Aspects of Abstraction in the Mesopotamian Mathematics

Matheus da Rocha Grasselli
Instituto de Física da Universidade de São Paulo
Rua do Matão, Travessa R, 748. Cidade Universitária
São Paulo, Brasil

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1 In search of a pre-Hellenic Mathematics

The idea of a Greek miracle, which attributes to the Hellenes the invention of politics, ethics and philosophy, insistently places mathematics, in whichever definition, as a "typically Greek fact", with no precedents in the Ancient World. This concept is, however, even contradicted by Greek sources: Thales of Miletus was called the "disciple of the Egyptians and the Babylonians" and Democritus of Abdera bragged that he was a geometer even better than the "cordlayers from Egypt". It was only after much work in transcription, translation and compilation of Egyptian and Mesopotamian mathematical texts, mostly done in the first decades of this century [8, 9, 12], that the birth of mathematics was shifted to centuries before the era of Thales and Pythagoras.

New unjustified dogmas, however, began to lead the opinions about the mathematical sciences from Egypt and Mesopotamia. It is argued to these days that all science that took place around the Nile and between the Tigris and Euphrates has had a pragmatic orientation, lacking general and abstract principles. In this way, the set of mathematical techniques applied by these people would not deserve the name of mathematical science, as we understand the term today and as it is naively believed to have been practiced by the Greeks.

In previous papers [10, 11], we have analyzed the appearance and development of mathematics in Egypt, relating its concrete flavour and its marked specificity with some political and social aspects of the Egyptian history. In this paper, we seek to present counterexamples to the thesis that abstract thinking was absent from the Babylonian mathematics.

It is certain that we cannot measure the quality of the mathematical production of a people, or of a school, or of a mathematician, by comparing it with our contemporary mathematics, much less by counting the number of times their discoveries still appear in our textbooks. Instead, we need to recognize the cultural context of the problems tackled and the solutions presented. But since it is impossible to reproduce today the way of thinking of millennia, centuries or even decades ago, we can and we must study the Babylonian mathematics with our own eyes, looking for the similarities, the differences, the shared or divergent problems and anxieties.

In this way, we seek in the ancient Babylonians not only predecessors for the Greeks, but for ourselves. Much to the surprise for some arrogant modern scientist, we seek for partners, collaborators in our own and original scientific discoveries.

2 On the number system

Our first considerations about the Mesopotamian mathematics are concerned with the number system found in their cuneiform tablets of mathematical content. We find, differently from other ancient civilizations, a positional system of basis 60. One can argue that the choice of such basis occurred by chance, or as the merge of a system of basis 10 with one of basis 6, or by any other "natural" process. A much more plausible hypothesis is that it had its origins from practical measurements, since the number 60, with its many divisors, is a comfortable choice in commercial activities. In this hypothesis, even though motivated by a utilitarian need, the choice of the basis took into account a general and intrinsic feature of numbers: the quantity of their divisors.

Regarding the invention of a positional system, at least one gradual implementation has been suggested [1, page 20]: from the habit of writing to the left, in larger digits, quantities that can be expressed in a "unit" 60 times greater than the previous one (such as 1 *talent*, which equals 60 *mannas*) one gradually shrinks the large digit until it has been reduced to an ordinary size, only translated in position with respect to the digit corresponding to the smaller "unit". A generalization of this principle, that is, that moving a number n positions to its left corresponds to multiplying its value by the n-th power of the basis, leads to the positional system for integers numbers. In this way, with only two different characters, a vertical wedge-shaped stroke for one unit one with the shape of an angular bracket representing 10, the Babylonians were able to write down any positive integer, however large, with the systematic repetition of symbols in different sexagesimal places.

At this point, it is interesting to note the absence of a definite role for a zero appearing at the right end of a number (in older tablets, not even an intermediate zero was recorded; after the Seleucid period, two oblique strokes started to be used to represent an empty intermediate sexagesimal place). For example, four vertical strokes could represent 4 or $4 \cdot 60$ or $4 \cdot 3600$ or $4 \cdot 60^n$, depending entirely on the context. But let us not be easily fooled, thinking we have found a serious flaw in the Babylonian mathematics, because it was precisely this ambiguity that led to their greatest scientific contribution: sexagesimal fractions. A simple extrapolation of what we just said shows that the same four vertical strokes can also represent $4/60$ or $4/3600$ or $4/60^n$. That is, any set of strokes could take values differing by a factor of 60^k, where k now is any integer, positive or negative. This fact makes sexagesimal fractions as easy to manipulate as our decimal fractions.

Here we might go back to the subject of the choice of a basis for the number system. A fraction of the form p/q can only be written in decimal form, that is, with a power of 10 in the denominator, if the prime factors of q are among the prime factors of 10, that is, 2 and 5. For instance, $1/9$ lacks a finite expansion in decimal fractions. For the basis 60, any fraction p/q for which the prime factors of q are in the set $\{2, 3, 5\}$ can be written in sexagesimal form. Clearly they are much more numerous than decimal fractions. The intensive use that Babylonian mathematicians made of such fractions compels us to say that these advantages were also taken into account in a judicious choice of basis for the number systems.

