# The investment game in incomplete markets.

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# Successes and Limitations of Real Options

- Real options accurately describe the value of flexibility in decision making under uncertainty.
- According to a recent survey, 26% of CFOs in North America "always or almost always" consider the value of real options in projects.
- This is due to familiarity with the option valuation paradigm in financial markets and its lessons.
- But most of the literature in Real Options is based on different combinations of the following unrealistic assumptions: (1) infinite time horizon, (2) perfectly correlated spanning asset, (3) absence of competition.
- Though some problems have long time horizons (30 years or more), most strategic decisions involve much shorter times.
- The vast majority of underlying projects are not perfectly correlated to any asset traded in financial markets.
- In general, competition erodes the value of flexibility.

# Alternatives

- The use of well–known numerical methods (e.g finite differences) allows for finite time horizons.
- As for the spanning asset assumption, the absence of perfect correlation with a financial asset leads to an incomplete market.
- Replication arguments can no longer be applied to value managerial opportunities.
- The most widespread alternative to replication in the decision-making literature is to introduce a risk-adjusted rate of return, which replaces the risk–free rate, and use dynamic programming.
- This approach lacks the intuitive understanding of opportunities as options.
- Finally, competition is generally introduced using game theory.
- Surprisingly, game theory is almost exclusively combined with real options under the hypothesis of risk-neutrality !

# A one–period investment model

- Consider a two–factor market where the discounted prices for the project $V$ and a correlated traded asset $S$ follow:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \tag{1}$$

  where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values $S_0, V_0$ and historical probabilities $p_1, p_2, p_3, p_4$.
- Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$.
- An investment opportunity is model as an option with discounted payoff $C_t = (V - e^{-rt} I)^+$, for $t = 0, T$.

# European Indifference Price

- The indifference price for the option to invest in the final period as the amount $\pi$ that solves the equation

$$\max_H E[U(x+H(S_T-S_0)] = \max_H E[U(x-\pi+H(S_T-S_0)] \quad (2)$$

- Denoting the two possible pay-offs at the terminal time by $C_h$ and $C_\ell$, the European indifference price is explicitly given by

$$\pi = g(C_h, C_\ell) \quad (3)$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined as

$$\begin{aligned} g(x_1, x_2) &= \frac{q}{\gamma} \log \left( \frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}} \right) \quad (4) \\ &\quad + \frac{1-q}{\gamma} \log \left( \frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}} \right), \end{aligned}$$

with

$$q = \frac{1-d}{u-d}.$$

# Early exercise

- When investment at time $t = 0$ is allowed, it is clear that immediate exercise of this option will occur whenever its exercise value $(V_0 - I)^+$ is larger than its continuation value $\pi^C$.
- That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT}I)^+, (\ell V_0 - e^{-rT}I)^+)\}.$$

# A multi–period model

- Consider now a continuous-time two–factor market of the form

$$
\begin{aligned}
dS_t &= (\mu_1 - r)S_t dt + \sigma_1 S_t dW \\
dV_t &= (\mu_2 - r)V_t dt + \sigma_2 V_t(\rho dW + \sqrt{1-\rho^2} dZ).
\end{aligned}
$$

- We want to approximate this market by a discrete–time processes $(S_n, V_n)$ following the one–period dynamics (1).
- This leads to the following choice of parameters:

$$
\begin{aligned}
u &= e^{\sigma_1 \sqrt{\Delta t}}, \qquad h = e^{\sigma_2 \sqrt{\Delta t}}, \\
d &= e^{-\sigma_1 \sqrt{\Delta t}}, \qquad \ell = e^{-\sigma_2 \sqrt{\Delta t}}, \\
p_1 + p_2 &= \frac{e^{(\mu_1 - r)\Delta t} - d}{u - d}, \quad p_1 + p_3 = \frac{e^{(\mu_2 - r)\Delta t} - \ell}{h - \ell} \\
\rho \sigma_1 \sigma_2 \Delta t &= (u-d)(h-\ell)[p_1 p_4 - p_2 p_3],
\end{aligned}
$$

supplemented by the condition $p_1 + p_2 + p_3 + p_4 = 1$.

