prove that a government willing to promptly intervene with enough subsidies and taxation policies prevents the economy from remaining permanently trapped at arbitrarily low levels of employment and profits, no matter how extreme the initial conditions are. It may be that stabilizing an unstable economy is too tall an order for the government sector, but destabilizing a stable crises is perfectly possible.

Chapter 9

Conclusion

In this thesis, we have explored several corners of macroeconomical modeling using dynamical systems. After reviewing the notion of stock flow consistency among Minsky models in Chapter 2, we familiarized the reader with the double entry book keeping framework that is necessary to guide any serious attempt of macroeconomical modeling.

In Chapter 3 we carefully discuss the influential Goodwin model, where we make our first contribution by proposing a non-linear Phillips curve which bounds the employment variable to the $(0, 1)$ domain, providing the solution in terms of a Lyapunov function. Chapter 4 discusses a novel extension with stochastic flavor. By introducing random fluctuations in the dynamics of productivity, we obtain a continuous extension of the Goodwin model for which we examine several interesting properties. After proving existence and almost surely uniqueness of solutions, we derive probabilistic estimates based on the Lyapunov function derived for the Goodwin model. More importantly, we show that every trajectory must indefinitely loop around a center without ever converging to any point in the closure of the domain. We end the chapter showing that when the volatility of the random noise is negligible, the solution is approximately that of the Goodwin system plus a martingale term.

The mathematical formalization of Minsky's Financial Instability Hypothesis is introduced in Chapter 5. To begin with, we perform local analysis, showing the existence of two very distinct fixed points, which we intuitively call "good" and "bad" (while also discarding some "ugly" ones). Their stability is shown to be guaranteed under usual assumptions, meaning that a solution can converge to either of them depending on how close to the both of them it starts. The basin of attraction of the good equilibrium is obtained numerically, clearly showing that for large debt levels, solutions seldom converge to the good equilibrium. We also specifically analyze regimes of low interest rate in Chapter 6. Perturbation techniques similar to the ones employed in Section 4.3, are used to suggest an approximate solution which is then fully solved analytically. Its accuracy is verified through numerical examples, confirming the expected restriction to finite time intervals. On a different direction, we propose an extension of the Keen model where we relax the assumption that capital projects are immediately developed. Once more, we perform local analysis verifying the existence of the key equilibrium points. This time, however, we find that when the average completion time of such projects rises beyond a first threshold value, the good equilibrium loses is stability giving place to a stable limit cycle. As the average time grows, so does the period of these cycles, until a second threshold is crossed and these limit cycles cease to exist. This behavior is completely novel, as so far we had only observed cycles in the zero-order solution of the approximate model in Chapter 6.

Perhaps our main contribution to macroeconomical theory lies in Chapter 8. After proposing an extension to the Keen model with government intervention in a very general setting, we proceed to study its many equilibria and their respective local stability. The many equilibria labeled as "bad" for being represented by negative exploding profits, with zero employment and wage share, are easily destabilized if the government stimulus subsidies are designed to be responsive enough under adverse conditions. Most significantly, we prove that under several mild conditions characterizing a responsive government, the model is uniformly weakly persistent with respect to both profits and employment. In other words, when the government is willing to act responsively, capitalists' profits and employment are guaranteed to never be trapped under arbitrarily low levels, regardless of the initial state of the system.

We believe that this thesis paves the road for a variety of extensions. Among the many future projects one could be interested in, we suggest a few. For instance, one can model prices through a dynamical equation designed to converge to an equilibrium price, which itself could be obtained through supply-demand arguments. Such a model should be able to answer questions regarding the effect of inflation in an economy, reproducing stylized scenarios such as stagflation, or even hyperinflation. Alternatively, stock prices could be introduced by allowing firms and/or banks to capitalize themselves by issuing stocks. These should then take part in the portfolio choice of households, who should take into consideration the risk/return profile of all the instruments available to them. Further down the road, one could enlarge the model by adding securitization, issued by special purpose vehicles, and study the effect of structured products, such as the stigmatized mortgage-backed securities, in the whole economy. This thesis provides a useful toolbox for a thorough analysis of any of the extensions suggested above. Even though one could argue that none of the ideas just proposed are intrinsically novel, the depth and precision that one could reach with the apparatus presented here would certainly be so.