3 Tablets of tables

Most of Babylonian mathematical texts consist of several tables of arithmetic manipulations. There are, for examples, tables for multiplication by different numbers (but not for all positive integers less than 60, as one might expect at first sight), much like our own school tables, which making use of the distributive property were enough to carry on quickly any given multiplication. Tables of inverses, mainly of regular numbers (that is, those for which the inverse can be expressed as a finite sum of sexagesimal fraction), combined with multiplication tables, took care of most divisions. Tables containing number raised to different exponents were used to perform compound interest calculations (with rates from 20 to 30% a year !), and so on and so forth.

For practically all daily operations, the Babylonian mathematicians, student or vendors could resort to these extensive and useful tables, combined with some ingenious algorithms, such as the one used to obtain the square root of a number, a good deal more efficient than the one taught to school children nowadays [3, page 21].

Some tables, however, contained material of less direct application, such as the tables of squares, cubes or with the quantity $n^2(n+1)$, for several values of n, integer or not. Those tables were used in another significant part of the Mesopotamian mathematical texts: the tablets containing list of problems and their solutions.

Before dealing with these texts, it is impossible to resist the temptation to cite the most extraordinary Babylonian mathematical table, called Plimpton 322 ([5, pp 35–37] and [7, pp 36–40]. It is a small table of 15 rows and 4 columns, with the first column to the right being a counting column (numbered 1 to 15) and the next two bringing apparently unrelated numbers. It is the fourth column, further to the left, which provides the key for understanding this table: if we call by c and a the numbers on the second and third column (from right to left) on a given row, then the fourth column contains the ratio c^2/b^2, where $c^2 = a^2 + b^2$. In other words, a and c are the sides of a right triangle with the fourth column corresponding to $\sec\theta$, where θ is the angle opposite to a. It is even more extraordinary if we consider that the values for $\sec\theta$ are arranged in decreasing order in θ, in approximately equal interval between the end points of 45 and 31 degrees.

One can hardly think of a practical application which demanded such a table, and the ways of obtaining the Pythagorean triples that it contains reside far beyond the scope of a purely pragmatic and utilitarian science.

4 Lists of Problems

Differently from what we find in the mathematical papyri from Egypt, such as the Rhind [4] and the Moscow papyri, where several routine problems, without a clear connection between them, are presented together with the recipes for their solutions, the Mesopotamian cuneiform texts present us with comprehensive lists of similar problems, carefully arranged with a pedagogical flavour of increasing order of difficulty.

This fact alone is already relevant to hypothesize that the Babylonians had a knowledge of the theoretical relationships holding between the problems. It makes no sense to think that practical needs were responsible to put similar problems together and sort them from the rest.

Yet another distinctive feature strengthens the hypothesis that the Babylonians were aware of the general rules behind the specific cases. The method used to solved a problem is generally presented in the form of successive operations applied to the numbers involved in the problem, the same steps being rigorously followed for all problems of the same kind. For instance, in solving an equation equivalent to $x^2 + px = q$, the escribe follows always the same sequence of steps on p and q, leading up to the solution $x = \sqrt{q + (p/2)^2} - p/2$ [1, page 23]. The fact that a general formula is never offered, with all the work being done on the numbers themselves, only shows a tradition of mathematical prose different from ours. For example, in another problem, to carry over the sum of the equations $xy + x - y = 183$ and $x + y = 27$, the escribe simply tells us that $27 + 183 = 210$ and goes on treating the equation $xy + 2x = 210$ [13, page 63].

Not only the same method appears to be always used to solved problems of the same type, but also a sort of general treatment is employed for several related problems. For instance, in systems of equations displaying the sum (or difference) between two unknowns, such as $x + y = a$, the text put $x = \frac{1}{2}a + w$ and $y = \frac{1}{2} - w$ and proceeds to calculate w from the condition expressed in the accompanying equation. The same procedure appears in systems with a larger number of equations (in some cases up to 10) and it is interesting to note that it is the same method used by the Greek algebraist Diophantus (III century A.D.).

Still on this method, it is appropriate to discuss the possible path taken for the discovery of the formula mentioned above for the solution of a quadratic equation of the form $x^2 + px = q$. It has been suggested [13, page 69] that the Babylonians were the first to use the ingenious trick of 'completing squares', in the way that the Arab algebraists would use much later. Thus,

$$\begin{aligned} x^2 + px + \left(\frac{p}{2}\right)^2 &= q + \left(\frac{p}{q}\right)^2 \\ \left(x + \frac{p}{q}\right)^2 &= q + \left(\frac{p}{q}\right)^2 \\ x &= \sqrt{q + \left(\frac{p}{q}\right)^2} - \frac{p}{2} \end{aligned}$$