# Numerical Experiments - Act I

- We now investigate how the exercise threshold varies with the different model parameters.
- The fixed parameters are

$$\begin{aligned} I &= 1, \quad r = 0.04, \quad T = 10 \\ \mu_1 &= 0.115, \quad \sigma_1 = 0.25, \quad S_0 = 1 \\ \sigma_2 &= 0.2, \quad V_0 = 1 \end{aligned}$$

- Given these parameters, the CAPM equilibrium expected rate of return on the project for a given correlation $\rho$ is

$$\bar{\mu}_2 = r + \rho \left( \frac{\mu_1 - r}{\sigma_1} \right) \sigma_2. \tag{5}$$

- The difference $\delta = \bar{\mu}_2 - \mu_2$ is the below–equilibrium rate–of–return shortfall and plays the role of a dividend rate paid by the project, which we fix at $\delta = 0.04$.

# Known Thresholds

- In the limit $\rho \to \pm 1$ (complete market), the closed–form expression for the investment threshold obtained in the case $T = \infty$ gives $V^*_{DP} = 2$.
- This should be contrasted with the NPV criterion (that is, invest whenever the net present value for the project is positive) which in this case gives $V^*_{NPV} = 1$.
- The limit $\gamma \to 0$ in our model corresponds to the McDonald and Siegel (1986) threshold, obtained by assuming that investors are averse to market risk but neutral towards idiosyncratic risk.
- For our parameters, the adjustment to market risks is accounted by CAPM and this threshold coincides with $V^*_{DP} = 2$

# Dependence on Correlation and Risk Aversion

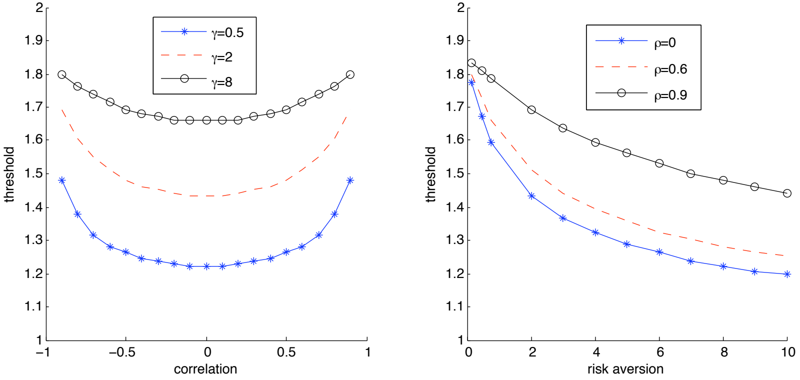



Figure: Exercise threshold as a function of correlation and risk aversion.

# Dependence on Volatility and Dividend Rate

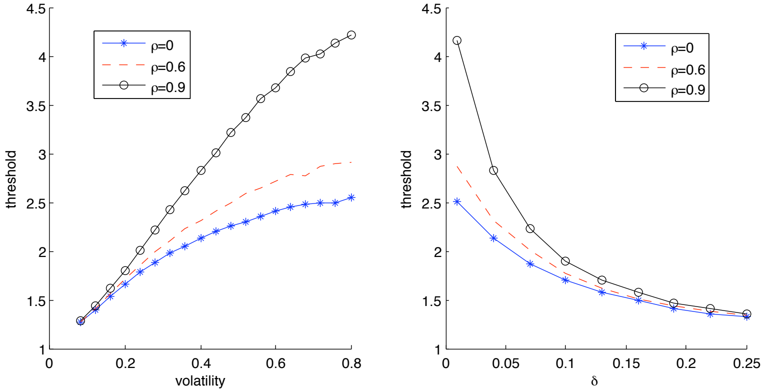



Figure: Exercise threshold as a function of volatility and dividend rate.

