Numerical methods for indifference pricing in stochastic volatility models

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1. Introduction

- Market Model: We consider two factor stochastic volatility models of the form

$$\begin{aligned} d\bar{S}_t &= \bar{S}_t[\mu(t, Y_t)dt + \sigma(t, Y_t)dW_t^1] \\ dY_t &= a(t, Y_t)dt + b(t, Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2] \end{aligned} \quad (1)$$

  with initial values $\bar{S}_0, Y_0 \geq 0$, for deterministic functions μ, a, b and independent one dimensional P–Brownian motions W_t^1 and W_t^2 with constant correlation $|\rho| \leq 1$.

- Optimal hedging portfolio: the strategy followed by an investor who, when faced with a (discounted) financial liability B maturing at a future time T, tries to solve the stochastic control problem

$$u(x) = \sup_{H \in \mathcal{A}} E\left[U\left(X_T - B\right) | X_0 = x\right], \qquad (2)$$

where X_T is the discounted terminal wealth obtained when investing $H_t \bar{S}_t$ dollars on the risky asset and $\eta_t C_t$ dollars in a riskless cash account with value C_t initialized at $C_0 = 1$ and governed by

$$dC_t = r_t C_t dt. \qquad (3)$$

- Utility function: $U(x) = -\frac{e^{-\gamma x}}{\gamma}$, where $\gamma > 0$ is the risk aversion parameter.

For self–financing portfolios, the wealth process satisfies

$$C_t X_t := H_t \bar{S}_t + \eta_t C_t = x + \int_0^t H_u d\bar{S}_u + \int_0^t \eta_u dC_u. \tag{4}$$

In addition to the self–financing condition, economic reasoning imposes further restrictions on the class $\mathcal{A}$ of admissible portfolios.

Finally, the liability B is assumed to be a random variable of the form $B = B(S_T, Y_T)$, for some (bounded) Borel function $B : \mathbb{R}_+^2 \to \mathbb{R}$.

The market model in terms of the discounted prices $S_t = \bar{S}_t / C_t$ is

$$\begin{aligned} dS_t &= S_t[(\mu(t, Y_t) - r)dt + \sigma(t, Y_t)dW_t^1], \\ dY_t &= a(t, Y_t)dt + b(t, Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2] \end{aligned} \qquad (5)$$

and it is immediate that the discounted wealth process satisfies

$$dX_t = H_t dS_t = H_t S_t[(\mu(t, Y_t) - r)dt + \sigma(t, Y_t)dW_t^1], \qquad (6)$$

so that the only relevant control in (2) is H_t, with the holdings in the cash account being determined by $\eta_t = X_t - H_t S_t$.

2. Utility based pricing

For such Markovian markets we can embed the optimal hedging problem (2) into the larger class of optimization problems defined by

$$u(t, x, s, y) = \sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(X_T - B(S_T, Y_T))|X_t = x], \qquad (7)$$

for $t \in (0, T)$, where $x \in \mathbb{R}$ denotes some arbitrary level of wealth, $\mathcal{A}_t$ denotes admissible portfolios starting at time t and $E_{t,s,y}[\cdot]$ denotes expectation with respect to the joint probability law at time t of the processes S_u, Y_u satisfying (5) for $u \geq t$, with initial condition $S_t = s$ and $Y_t = y$.

Suppose that (7) has an optimizer H_t^B, that is, assume that

$$u(t,x,s,y) = E_{t,s,y}[U(x + (H^B \cdot S)_t^T - B(S_T, Y_T))],$$

Define the certainty equivalent for the claim B at time t as the process $c_t^B = c^B(t,x,s,y)$ satisfying the equation

$$U(x - c_t^B)) = E_{t,s,y}[U(x + (H^B \cdot S)_t^T - B(S_T, Y_T))]. \qquad (8)$$

If we set $B = 0$, then the optimal hedging problem becomes the Merton optimal investment problem and we denote the certainty equivalent by $c_t^0 = c^0(t,x,s,y)$.

The indifference price for the claim B is defined to be solution $\pi^B = \pi^B(t,x,s,y)$ to the equation

$$\sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(x+(H\cdot S)_t^T] = \sup_{H \in \mathcal{A}_t} E_{t,s,y}[U(x+\pi^B+(H\cdot S)_t^T - B(S_T, Y_T)]. \tag{9}$$

From the definition of the certainty equivalent, we see that this equation is equivalent to

$$U(x - c_t^0) = U(x + \pi^B - c_t^B), \tag{10}$$

so that the indifference price is given by

$$\pi^B(t,x,s,y) = c^B(t, x + \pi^B(t,x,s,y), s, y) - c^0(t,x,s,y). \tag{11}$$

The advantage of using an exponential utility is that we can factorize the value function $u(t,x,s,y)$ in (7) as

$$u(t,x,s,y) = -e^{-\gamma x} \inf_{H \in \mathcal{A}_t} E_{t,s,y}\left[e^{-\gamma\left((H\cdot S)_t^T - B(S_T, Y_t)\right)} \right] =: U(x)v(t,s,y). \tag{12}$$

It follows directly from (8) that the certainty equivalent is wealth independent and given by

$$c^B(t,s,y) = \frac{1}{\gamma} \log v(t,s,y), \tag{13}$$

and analogously for the Merton problem. Thus the indifference price process for the claim B obtained from an exponential utility is given by

$$\pi^B(t,s,y) = c^B(t,s,y) - c^0(t,s,y) = \frac{1}{\gamma} \log \frac{v(t,s,y)}{v^0(t,s,y)}. \tag{14}$$

3. The HJB approach

Direct substitution of (12) into the HJB problem for the value function $u(t, x, s, y)$ leads to an optimizer of the form

$$H_t^B = h^B(t, s, y) = \frac{1}{\gamma}\frac{\partial_s v}{v} + \frac{b\rho}{\gamma s\sigma}\frac{\partial_y v}{v} + \frac{(\mu - r)}{\gamma s\sigma^2}. \tag{15}$$

The partial differential equation satisfied by the optimal function $v(t, s, y)$ is then

$$\begin{aligned}\partial_t v \quad &+ \quad \frac{1}{2}\left(s^2\sigma^2\partial_{ss}^2 v + 2b\rho s\sigma\partial_{ys}^2 v + b^2\partial_{yy}^2 v\right) + \left[a - \frac{b\rho(\mu - r)}{\sigma}\right]\partial_y v \\ &-\frac{1}{2}\left[\frac{1}{v}(b\rho\partial_y v + s\sigma\partial_s v)^2 + \frac{(\mu - r)^2}{\sigma^2}v\right] = 0, \end{aligned} \tag{16}$$

subject to the terminal condition $v(T, s, y) = e^{\gamma B(s,y)}$.