Appendices

Appendix A

Stochastic Dynamical Systems Toolbox

Consider the n-dimensional stochastic process $\vec{x}_t$, with dynamics

$$d\vec{x}_t = \vec{b}(t, \vec{x}_t)dt + \sum_{r=1}^{k} \sigma_r(t, \vec{x}_t)\, d\vec{W}_r(t) \tag{A.1}$$

where $W_1(t), W_2(t), \cdots, W_r(t)$ are mutually independent Brownian motions. The next couple of results are Theorem 3.5, and Corollary 3.1 from [Kha12].

Theorem A.1. *Assume that*

$$\begin{aligned} ||b(s,\vec{x}) - b(s,\vec{y})|| + \sum_{r=1}^{k} ||\sigma_r(s,\vec{x}) - \sigma_r(s,\vec{y}|| &\leq B\, ||\vec{x} - \vec{y}|| \\ ||b(s,\vec{x})|| + \sum_{r=1}^{k} ||\sigma_r(s,\vec{x})|| &\leq B(1 + ||\vec{x}||) \end{aligned} \tag{A.2}$$

hold in every cylinder $\mathbb{R}^+ \times \{||\vec{x}|| < R\}$, for any $R > 0$. Moreover, suppose that there exists a nonnegative function $V \in C^2$ on the domain D such that for some constant

$c > 0$

$$\mathcal{L}V \leq cV, \tag{A.3}$$

$$V_R = \inf_{||\vec{x}||>R} V(t,\vec{x}) \to \infty \text{ as } R \to \infty \tag{A.4}$$

Then:

1. *For every random variable $\vec{x}(t_0)$ independent of the process $W_r(t) - W_t(t_0)$ there exists a solution $\vec{x}(t)$ of* (A.1) *which is an almost surely continuous stochastic process and is unique up to equivalence;*

2. *This solution is a Markov process whose Feller transition probability function $P(s, \vec{x}_s, t, A)$ is defined for $t > s$ by the relation $P(s, x, t, A) = \mathbb{P}\left[\vec{x}^{s,\vec{x}_s}(t) \in A\right]$, where $\vec{x}^{s,\vec{x}_s}(t)$ is a solution of the equation*

$$\vec{x}^{s,\vec{x}_s}(t) = \vec{x}_s + \int_s^t \vec{b}(u, \vec{x}^{s,\vec{x}_s}(u)\, du + \sum_{r=1}^{k} \int_s^t \sigma_r(u, \vec{x}^{s,\vec{x}_s}(u))\, dW_r(u) \tag{A.5}$$

3. *If the functions $\vec{b}(t,\vec{x})$ and $\vec{\sigma}_r(t,\vec{x})$ are independent of t, then the transition probability function of the corresponding Markov process is time-homogeneous; and if the coefficients are T-periodic in t, then the transition probability is T-periodic.*

Corollary A.1. *Consider an increasing sequence of open sets (D_n) whose closure are contained in D and such that $\bigcup_n D_n = D$. Suppose that in every cylinder $\mathbb{R}^+ \times D_n$ the coefficients $\vec{b}$ and $\vec{\sigma}_r$ satisfy conditions* (A.2) *and there exists a function $V(t,\vec{x})$, twice continuously differentiable in $\vec{x}$ and continuously differentiable in t in the domain $\mathbb{R}^+ \times D$, which satisfies condition* (A.3) *and the condition*

$$\inf_{t>0, \vec{x}\in D\setminus D_n} V(t,\vec{x}) \to \infty \text{ as } n \to \infty \tag{A.6}$$

then the conclusion of Theorem A.1 holds provided that also $\mathbb{P}\left[\vec{x}(t_0) \in D\right] = 1$*. Moreover, the solution satisfies the relation*

$$\mathbb{P}\left[\vec{x}(t) \in D\right] = 1 \textit{ for all } t \geq t_0 \tag{A.7}$$

The next Theorem is a slightly more general version of the Corollary above, though applied to the stochastic dynamical system 4.2.