# Dependence on Time to Maturity

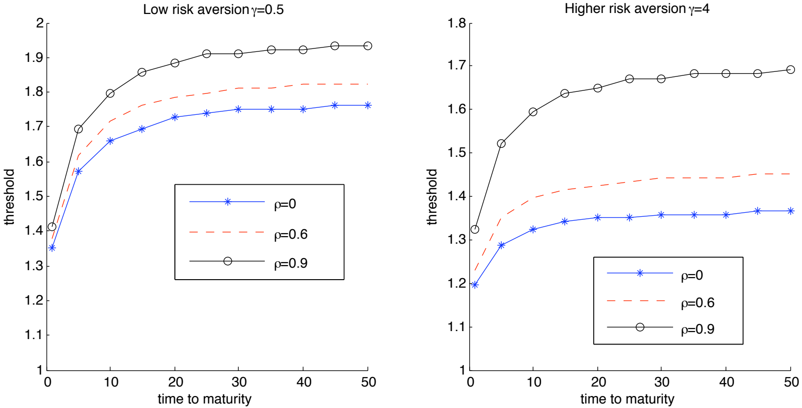



Figure: Exercise threshold as a function of time to maturity.

# Values for the Option to Invest

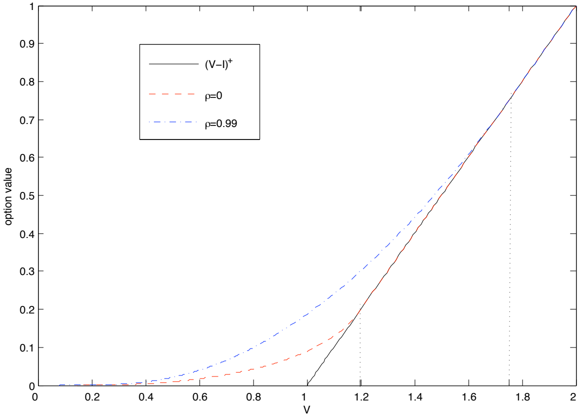



Figure: Option value as a function of underlying project value. The threshold for $\rho = 0$ is 1.1972 and the one for $\rho = 0.99$ is 1.7507.

# Suspension, Reactivation and Scrapping

- Let us denote the value of an idle project by $F^0$, an active project by $F^1$ and a mothballed project by $F^M$.
- Then

$$
\begin{aligned}
F^0 &= \text{option to invest at cost } I \\
F^1 &= \text{cash flow } + \text{option to mothball at cost } E_M \\
F^M &= \text{cash flow } + \text{option to reactivate at cost } R \\
&\quad + \text{option to scrap at cost } E_S
\end{aligned}
$$

- We obtain its value on the grid using the recursion formula

$$F^k(i,j) = \max\{\text{continuation value, possible exercise values}\}.$$

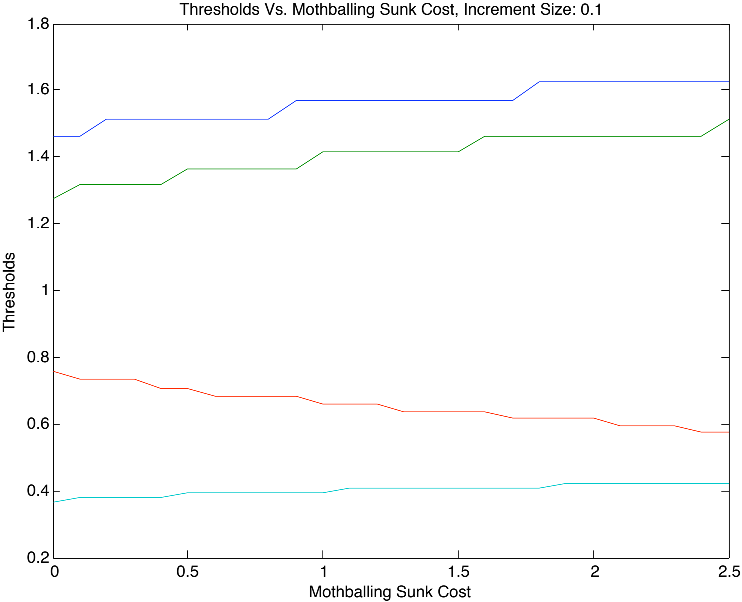
- As before, the decisions to invest, mothball, reactivate and scrap are triggered by the price thresholds $P_S < P_M < P_R < P_H$.