From (13), we find that the certainty equivalent process $c^B(t,s,y)$ is a solution to the partial differential equation

$$\partial_t c^B + \frac{1}{2}(s^2\sigma^2\partial^2_{ss}c^B + 2s\sigma b\rho\,\partial^2_{sy}c^B + b^2\partial^2_{yy}c^B) + \left[a - \frac{b\rho(\mu-r)}{\sigma}\right]\partial_y c^B$$
$$-\frac{(\mu-r)^2}{2\gamma\sigma^2} + \frac{\gamma}{2}b^2(1-\rho^2)(\partial_y c^B)^2 = 0, \qquad (17)$$

with terminal condition $c^B(T,s,y) = B(s,y)$. The partial differential equation satisfied by $c^0(t,s,y)$, the certainty equivalent for Merton's problem, is identical to (17), but with the terminal condition $c^0(T,s,y) = 0$.

Using (15), the optimal portfolio can be obtained in terms of the certainty equivalent process by

$$h^B(t,s,y) = \partial_s c^B + \frac{b\rho}{s\sigma}\partial_y c^B + \frac{(\mu-r)}{\gamma s\sigma^2}. \qquad (18)$$

For pure volatility claims of the form $B = B(Y_T)$, the equation for the certainty equivalent $c_t^B = c^B(t,y)$ is reduced to

$$\partial_t c^B + \left[a - \frac{b\rho(\mu - r)}{\sigma}\right]\partial_y c^B + \frac{1}{2}b^2\partial_{yy}^2 c^B - \frac{(\mu-r)^2}{2\gamma\sigma^2} + \frac{\gamma}{2}b^2(1-\rho^2)(\partial_y c^B)^2 = 0, \tag{19}$$

subject to the terminal condition $c^B(T,y) = B(y)$. Following Zariphopoulou (2001) we now use the transformation

$$c^B(t,y) = \frac{1}{\gamma(1-\rho^2)}\log f(t,y), \tag{20}$$

to reduce (19) to the linear parabolic final value problem

$$\begin{aligned}\partial_t f + \frac{1}{2}b^2\partial^2_{yy}f + \left[a - \frac{b\rho(\mu - r)}{\sigma}\right]\partial_y f - \frac{(1-\rho^2)(\mu - r)^2}{2\sigma^2}f &= 0, \\ f(T, y) = e^{\gamma(1-\rho^2)B(y)}. \qquad (21)\end{aligned}$$

We can use the Feynman–Kac formula to represent the solution to the problem above as

$$f(t, y) = \widetilde{E}_{t,y}\left[e^{-\int_t^T R(s,Y_s)ds} e^{\gamma(1-\rho^2)B(Y_T)}\right], \qquad (22)$$

where we define

$$R(t, y) = \frac{(1-\rho^2)(\mu(t, y) - r)^2}{2\sigma(t, y)^2}, \qquad (23)$$

and $\widetilde{E}_{t,y}[\cdot]$ denotes the expectation with respect to the probability law at time $s = t$ of the solution to

$$\begin{aligned} dY_s &= \left[a - \frac{b(\mu - r)\rho}{\sigma}\right] ds + b\left[\rho d\widetilde{W}_s^1 + \sqrt{1-\rho^2} d\widetilde{W}_s^2\right], \\ Y_t &= y \end{aligned} \tag{24}$$

for a pair of independent one dimensional $\widetilde{P}$–Brownian motions $\widetilde{W}_t^1, \widetilde{W}_t^2$, for a probability measure $\widetilde{P}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]})$. If we further required S to be a $\widetilde{P}$ martingale, the comparison with (5) leads to the identification

$$\begin{aligned} d\widetilde{W}_t^1 &= dW_t^1 + \widetilde{\lambda}_t^1 dt \\ d\widetilde{W}_t^2 &= dW_t^2, \end{aligned} \tag{25}$$

where

$$\widetilde{\lambda}_t^1 = \frac{\mu(t, Y_t) - r}{\sigma(t, Y_t)}. \tag{26}$$

4. Reciprocal affine models

We now take μ and r to be constants and $\sigma(t, Y_t) = \sqrt{Y_t}$, so that (23) becomes

$$R_t = R(t, Y_t) = \frac{(1-\rho^2)(\mu - r)^2}{2Y_t}, \tag{27}$$

which we postulate to be a CIR process. Since our calculations are going to take place under the measure $\widetilde{P}$, we specify the dynamics for R_t as

$$dR_t = \widetilde{\alpha}(\widetilde{\kappa} - R_t)dt + \beta\sqrt{R_t}\left[\rho d\widetilde{W}_t^1 + \sqrt{1-\rho^2}d\widetilde{W}_t^2\right], \tag{28}$$

for constants $\widetilde{\alpha}, \widetilde{\kappa}, \beta > 0$ with $4\widetilde{\alpha}\widetilde{\kappa} > \beta^2$.

It follows from (25) that the dynamics of R_t under the economic measure P is

$$dR_t = \alpha(\kappa - R_t)dt + \beta\sqrt{R_t}\left[\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2\right], \tag{29}$$

where $\alpha = \left(\widetilde{\alpha} - \beta\rho\sqrt{\frac{2}{1-\rho^2}}\right)$ and $\alpha\kappa = \widetilde{\alpha}\widetilde{\kappa}$.