Theorem A.2. *Consider an increasing sequence of open sets* (D_n) *whose closure are contained in* D *and such that* $\bigcup_n D_n = D$*. Assume that in each set* D_n*, the drift and volatility coefficients of system* (4.2) *are Lipschitz and sub-linear functions. Assume that there exists a function* $\mathcal{V}(\omega, \lambda, t) \in C^{2,1}(D \times \mathbb{R}_+)$ *such that*

$$\mathcal{L}\mathcal{V}(\omega, \lambda, t) \leq k_1 \mathcal{V}(\omega, \lambda, t) + k_2 \qquad \textit{on } D \times \mathbb{R}_+ \tag{A.8}$$

for some $k_1, k_2 \in \mathbb{R}_+$*, and*

$$\inf_{D \setminus D_n} \mathcal{V}(\omega, \lambda, t) \to \infty \textit{ as } n \to \infty \tag{A.9}$$

then the system (4.2) *possesses a unique almost surely continuous regular solution for* $(\omega_0, \lambda_0) \in L^0(D, \mathcal{F}_0)$ *satisfying* $\mathbb{P}\left[(\omega_t, \lambda_t) \in D\right] = 1$ *for all* $t \geq 0$*.*

Proof. The proof follows from applying Corollary A.1 with the function $V = \mathcal{V} + k_2/k_1$, for which we have

$$\begin{aligned} \mathcal{L}V &= \mathcal{L}\left(\mathcal{V} + k_2/k_1\right) = \mathcal{L}\mathcal{V} \leq k_1 \mathcal{V} + k_2 \leq k_1 \left(\mathcal{V} + k_2/k_1\right) \\ &= k_1 V \end{aligned} \tag{A.10}$$

Observe that condition (A.2) holds from the local Lipschitz and sub-linearity assumptions on the drift and the volatility, while $V(\omega, \lambda, t)$ satisfies conditions (A.3) and (A.6), immediately giving us the desired result. ∎

The next result is Theorem 3.9 from [Kha12], simply adapted to the notation of Chapter 4.

Theorem A.3. *Let $(\omega_t, \lambda_t)_{t\geq 0}$ be a regular process in D, with $(\omega_0, \lambda_0) \in U$, for some $U \subset D$. Assume that there exists a function $V(t, \omega, \lambda) \in C^{1,2,2}(\mathbb{R}_+ \times U)$ verifying*

$$\mathcal{L}V(s, \omega, \lambda) \leq -f(s) \tag{A.11}$$

where $f(s) \geq 0$ and $\lim_t \int_0^t f(s)\, ds = +\infty$. Then (ω_t, λ_t) exits the region U in finite time $\mathbb{P} -$ a.s..