# Numerical Experiments - Act II

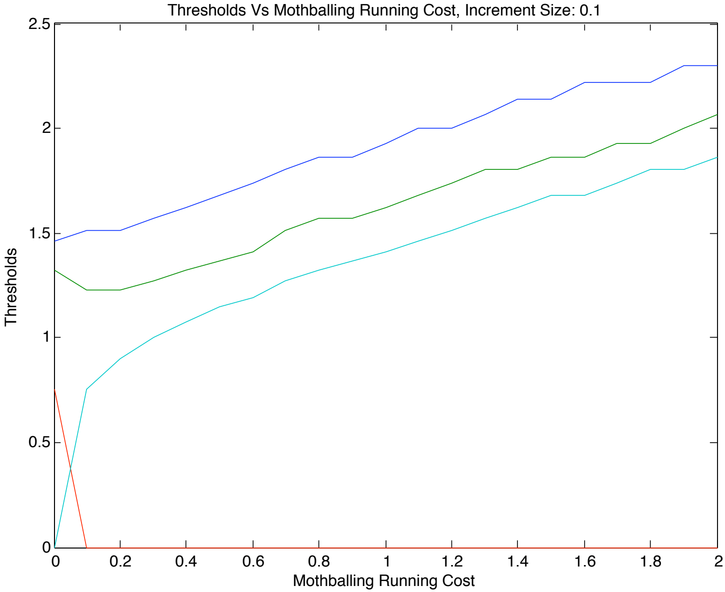
- We calculate these thresholds by keeping track of three simultaneous grids of option values.
- The fixed parameters now are

$$
\begin{aligned}
\mu_1 &= 0.12, \quad \sigma_1 = 0.2, \quad S_0 = 1 \\
\sigma_2 &= 0.2, \quad V_0 = 1 \\
r &= 0.05, \quad \delta = 0.05, \quad T = 30 \\
I &= 2, \quad R = 0.79, \quad E_M = E_S = 0 \\
C &= 1, \quad m = 0.01 \\
\rho &= 0.9, \quad \gamma = 0.1
\end{aligned}
$$