We then obtain from the Itô formula that

$$a(t, Y_t) = \alpha Y_t + \frac{2(\beta^2 - \alpha\kappa)}{(1-\rho^2)(\mu - r)^2} Y_t^2, \tag{30}$$

$$b(t, Y_t) = -\left(\frac{2}{1-\rho^2}\right)^{1/2} \frac{\beta}{(\mu - r)} Y_t^{3/2}. \tag{31}$$

5. Pricing and hedging formulas

We need to compute expressions of the form

$$I := \widetilde{E}_t \left[e^{-\int_t^T R_s ds} g(R_T) \right], \tag{32}$$

for functions $g : \mathbb{R}^+ \to \mathbb{R}$. Provided its Fourier transform is well defined and invertible, we can express g as

$$g(R) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iuR} \widehat{g}(u) du, \tag{33}$$

where

$$\widehat{g}(u) = \int_{-\infty}^{\infty} e^{iuR} g(R) dR. \tag{34}$$

Exchanging the order of integration, we have

$$I = I(R_t, t, T) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(u)\widehat{g}(u)du, \tag{35}$$

where Ψ can be computed as

$$\begin{aligned}\Psi(u) = \Psi(u, R_t, t, T) &:= \widetilde{E}_t\left[e^{-\int_t^T R_s ds} e^{-iuR_T}\right] \\ &= \exp[M(u,t,T) + N(u,t,T)R_t]. \end{aligned}\tag{36}$$

Here

$$\begin{aligned} N(u) = N(u,t,T) &= \frac{(b_2+iu)b_1 - (b_1+iu)b_2 e^{\Delta(t-T)}}{(b_2+iu) - (b_1+iu)e^{\Delta(t-T)}}, \\ M(u) = M(u,t,T) &= \frac{-2\alpha\kappa}{\beta^2}\log\left(\frac{b_2+iu}{b_2-N}\right) + \alpha\kappa b_1(t-T), \end{aligned}\tag{37}$$

with $b_2 > b_1$ being the two roots of $x^2 - \frac{2\widetilde{\alpha}}{\beta^2}x - \frac{2}{\beta^2}$ and $\Delta = \sqrt{\widetilde{\alpha}^2 + 2\beta^2}$.

Setting $g(R_T) = e^{\gamma(1-\rho^2)B(R_T)}$, we obtain from (14), (20) and (22) that the indifference price of the volatility claim $B = B(R_T)$ is simply

$$\begin{aligned} \pi^B &= \frac{\delta}{\gamma} \log \left[\frac{\widetilde{E}_t \left[e^{-\int_t^T R_s ds} e^{\gamma(1-\rho^2)B(R_T)} \right]}{\widetilde{E}_t [e^{-\int_t^T R_s ds}]} \right] \\ &= \frac{1}{\gamma(1-\rho^2)} \log \left[\frac{I(R_t, t, T)}{\Psi(0, R_t, t, T)} \right]. \end{aligned} \tag{38}$$

The number of shares of stock to be held in order to optimally hedge against the claim B is

$$\begin{aligned} h^B(t,y) &= \frac{1}{\gamma s}\left[\frac{b\rho}{\gamma(1-\rho^2)\sqrt{y}}\frac{\partial \log I}{\partial y} + \frac{(\mu - r)}{\gamma y}\right] \qquad (39) \\ &= \frac{1}{\gamma s}\frac{(\mu-r)}{y}\left[\frac{\beta\rho}{\sqrt{2(1-\rho^2)}}\frac{\int_{-\infty}^{\infty}\Psi(u)N(u)\widehat{g}(u)du}{\int_{-\infty}^{\infty}\Psi(u)\widehat{g}(u)du} + 1\right], \end{aligned}$$