Appendix B

XPPAUT Instructions

B.1 AUTO instructions for Figures 7.2 – 7.4

We used the following .ode file

```
######################################
## keen_erlang.ode
######################################
## EQUATIONS:
omega'=omega*(Phi(lambda)-alpha)
tan_lambda'=(1+tan_lambda^2)*PI*lambda*(g_Y-alpha-beta)
d'=Kappa(pi_n)-pi_n-d*g_Y
theta_[75..2]'=(n/tau)*(theta_[j-1]-theta_[j])-theta_[j]*g_Y
theta_1'=Kappa(pi_n)-theta_1*(n/tau+g_Y)

## ALIAS
lambda=atan(tan_lambda)/PI+0.5
pi_n=1-omega-r*d
theta_n=theta_75
```

```
g_Y=n/tau/nu*theta_n-delta
phi0=(alpha*(1-lambda_eq)^2-Phi_min)/(1-(1-lambda_eq)^2)
phi1=phi0+Phi_min
kappa_eq=nu*(alpha+beta+delta)
kappa1=(kappa_U-kappa_L)/pi
kappa0=kappa_U-kappa1*pi/2
kappa2=kappa_pri*(1+tan((kappa_eq-kappa0)/kappa1)^2)/kappa1
kappa3=tan((kappa_eq-kappa0)/kappa1)-kappa2*pi_eq

aux auxLambda=atan(tan_lambda)/PI+0.5
aux auxTau=tau

## FUNCTIONS
Phi(u)=phi1/(1-u)^2-phi0
Kappa(u)=kappa0+kappa1*atan(kappa2*u+kappa3)

## PARAMETERS:
par tau=0.01,r=0.03,nu=3,alpha=0.025,beta=0.02,delta=0.01
par lambda_eq=0.96,Phi_min=-0.04
par pi_eq=0.16,kappa_pri=500,kappa_L=0,kappa_U=1
par n=75

## INITIAL CONDITIONS:
init omega=0.8366
init tan_lambda=7.916
init d=0.1137
init theta_[1..75]=0
```

```
## XPP SETUP
@ meth=gear
@ total=200,dt=0.01,bounds=1e10
@ maxstor=2000000,back=white

done
```

The sequence of commands below produce the bifurcation diagram shown in Figures 7.2 – 7.4:

1. open the file keen_erlang.ode;
2. integrate it once, through, (I)nitialconds, (G)o;
3. integrate it a couple more times, to make sure it converges and stabilizes at the equilibrium: (I)nitialconds, (L)ast, twice;
4. open AUTO: (F)ile, (A)uto;
5. change the axis to properly accommodate the diagram that will follow: **(A)xes**, **h(I)-lo**: Xmin=0.03, Ymin=0.83, Xmax=0.06, Ymax=0.845;
6. change the (N)umerics to Nmax=2000, Ds=0.0001, Dsmin=0.00005, Dsmax=0.0005, Par Min=0, Par Max=0.1;
7. **(R)un**, **(S)teady State**;

8. change the (N)umerics to Nmax=200, Ds=0.01, Dsmin=0.005, Dsmax=0.05;

9. **(G)rab** the (HB) point;

10. **(R)un**, **(P)eriodic**. **(A)BORT** if AUTO stops responding;

11. **(F)ile**, **(P)ostscript** to save the diagram;

12. **(A)xes**, **(P)eriod**: : Xmin=0.03, Ymin=0, Xmax=0.06, Ymax=1.3;

13. **(F)ile**, **(P)ostscript** to save the diagram;

B.2 Brute-force diagram for Figure 7.5

The following code was used

```
#####################################
## keen_delay.ode
#####################################
## EQUATIONS:
omega'=omega*(Phi(lambda)-alpha)
tan_lambda'=(1+tan_lambda^2)*PI*lambda*(g_Y-alpha-beta)
d'=kappa(pi_n)-pi_n-d*g_Y
z'=z*(g_Y_d-g_Y)

## ALIAS
```

```
lambda=atan(tan_lambda)/PI+0.5
pi_n=1-omega-r*d
pi_n_d=1-delay(omega,tau)-r*delay(d,tau)
pi_n_2d=1-delay(omega,2*tau)-r*delay(d,2*tau)
g_Y=kappa(pi_n_d)/nu*z-delta
g_Y_d=kappa(pi_n_2d)/nu*delay(z,tau)-delta
phi0 = (alpha*(1-lambda_eq)^2-Phi_min)/(1-(1-lambda_eq)^2)
phi1 = phi0+Phi_min
kappa_eq=nu*(alpha+beta+delta)
kappa1 = (kappa_U-kappa_L)/PI
kappa0 = kappa_U-kappa1*PI/2
kappa2 = kappa_pri*(1+tan((kappa_eq-kappa0)/kappa1)^2)/kappa1
kappa3 = tan((kappa_eq-kappa0)/kappa1)-kappa2*pi_eq


aux auxLambda=atan(tan_lambda)/PI+0.5
aux auxTau=tau
#aux auxG_Y=g_Y
#aux auxPi=pi_n
#aux auxPi_d=pi_n_d
#aux omega_d=delay(omega,tau)
#aux rd_d=r*delay(d,tau)
## FUNCTIONS
Phi(u)=phi1/(1-u)^2-phi0
Kappa(u)=kappa0+kappa1*atan(kappa2*u+kappa3)

## PARAMETERS:
par tau=0.04,r=0.03,,nu=3,alpha=0.025,beta=0.02,delta=0.01
```

```
par lambda_eq=0.96,Phi_min=-0.04
par pi_eq=0.16,kappa_pri=500,kappa_L=0,kappa_U=1
## INITIAL CONDITIONS:
omega(0)=0.8366171
tan_lambda(0)=7.915815
d(0)=0.1127582
z(0)=0.9995501
init omega=0.836171
init tan_lambda=7.915815
init d=0.1127582
init z=0.9995501

## XPP SETUP
@ meth=qualrk4
@ total=150,delay=1,dt=1e-3,bounds=1e10,tol=1e-10
@ trans=100
@ maxstor=2000000,back=white

## POINCARE MAP SET UP:
@ poimap=section,poivar=t,poipln=1
## range set up
@ range=1, rangeover=tau, rangestep=2000
@ rangelow=0.03, rangehigh=0.06, rangereset=no

## STORAGE
@ output=bruteforce_KeenDelay_tau.dat
done
```