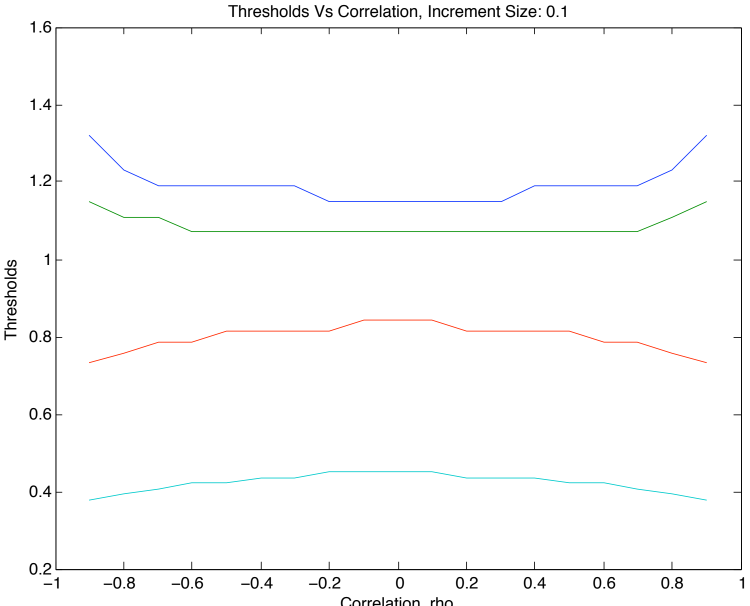
# Dependence on Mothballing Sunk Cost



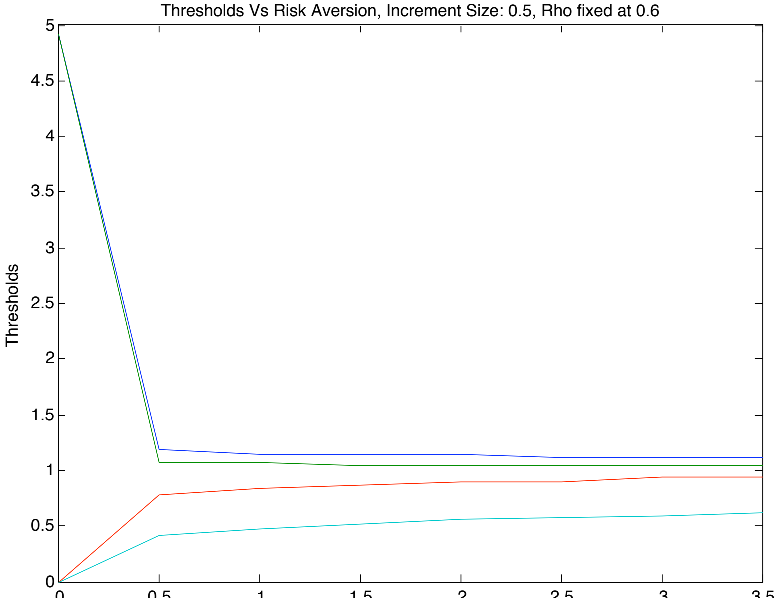
# Dependence on Mothballing Running Cost



# Dependence on Correlation



# Dependence on Risk Aversion



# Combining options and games

- For a systematic application of both real options and game theory in strategic decisions, we consider the following rules:
    1. Outcomes of a given game that involve a "wait–and–see" strategy should be calculated by option value arguments.
    2. Once the solution for a given game is found on a decision node, its value becomes the pay-off for an option at that node.
- In this way, option valuation and game theoretical equilibrium become dynamically related in a decision tree.

# One–period expansion option under monopoly

- Suppose now that a firm faces the decision to expand capacity for a product with uncertain demand:

$$Y_1 = \begin{cases} hY_0 & \text{with probability } p \\ \ell Y_0 & \text{with probability } 1-p \end{cases}, \qquad (6)$$

  correlated with a traded asset
- The expansion requires a sunk cost $I$.
- The state of the firm after the investment decision at time $t_k$ is

$$x(k) = \begin{cases} 1 & \text{if the firm invests at time } t_k \\ 0 & \text{if the does not invest at time } t_k \end{cases} \qquad (7)$$

- The cash flow per unit demand for the firm is denoted by $D_{x(k)}$.

# The NPV solution

- If no expansion occurs at time $t_0$, then the value of the project at time $t_0$ is

$$v_{out} = D_0 Y_0 + g(D_0 h Y_0, D_0 \ell Y_0) = D_0 Y_0 + \pi_0(D_0 Y_1).$$

- If expansion occurs, then the value of the project at time $t_0$ is

$$v_{in} = (D_1 Y_0 - I) + g(D_1 h Y_0, D_1 \ell Y_0) = D_1 Y_0 + \pi_0(D_1 Y_1).$$

- textcolorredIf the decision needs to be taken at time $t_0$, then according to NPV the firm should expand provided $v_{in} \geq v_{out}$, that is, if the sunk cost $I$ is smaller then

$$I^{NPV} = (D_1 - D_0) Y_0 + (\pi_0(D_1 Y_1) - \pi_0(D_0 Y_1)). \tag{8}$$

# The RO solution

- By contrast, if the decision to invest can be postponed until time $t_1$, then the value of the project when no investment occurs at time $t_0$ is

$$v_{wait} = D_0 Y_0 + \pi_0(C_1),$$

where $C_1$ denotes the random variable

$$C_1 = C_1(Y_1) = \max\{D_0 Y_1, D_1 Y_1 - I\} \geq D_0 Y_1.$$

- Accordingly, the firm should invest at time $t_0$ provided $v_{in} \geq v_{wait}$, that is, if the sunk cost is smaller than

$$I^{RO} = (D_1 - D_0) Y_0 + (\pi_0(D_1 Y_1) - \pi_0(C_1)). \tag{9}$$

- Since the function $g$ is non-decreasing in each of its arguments,

$$I^{NPV} - I^{RO} = \pi_0(C_1) - \pi_0(D_0 Y_1) \geq 0. \tag{10}$$

- That is, according to RO, the firm is less likely to expand at time $t_0$.