whereas the number of shares held in the Merton portfolio is

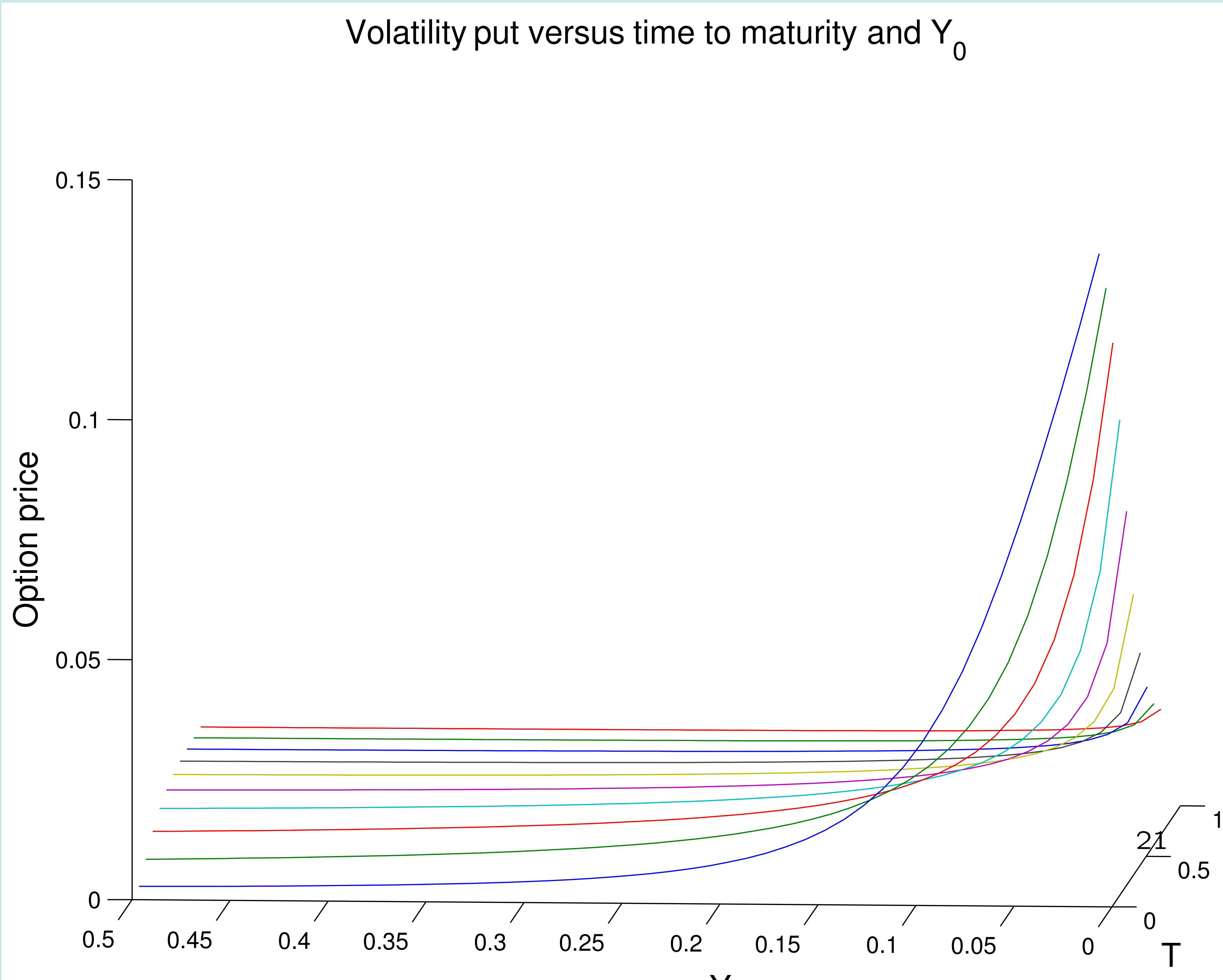
$$\begin{aligned} h^0(t,y) &= \frac{1}{s}\left[\frac{b\rho}{\gamma(1-\rho^2)\sqrt{y}}\frac{\partial \log \Psi(0)}{\partial y} + \frac{(\mu - r)}{\gamma y}\right] \\ &= \frac{1}{\gamma s}\frac{(\mu-r)}{y}\left[\frac{\beta\rho}{\sqrt{2(1-\rho^2)}}N(0) + 1\right]. \qquad (40) \end{aligned}$$

6. Numerical results (act I)

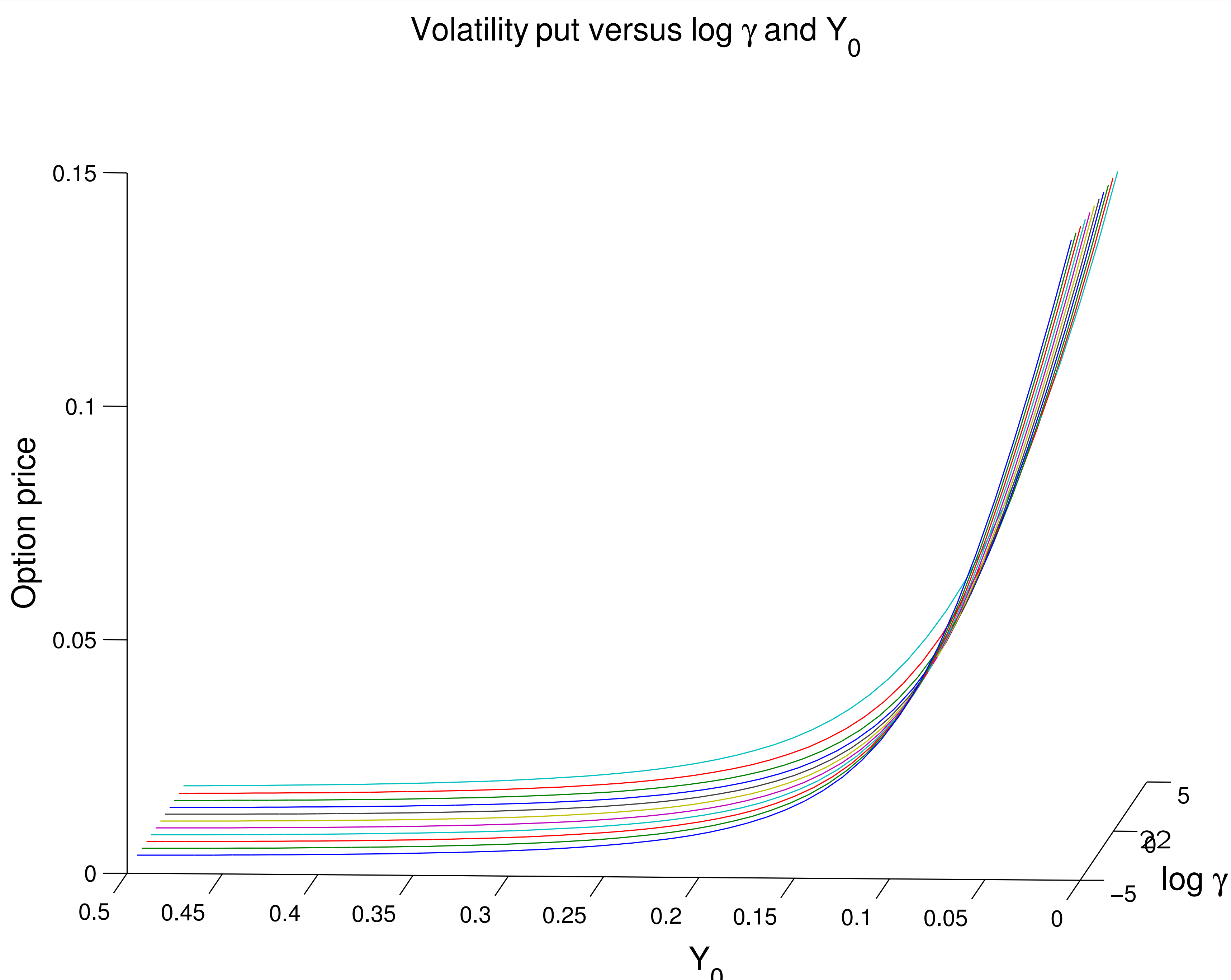
We illustrate the range of possibilities for model parameters fixed at reasonable values:

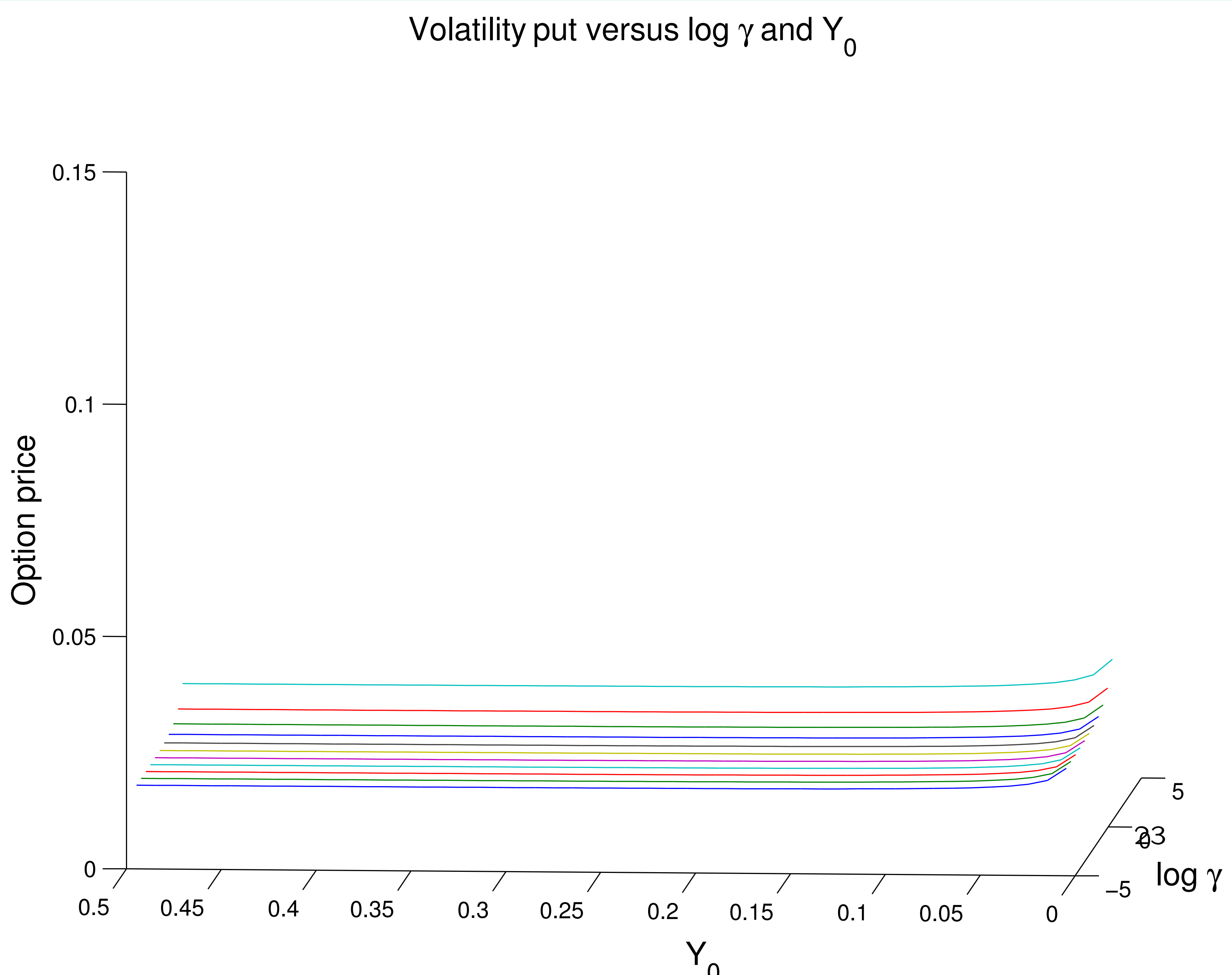
$$\begin{aligned} \alpha &= 5, \quad \beta = 0.04, \quad \kappa = 0.001, \\ \mu &= 0.04, \quad r = 0.02, \quad \rho = 0.5 \end{aligned}$$

and initial squared volatility ranging in the interval $[0, 0.5]$. With these parameters the squared volatility process has a mean reversion time of approximately two months and an equilibrium distribution with expected value approximately 40%. We calculate the price of a put option on volatility with payoff $(0.15 - \sigma_T^2)^+$. When not mentioned the risk aversion parameter is set to $\gamma = 1$.