The above code was executed in silent mode, that is, through the command

```
xppaut -silent keen_delay.ode
```

The output was then treated handle through the following **R** code

```
##########################
## bruteforce_keen_delay.R
## Author: Bernardo R. C. da Costa Lima
## Created: 13 Jun 2013
## Read data file created by xppaut and
## create pdf picture of brute force bifurcation diagram wrt tau
## for the Keen model with construction delay.
##########################
## read data
rawdata <- read.table("bruteforce_KeenDelay_tau.dat")

## define variables
time <- rawdata[,1]
omega <- rawdata[,2]
tau <-rawdata[,7]

## plot set up
pdf("keen_delay_bruteforce_tau_n_inf.pdf",width=10, height=10/1.62)
par(mar=c(4,4,2,0.5))
plot(tau,omega, type="p",
 xlab=expression(paste(tau)),
 ylab=expression(paste(omega)),
 col="blue", pch=".", cex=1, ylim=c(0.83,0.845))
dev.off()
```

with the following console command

```
R CMD BATCH --vanilla bruteforce_keen_delay.R
```

producing the desired pdf figure.

Appendix C

Persistence Definitions

Let $\Phi(t,x) : \mathbb{R}^+ \times X \to X$ be the semiflow generated by a differential system with initial values $x \in X$. For a nonnegative functional ρ from X to $\mathbb{R}^+$, we say

- Φ is $\rho-$ uniformly strongly persistent (USP) if there exists an $\varepsilon > 0$ such that $\liminf_{t\to\infty} \rho(\Phi(t,x)) > \varepsilon$ for any $x \in X$ with $\rho(x) > 0$.

- Φ is $\rho-$ uniformly weakly persistent (UWP) if there exists an $\varepsilon > 0$ such that $\limsup_{t\to\infty} \rho(\Phi(t,x)) > \varepsilon$ for any $x \in X$ with $\rho(x) > 0$.

- Φ is $\rho-$ strongly persistent (SP) if $\liminf_{t\to\infty} \rho(\Phi(t,x)) > 0$ for any $x \in X$ with $\rho(x) > 0$.

- Φ is $\rho-$ weakly persistent (WP) if $\limsup_{t\to\infty} \rho(\Phi(t,x)) > 0$ for any $x \in X$ with $\rho(x) > 0$.

As an example, consider the Goodwin model (3.12). From Chapter 3, we know that the solution passing through the initial condition (ω_0, λ_0) satisfies the equation (3.20).