# A multi-period investment game

- Consider two firms $L$ and $F$ each operating a project with an option to re-invest at cost $I$ and increase cash–flow according to an uncertain demand

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW.$$

- Suppose that the option to re-invest has maturity $T$, let $t_m$, $m = 0, \ldots, M$ be a partition of the interval $[0, T]$ and denote by $(x_L(t_m), x_F(t_m) \in \{(0,0),(0,1),(1,0),(1,1)\}$ the possible states of the firms *after* a decision has been at time $t_m$.
- Let $D_{x_i(t_m)x_j(t_m)}$ denote the cash–flow per unit of demand of firm $i$.
- Assume that $D_{10} > D_{11} > D_{00} > D_{01}$.
- We say that there is FMA is $(D_{10} - D_{00}) > (D_{11} - D_{01})$ and that there is SMA otherwise.

# Derivation of project values (1)

- Let $V_i^{(x_i(t_{m-1}),x_j(t_{m-1}))}(t_m, y)$ denote the project value for firm $i$ at time $t_m$ and demand level $y$.
- Denote by $v_i^{(x_i(t_m),x_j(t_m))}(t_m, y)$ the continuation values:

$$
\begin{aligned}
v_i^{(1,1)}(t_m, y) &= D_{11} y \Delta t + \frac{g(V_i^{(1,1)}(t_{m+1}, y^u), (V_i^{(1,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_L^{(1,0)}(t_m, y) &= D_{10} y \Delta t + \frac{g(V_L^{(1,0)}(t_{m+1}, y^u), (V_L^{(1,0)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_L^{(0,1)}(t_m, y) &= D_{01} y \Delta t + \frac{g(V_L^{(0,1)}(t_{m+1}, y^u), (V_L^{(0,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_F^{(1,0)}(t_m, y) &= D_{01} y \Delta t + \frac{g(V_F^{(1,0)}(t_{m+1}, y^u), (V_F^{(1,0)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_F^{(0,1)}(t_m, y) &= D_{10} y \Delta t + \frac{g(V_F^{(0,1)}(t_{m+1}, y^u), (V_F^{(0,1)}(t_{m+1}, y^d))}{e^{r\Delta t}} \\
v_i^{(0,0)}(t_m, y) &= D_{00} y \Delta t + \frac{g(V_i^{(0,0)}(t_{m+1}, y^u), (V_i^{(0,0)}(t_{m+1}, y^d))}{e^{r\Delta t}}
\end{aligned}
$$

# Derivation of project values (2)

- For fully invested firms, the project values are simply given by

$$V_i^{(1,1)}(t_m, y) = v_i^{(1,1)}(t_m, y).$$

- Now consider the project value for firm $F$ when $L$ has already invested and $F$ hasn't:

$$V_F^{(1,0)}(t_m, y) = \max\{v_F^{(1,1)}(t_m, y) - I, v_F^{(1,0)}(t_m, y)\}.$$

- Similarly, the project value for $L$ when $F$ has invested and $L$ hasn't is

$$V_L^{(0,1)}(t_m, y) = \max\{v_L^{(1,1)}(t_m, y) - I, v_L^{(0,1)}(t_m, y)\}.$$

# Derivation of project values (3)

- Next consider the project value for $L$ when it has already invest and $F$ hasn't:

$$V_L^{(1,0)}(t_m, y) = \begin{cases} v_L^{(1,1)}(t_m, y) \text{ if } v_F^{(1,1)}(t_m, y) - I > v_F^{(1,0)}(t_m, y), \\ v_L^{(1,0)}(t_m, y) \text{ otherwise.} \end{cases}$$