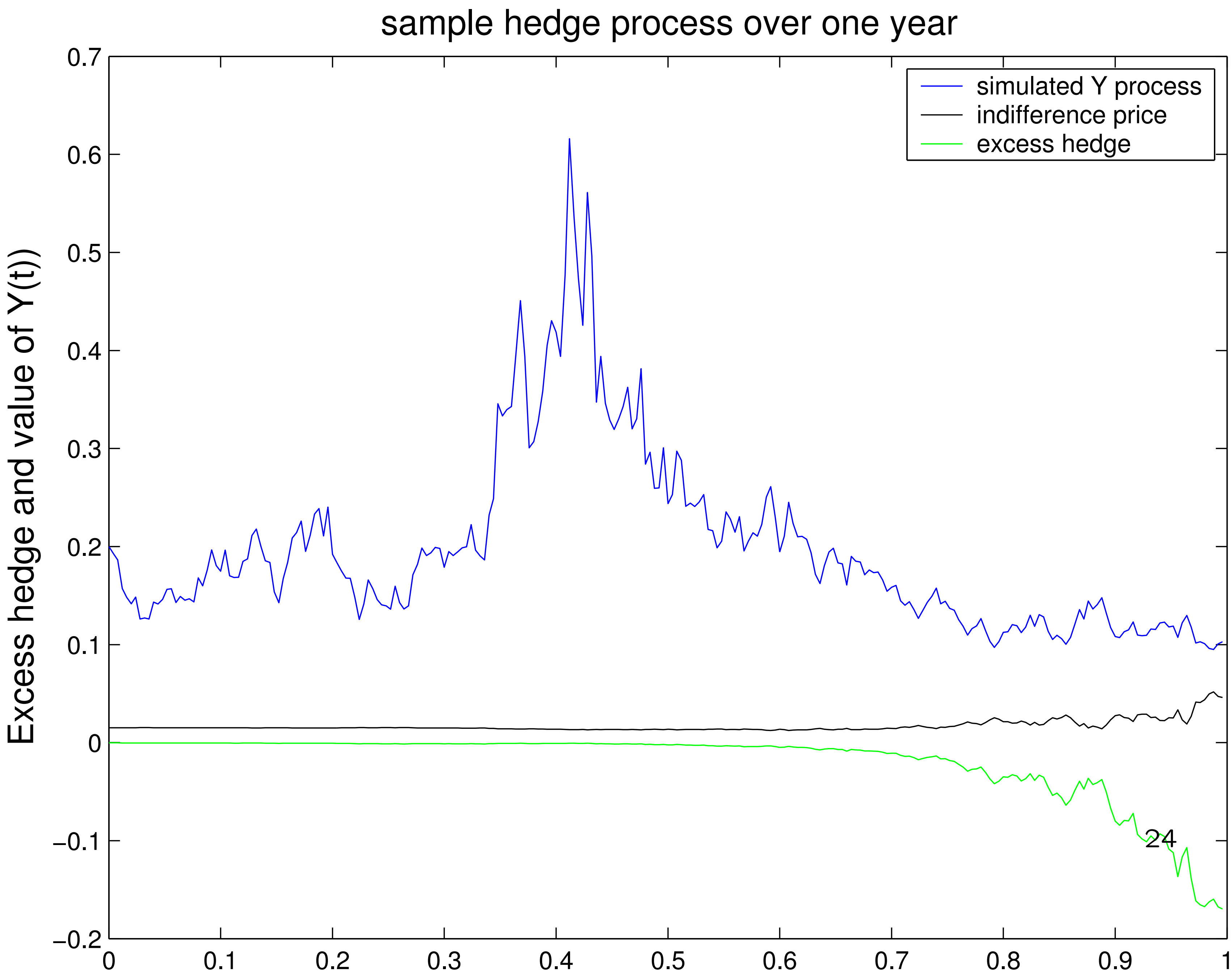


Volatility put versus time to maturity and Y_0
Option price
0.15
0.1
0.05
0
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
T
1
0.5
0
21


Volatility put versus log γ and Y_0
Option price
0.15
0.1
0.05
0
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Y_0
log γ
5
2
0
−2
−5



Volatility put versus log γ and Y_0
Option price
0.15
0.1
0.05
0
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Y_0
5
3
0
−5
log γ


sample hedge process over one year
Excess hedge and value of Y(t))
simulated Y process
indifference price
excess hedge
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

7. The Monte Carlo approach

We now consider discrete time hedgings, where the portfolio processes have the form

$$H_t = \sum_{k=1}^{K} H_k \, \mathbf{1}_{(t_{k-1},t_k]}(t) \tag{41}$$

where each H_k is an $\mathbb{R}^d$–valued $\mathcal{F}_{k-1}$ random variable. We take the discrete time partition of the interval $[0,T]$ to be of the form

$$t_0 = 0 < t_1 = \frac{T}{K} < \ldots < t_k = \frac{kT}{K} \ldots < t_K = T$$

and use the notation $S_j := S_{t_j}$ for discrete time stochastic processes.

The discounted wealth for self–financing portfolios is

$$X_j = x + (H \cdot S)_j, \tag{42}$$

with the notation $(H \cdot S)_k^j := (H \cdot S)_j - (H \cdot S)_k$, where

$$(H \cdot S)_j := \sum_{k=1}^{j} H_k \Delta S_k \tag{43}$$

and $\Delta S_k := S_k - S_{k-1}$.

Now the dynamic programming problem for the optimal hedge falls into K subproblems

$$u_{k-1}(x) = \sup_{H_k \in \mathcal{F}_{k-1}} E_{k-1}[u_k(x + H_k \Delta S_k)], \tag{44}$$

for $k = K, K-1, \ldots, 1$, with $u_K(x) = U(x - B)$. Similarly, the certainty equivalent value process $c_k^B(x)$ is defined iteratively by

$$U(x - c_{k-1}^B(x)) = \sup_{H_k \in \mathcal{F}_{k-1}} E_{k-1}[U(x + H_k \Delta S_k - c_k^B(x + H_k \Delta S_k)] \tag{45}$$

with $c_K^B(x)$ taken equal to the terminal discounted claim B.

In our Markovian setting and with an exponential utility, the solution of (44) and (45) as well as the optimal allocation H^B have the form wealth independent form

$$\begin{aligned} u_k &= g_k(S_k, Y_k) && (46) \\ c_k^B &= c_k(S_k, Y_k) && (47) \\ H_{k+1}^B &= h_{k+1}(S_k, Y_k) && (48) \end{aligned}$$

for (deterministic) Borel scalar functions $\{g_k, c_k\}_{k=0}^{K-1}$ and $\mathbb{R}^d$–valued functions $\{h_{k+1}\}_{k=0}^{K-1}$ on the state space $\mathcal{S} = \mathbb{R}_+^2$.