The closed periodic orbits implied by this equation are shown in Figure 3.1. Recalling that $\pi = 1 - \omega$ for this model, observe that ω remains bounded on each orbit, so that $\liminf_{t\to\infty} \exp(1-\omega) > 0$ and the system is $e^{\pi}-$ strongly persistent. However, since the bound on ω can be made arbitrarily large by changing the initial conditions, we see that the system is not e^π – uniformly strongly persistent. Finally, we see in Figure 3.1 that ω becomes smaller than the equilibrium value $\overline{\omega}$ infinitely often, regardless of the initial conditions. Therefore, taking $\varepsilon < \exp(1 - \overline{\omega})$ shows that the system is e^π – uniformly weakly persistent. The exact same arguments show that the Goodwin model (3.12) is λ–SP, λ–UWP, but not λ–USP.

For the Keen model without government intervention defined in (5.7) the situation is less satisfactory. Whenever the conditions for local stability of the bad equilibrium (5.20) are satisfied, we cannot have either λ or e^π persistence of any form, since initial conditions sufficiently close to the bad equilibrium will necessarily lead to $\lambda = e^\pi = 0$ asymptotically.

Bibliography

[AC08] L. Aguiar-Conraria. A note on the stability properties of Goodwin's predator—prey model. *Review of Radical Political Economics*, 40(4):518–523, 2008.

[AD80] W. R. Anderson D. On the spread of a disease with gamma distributed latent and infectious periods. *Biometrika*, 67(1):191–198, April 1980.

[BT68] W. C. Brainard and J. Tobin. Pitfalls in financial model building. *The American Economic Review*, 58(2):99–122, 1968.

[CGH13] B. Costa Lima, M. R. Grasselli, and A. N. Huu. Stochastic orbits of the Goodwin-Lotka-Volterra model with Brownian noise. In preparation, 2013.

[CGWW13] B. Costa Lima, M. R. Grasselli, X.-S. Wang, and J. Wu. Destabilizing a stable crisis: persistence and government intervention in macroeconomics. Submitted to Journal of Economic Dynamics and Control, 2013.

[Chi90] C. Chiarella. *The elements of a nonlinear theory of economic dynamics*. Lecture notes in economics and mathematical systems. Springer-Verlag, 1990.

[Des73] M. Desai. Growth cycles and inflation in a model of the class struggle. *Journal of Economic Theory*, 6(6):527–545, December 1973.

[DGG92] D. Delli Gatti and M. Gallegati. Imperfect information, corporate finance, debt commitments and business fluctuations. *Financial Conditions and Macroeconomic Performance: Essays in Honor of Hyman P. Minsky, Armonk: ME Sharpe*, 1992.

[DGGG90] D. Delli Gatti, M. Gallegati, and L. Gardini. Real accumulation and financial instability: A model of profit flows, debt commitments and capital asset prices. *Studi Economici*, 41(2):101–126, 1990.

[DGGM98] D. Delli Gatti, M. Gallegati, and H. Minsky. Financial institutions, economic policy, and the dynamic behavior of the economy. *Economic Policy, and the Dynamic Behavior of the Economy (October 1994). Jerome Levy Economics Institute Working Paper*, (126), 1998.

[DHMP06] M. Desai, B. Henry, A. Mosley, and M. Pemberton. A clarification of the Goodwin model of the growth cycle. *Journal of Economic Dynamics and Control*, 30(12):2661–2670, December 2006.

[DN88] V. De Nicolò. Learning by doing and cyclical growth. In G. Ricci and K. Velupillai, editors, *Growth Cycles and Multisectoral Economics: the Goodwin Tradition*, volume 309 of *Lecture Notes in Economics and Mathematical Systems*, pages 62–82. Springer Berlin Heidelberg, 1988.

[DS05] C. H. Dos Santos. A stock-flow consistent general framework for formal Minskyan analyses of closed economies. *Journal of Post Keynesian Economics*, 27(4):712–735, 2005.

[EMT08] J. Eatwell, T. Mouakil, and L. Taylor. Liquidity, leverage and the impact of sub-prime turbulence. In *Centre for Financial Analysis and Policy, Judge Business School, Cambridge, 4th Cambridge-Princeton conference*, 2008.