We can argue, however, about the possible change of variables $y = x + p$ (a common device for Babylonian algebraists), which would convert the original equation to the system

$$y - x = p \qquad \text{and} \qquad xy = p.$$

Now for this system, the Babylonians were absolutely comfortable in putting $y = w + \frac{1}{2}p$ and $x = w - \frac{1}{2}p$ to obtain [8, page 280]

$$\begin{aligned} w^2 - \left(\frac{p}{2}\right)^2 &= q \\ w &= \sqrt{q + \left(\frac{p}{2}\right)^2}, \end{aligned}$$

implying

$$x = \sqrt{q + \left(\frac{p}{q}\right)^2} - \frac{p}{2}.$$

We might wonder at this point how did the Mesopotamians treat problems involving quadratic equations of the general form $ax^2 + bx + c = 0$. The question is important and naturally leads us to discuss another facet of their algebraic talents. For these cases, the escribe multiplies the whole equation by a, obtaining $(ax)^2 + b(ax) + ac = 0$, and proceeds solving this new equation in the variable $y = ax$, which is clearly in their preferred form $y^2 + py = q$. After solving this equation, the text offers the value for the original unknown x. We can observe an obvious theoretical sophistication involved in the solution of this problem, through the transformation of an equation into an equivalent one and then the recognition that, in the different variable, the new equation falls into a class already extensively treated elsewhere.

In the same direction, we find several cubic equations reduced to the form $x^3 + x^2 = a$ [8, page 193], for which one can easily find the solution from one of the tables displaying the values for $n^2(n+1)$, even when a is not listed in the table, a frequent situation, for which the Babylonians then recured to linear interpolation. As a final example of the level generality achieved by the Mesopotamians in the solution of algebraic equations, we observe that they routinely solved equations of the form $ax^4 + bx^2 = c$ or $ax^8 + bx^4 = c$ by reducing it to $ay^2 + by = c$, where $y = x^2$ or $y = x^4$, respectively.

We have presented all the equations using our own modern symbols, without much care for how they were actually stated in the Mesopotamian texts. A brief analysis of these statements offers yet another indication to the path for abstraction that we claim was present in their mathematics. When referring to terms which would correspond to our x^3, x^2 and x, a Babylonian escribe uses respectively the words for "volume", "area" and "length". At first sight, we might assume that they were just a direct representation of what is encountered in the practical daily life, thence all the problems would correspond to concrete situations. But then what we see are areas added to lengths and volumes, or areas multiplied by each other, with no intelligible relation to the world of practical problems. It is what I decided to call "algebraic surrealism": like a surrealist painting, the elements used in the composition of a scene are concrete, but the situation then formed, being often absurd and incompatible with reality, bears no responsibility to represent any concretness.

Even in more conventional geometric problems, what we find are applications of arithmetics and algebra to the treatment of geometric figures, an order that would be later reverted by the plastic sense applied by the Greeks to their mathematics. In this vein, we find for instance the many problems solved by a systematic application of the Pythagoras theorem (circa 1700 B.C [8, page 22] and [1, page 25]), or the (correct) calculations of the areas for a variety of triangles and trapezoids.

5 Final remarks: reservations and influences.

The aspect which most distinguishes the Babylonian mathematics from that later done by the Greeks comes precisely from Geometry: nowhere in Babylonian texts we find the preoccupation in sorting exact results from approximations. With no particular warning, they use $6r$ and $3r^2$ for the perimeter and the area of a circle of radius r (compare with [3, page 28]). In other texts, the volume for the trunk of a square pyramid having bases with areas a^2 and b^2 and height h is inadvertently given as

$$V = \left(\frac{a+b}{2}\right)^2 h,$$

although Neugebauer offers a text [13, page 75] with the formula

$$V = \left[\left(\frac{a+b}{2}\right)^2 + \frac{1}{3}\left(\frac{a-b}{2}\right)^2\right] h,$$

which is correct and equivalent to the one given by the Egyptians in the Moscow papyrus.

But let us not forget that Geometry was not the primary concern of the Babylonians, a position certainly occupied by Algebra and Arithmetics. It is this interest in questions concerning numbers, allied with the enormous computational power obtained from the positional sexagesimal number system, which would propel the numerological worship of the Pythagorean school. The Greeks would also receive from the Mesopotamians the habit, later transformed into almost a dogma, of avoiding speculations about infinite quantities, infinite procedures, infinite repetitions. This was evident, for example, in the absence of a representation for the inverse of 7, which in the form of a sexagesimal fraction is infinitely repeating with period of three sexagesimal positions [3, page 22].

And talking about absences, we have to notice the lack of any mention of a proof for the possible theorems and procedures used by the Babylonians. Maybe this was systematic carelessness; more likely the concept of a proof was inexistent to them. It seems that, as far as recognizing the necessity of proof, we are in effect facing a novelty of Greek origin.

Far from a mere grouping of rules and techniques with a strictly utilitarian orientation, the Mesopotamian mathematics presents itself to us as a dense, elaborate and remarkably durable activity. We might even point to it as one of the unifying factors for the many groups of people that inhabited the Mesopotamia, alongside with the cuneiform writing and their mythology and religion. Going back to the idea sketched in our introduction, we find in it the same elements present in the mathematical sciences of any subsequent civilization. Therefore, analyzing them is a way of understanding the scientific activity of any other era, to the extent that any investigation of the past is an interior investigation, even of a mathematical past.

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