The exponential utility allocation algorithm

We want an algorithm which will generate an approximate trading rule, based on a data set

$$\{(S_k^i, Y_k^i)\}_{i=1,\ldots,N;k=0,\ldots,K}$$

where $(S_k^i, Y_k^i) \in \mathbb{R}^n$ denotes the state of the ith sample path at time $t_k = kT/K$ for the processes given by (5). In the special case of an exponential utility, the theoretical optimal rule

$$H_{k+1}^B = h_k(S_k^i, Y_k^i)$$

in (48) depends only on the directly observed data $\{S_k^i, Y_k^i\}$ and is independent of the wealth X_k^i. For this reason our algorithm is at this point restricted to exponential utility functions, and we take $\gamma = 1$ for simplicity.

1. Step $k = K$: The final optimal allocation is the $\mathcal{F}_{K-1}$–random variable H_K^B which solves

$$\min_{H_K \in \mathcal{F}_{K-1}} E[\exp(-H \cdot \Delta S_K + B)]. \tag{49}$$

Since the solution is known to be given by $H_K^B = h_K(S_{K-1}, Y_{K-1})$ for some deterministic function $h_K \in \mathcal{B}(\mathcal{S})$ (the set of Borel functions on $\mathcal{S}$), we write this as

$$\min_{h \in \mathcal{B}(\mathcal{S})} E[\exp(-h(S_{K-1}, Y_{K-1}) \cdot \Delta S_K + B)]. \tag{50}$$

On a finite set of data, we can pick an R–dimensional subspace $\mathcal{R}(\mathcal{S}) \subset \mathcal{B}(\mathcal{S})$ of functions on $\mathcal{S}$ and attempt to "learn" a suboptimal solution

$$\arg\min_{h \in \mathcal{R}(\mathcal{S})} E[\exp(-h(S_{K-1}, Y_{K-1}) \cdot \Delta S_K + B)].$$

By the central limit theorem, the expectation above can be approximated by the finite sample estimate

$$\Psi_K(h) = \frac{1}{N} \sum_{i=1}^{N} \ \exp\Big(-h(S^i_{K-1}, Y^i_{K-1}) \cdot \Delta S^i_K + B(S^i_K, Y^i_K) \Big) \tag{51}$$

This leads to the estimator $h_K^{\mathcal{R}}$ based on $\{S^i_k, Y^i_k\}$ and the choice of subspace $\mathcal{R}$ defined by

$$h_K^{\mathcal{R}} = \operatorname*{arg\,min}_{h \in \mathcal{R}(\mathcal{S})} \Psi_K(h) \tag{52}$$

2. Inductive step for $k = K-1, \ldots, 2$: The estimate $h_k^{\mathcal{R}}$ of the optimal rule h_k, for the intermediate time steps $2 \le k < K-1$ is determined inductively given the estimates $h_{k+1}^{\mathcal{R}}, \ldots, h_K^{\mathcal{R}}$. It is defined to be

$$h_k^{\mathcal{R}} = \operatorname*{arg\,min}_{h \in \mathcal{R}(\mathcal{S})} \Psi_k(h; h_{k+1}^{\mathcal{R}}, \ldots, h_K^{\mathcal{R}}) \tag{53}$$

where

$$\Psi_k(h) = \frac{1}{N} \sum_{i=1}^{N} \exp\left(-h(S_k^i, Y_k^i) \cdot \Delta S_{k+1}^i + c_k^i(h_{k+1}^{\mathcal{R}}, \ldots, h_K^{\mathcal{R}}, S_K^i, Y_K^i)\right), \tag{54}$$

with

$$c_k^i(h_{k+1}^{\mathcal{R}}, \ldots, h_K^{\mathcal{R}}, S_K^i, Y_K^i) = B(S_K^i, Y_K^i) - \sum_{j=k+1}^{K} h_j^{\mathcal{R}}(S_{j-1}^i, Y_{j-1}^i) \cdot \Delta S_j^i \tag{55}$$

3. Final step $k = 1$: This step is degenerate since the initial values (S_0, Y_0) are constant over the sample. Therefore we determine the optimal constant vector $h_1 \in \mathbb{R}^d$ by solving

$$h_1 = \operatorname*{arg\,min}_{h \in \mathbb{R}^d} \Psi_1(h; h_2^{\mathcal{R}}, \ldots, h_K^{\mathcal{R}}) \tag{56}$$

Finally, the optimal value

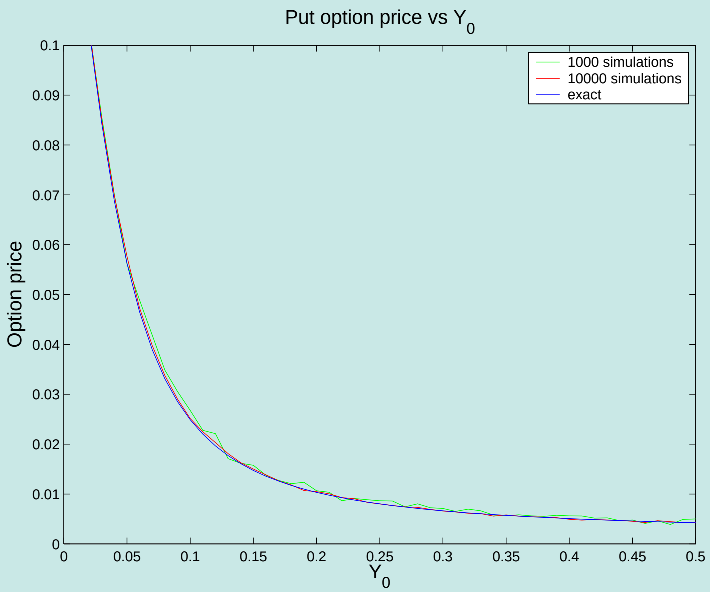
$$\Psi_1 = \frac{1}{N} \sum_{i=1}^{N} \exp\left(-h_1(S_0, Y_0) - \sum_{j=2}^{K} h_j^{\mathcal{R}}(S_{j-1}^i, Y_{j-1}^i) \cdot \Delta S_j^i + B(S_K^i, Y_K^i) \right),$$

is an estimate of the quantity $\exp(c_0^B)$, where c_0^B is the certainty equivalent value of the claim B at time $t = 0$