[FK92] M. Farkas and M. Kotsis. Modelling predator-prey and wage-employment dynamics. *Dynamic Economic Models and Optimal Control. Elsevier Science Publishers, Amsterdam*, pages 513–526, 1992.

[FM98] L. Fanti and P. Manfredi. A Goodwin-type growth cycle model with profit-sharing. *ECONOMIC NOTES-SIENA-*, pages 371–402, 1998.

[FS91] R. Franke and W. Semmler. A dynamical macroeconomic growth model with external financing of firms: A numerical stability analysis. *Nicholas Kaldor and Mainstream Economics: Confrontation or Convergence*, pages 335–59, 1991.

[GCL12] M. R. Grasselli and B. Costa Lima. An analysis of the Keen model for credit expansion, asset price bubbles and financial fragility. *Mathematics and Financial Economics*, 2012.

[GL07] W. Godley and M. Lavoie. Monetary economics. An integrated approach to credit, money, income, production and wealth., 2007.

[God04] W. Godley. *Towards a Reconstruction of Macroeconomics Using a Stock Flow Consistent (SFC) Model.* Working paper (Cambridge Endowment for Research in Finance). Cambridge Endowment for Research in Finance, 2004.

[Goo67] R. M. Goodwin. A growth cycle. In C. H. Feinstein, editor, *Socialism, Capitalism and Economic Growth*, pages 54–58. Cambridge University Press, 1967.

[Goo91] R. M. Goodwin. *Economic evolution, chaotic dynamics and the Marx-Keynes-Schumpeter system*, pages 138–52. Edward, 1991.

[Har00] D. Harvie. Testing Goodwin: Growth cycles in ten OECD countries. *Cambridge Journal of Economics*, 24(3):349–76, May 2000.

[HSD04] M. Hirsch, S. Smale, and R. Devaney. *Differential equations, dynamical systems, and an introduction to chaos.* Pure and applied mathematics. Academic Press, 2004.

[Hsu83] S.-B. Hsu. A remark on the period of the periodic solution in the Lotka-Volterra system. *Journal of Mathematical Analysis and Applications*, 95(2):428 – 436, 1983.

[Ise88] D. L. Isenberg. Is there a case for Minsky's financial fragility hypothesis in the 1920s? *Journal of Economic Issues*, 22(4), 1988.

[Ise94] D. L. Isenberg. Financial fragility and the great depression: New evidence on credit growth in the 1920s. *New Perspectives in Monetary Macroeconomics*, pages 201–229, 1994.

[Kee95] S. Keen. Finance and economic breakdown: Modeling Minsky's "Financial Instability Hypothesis". *Journal of Post Keynesian Economics*, 17(4):pp. 607–635, 1995.

[Kee98] S. Keen. The nonlinear dynamics of debt deflation. *Complexity International*, 6, 1998.

[Kee09] S. Keen. Household debt: The final stage in an artificially extended ponzi bubble. *Australian Economic Review*, 42(3):347–357, 2009.

[Kha12] R. Khasminskii. *Stochastic Stability of Differential Equations.* Applications of Mathematics. Springer Berlin Heidelberg, 2012.

[Lav08] M. Lavoie. Towards a post-Keynesian consensus in macroeconomics: Reconciling the Cambridge and Wall Street views. In *12th Conference of the Research Network Macroeconomics and Macroeconomic Policy, "Macroeconomic Policies on Shaky foundations"-Whither Mainstream Economics*, 2008.

[LG01] M. Lavoie and W. Godley. Kaleckian models of growth in a coherent stock-flow monetary framework: a Kaldorian view. *Journal of Post Keynesian Economics*, pages 277–311, 2001.

[Lot25] A. J. Lotka. *Elements of physical biology*. Williams & Wilkins Baltimore, 1925.

[Med79] A. Medio. *Teoria nonlineare del ciclo economico*. Ricerche economiche. Il mulino, 1979.

[MF01] P. Manfredi and L. Fanti. Gestation lags and efficiency wage mechanisms in a Goodwin-type growth cycle model. *Studi Economici*, 2001.

[Min82] H. P. Minsky. *Can 'it' happen again ?* M E Sharpe, 1982.

[MMR02] X. Mao, G. Marion, and E. Renshaw. Environmental brownian noise suppresses explosions in population dynamics. *Stochastic Processes and their Applications*, 97(1):95 – 110, 2002.

[Pal96] T. I. Palley. *Post Keynesian Economics: Debt, Distribution and the Macro Economy*. Macmillan Press London, 1996.

[Rad05] M. P. Radke. *Explaining financial crises*. Peter Lang, 2005.

[Sat85] Y. Sato. Marx-Goodwin growth cycles in a two-sector economy. *Zeitschrift fr Nationalkonomie*, 45(1):21–34, 1985.

[Shr04] S. E. Shreve. *Stochastic calculus for finance. II*. Springer Finance. Springer-Verlag, New York, 2004. Continuous-time models.

[Sko81] P. Skott. On the Kaldorian saving function. *Kyklos*, 34(4):563–581, 1981.

[Sko88] P. Skott. Finance, saving and accumulation. *Cambridge Journal of Economics*, 12(3):339–354, 1988.

[Sko89a] P. Skott. *Conflict and effective demand in economic growth*. Cambridge University Press, 1989.

[Sko89b] P. Skott. Effective demand, class struggle and cyclical growth. *International Economic Review*, pages 231–247, 1989.

[Sko94] P. Skott. *On the modelling of systemic financial fragility*. Institute of Economics, University of Aarhus, 1994.

[SMJ98] J. G. Simmonds and J. E. Mann Jr. *A first look at perturbation theory*. Courier Dover Publications, 1998.

[Sol90] R. M. Solow. Goodwin's growth cycle: reminiscence and rumination. In K. Velupillai, editor, *Nonlinear and Multisectoral Macrodynamics*. Macmillan, London, 1990.

[Son72] H. Sonnenschein. Market excess demand functions. *Econometrica*, 40(3):pp. 549–563, 1972.

[Spo95] M. C. Sportelli. A Kolmogoroff generalized predator-prey model of Goodwin's growth cycle. *Journal of Economics*, 61(1):35–64, February 1995.

[SR08] P. Skott and S. Ryoo. Macroeconomic implications of financialisation. *Cambridge Journal of Economics*, 32(6):827–862, 2008.

[ST11a] J. Slane and S. Tragesser. Analysis of periodic nonautonomous inhomogeneous systems. *Nonlinear Dynamics and Systems Theory*, 11(2):183–198, 2011.

[ST11b] H. L. Smith and H. R. Thieme. *Dynamical systems and population persistence*, volume 118. American Mathematical Soc., 2011.

[Tay85] L. Taylor. A stagnationist model of economic growth. *Cambridge Journal of Economics*, 9(4):383–403, 1985.

[Tir85] J. Tirole. Asset bubbles and overlapping generations. *Econometrica*, 53(6):pp. 1499–1528, 1985.

[TO85] L. Taylor and S. A. O'Connell. A Minsky crisis. *The Quarterly Journal of Economics*, 100(Supplement):871–885, 1985.

[vdP85] F. van der Ploeg. Classical growth cycles. *Metroeconomica*, 37(2):221–230, 1985.

[VM06] R. Veneziani and S. Mohun. Structural stability and Goodwin's growth cycle. *Structural Change and Economic Dynamics*, 17(4):437–451, December 2006.

[Vol27] V. Volterra. *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi*. C. Ferrari, 1927.

[ZDS04] G. Zezza and C. H. Dos Santos. The role of monetary policy in post-Keynesian stock-flow consistent macroeconomic growth models: preliminary results. *Central Banking in the Modern World: Alternative Perspectives*, pages 183–208, 2004.

[Zez08] G. Zezza. US growth, the housing market, and the distribution of income. *Journal of Post Keynesian Economics*, 30(3):375–401, 2008.