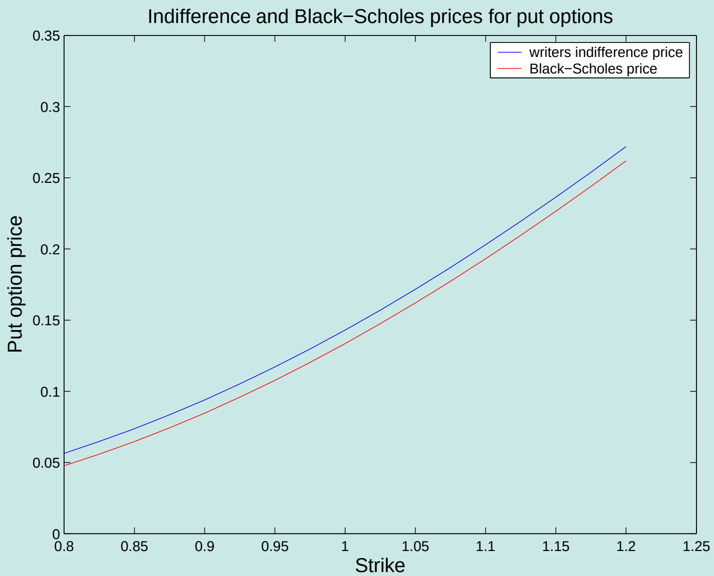
8. Numerical results (act II)

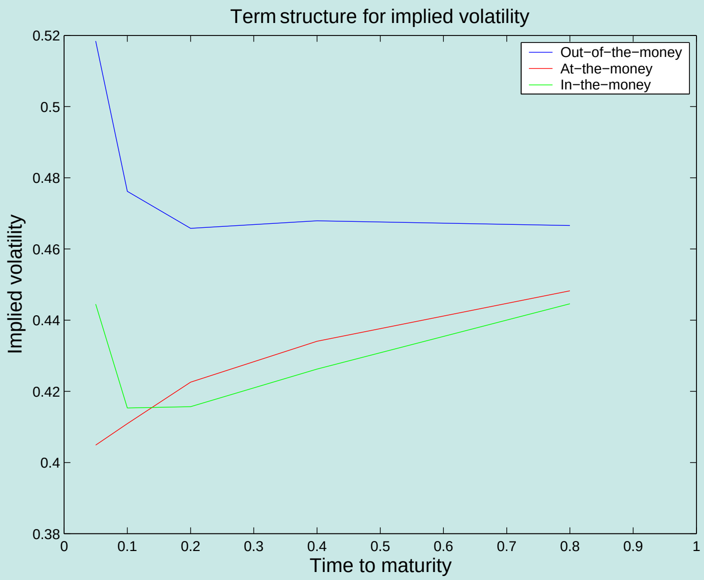
We run the algorithm with the same model parameters as before (in particular $\gamma = 1$). To account for the portfolio dependence in both S_t and Y_t we took $\mathcal{R}(\mathcal{S})$ to be the six–dimensional space spanned by the functions $\{1, y, y^2, s, sy, s^2\}$.

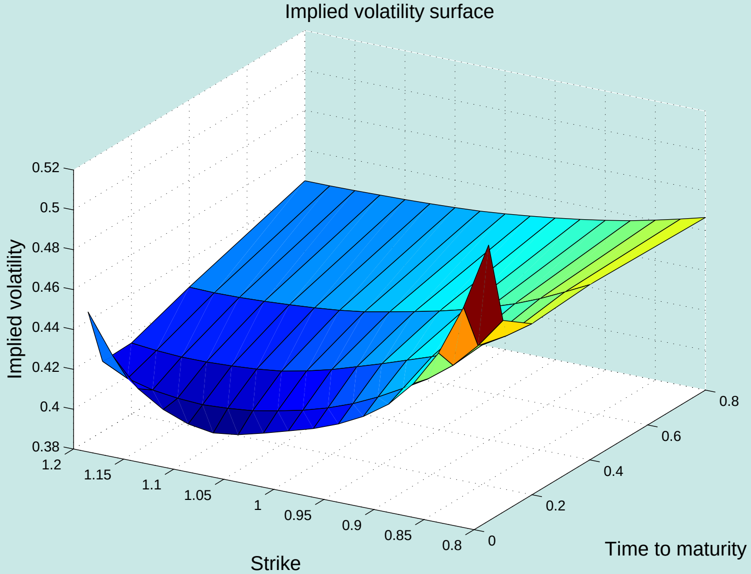
We first applied the allocation algorithm to a volatility put option with payoff $(0.15 - \sigma_T^2)^+$ and time to maturity at $T = 0.2$ and computed the indifference prices with Y_0 varying in the interval $[0, 0.5]$.


Put option price vs Y_0
Option price
Y_0
1000 simulations
10000 simulations
exact
0.1
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

Next we consider a put option on the stock, that is, with payoff $(K - S_T)^+$. The following pictures show the indifference prices and implied volatility surface and term structure obtained with $N = 10000$ simulations.


Indifference and Black−Scholes prices for put options
writers indifference price
Black−Scholes price
Put option price
Strike
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.8
0.85
0.9
0.95
1
1.05
1.1
1.15
1.2
1.25


Term structure for implied volatility
Out−of−the−money
At−the−money
In−the−money
Implied volatility
Time to maturity
0.52
0.5
0.48
0.46
0.44
0.42
0.4
0.38
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


Implied volatility surface
Implied volatility
0.52
0.5
0.48
0.46
0.44
0.42
0.4
0.38
1.2
1.15
1.1
1.05
1
0.95
0.9
0.85
0.8
Strike
0
0.2
0.4
0.6
0.8
Time to maturity