- Similarly, the project value for $F$ when it has already invest and $L$ hasn't is

$$V_F^{(0,1)}(t_m, y) = \begin{cases} v_F^{(1,1)}(t_m, y) \text{ if } v_L^{(1,1)}(t_m, y) - I > v_L^{(0,1)}(t_m, y), \\ v_F^{(0,0)}(t_m, y) \text{ otherwise.} \end{cases}$$

# Derivation of project values (4)

- Finally, the project values $V_i^{(0,0)}$ are obtained as a Nash equilibrium, since both firms still have the option to invest.
- The pay-off matrix for the game is

| | | Firm F: Invest | Firm F: Wait |
|---|---|---|---|
| Firm L | Invest | $(v_L^{(1,1)} - I, v_F^{(1,1)} - I)$ | $(v_L^{(1,0)} - I, v_F^{(1,0)})$ |
| | Wait | $(v_L^{(0,1)}, v_F^{(0,1)} - I)$ | $(v_L^{(0,0)}, v_F^{(0,0)})$ |

# FMA: dependence on risk aversion.

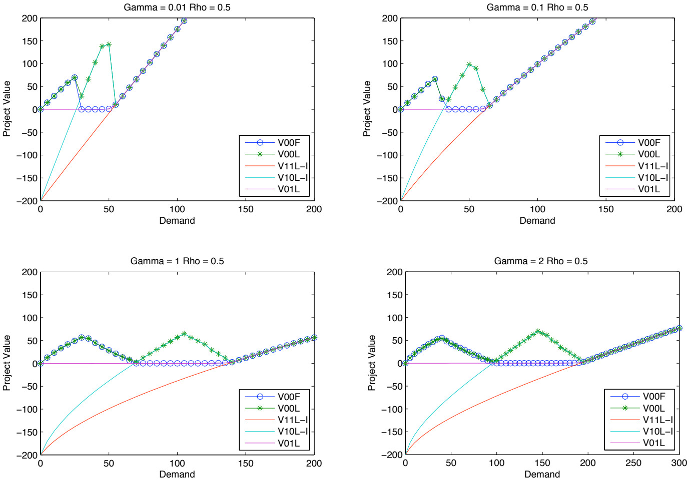



Figure: Project values in FMA case for different risk aversions.

# FMA: dependence on correlation.

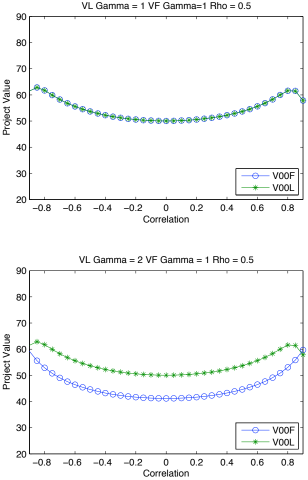



Figure: Project values in FMA case as function of correlation.

# SMA: dependence on risk aversion

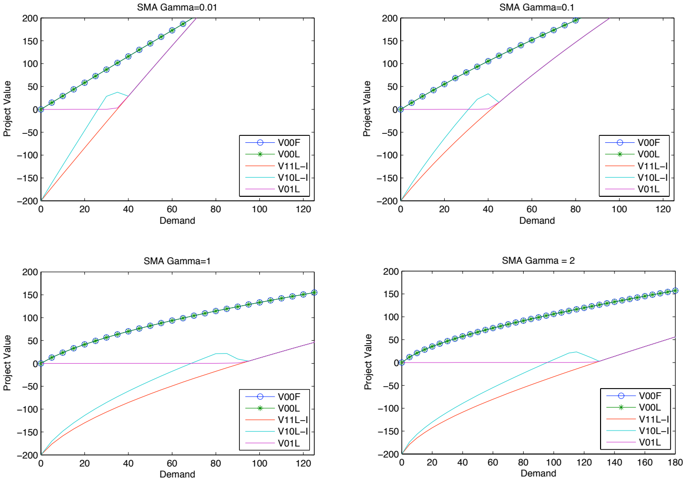



Figure: Project values in SMA case for different risk aversions.

# SMA: dependence on correlation.

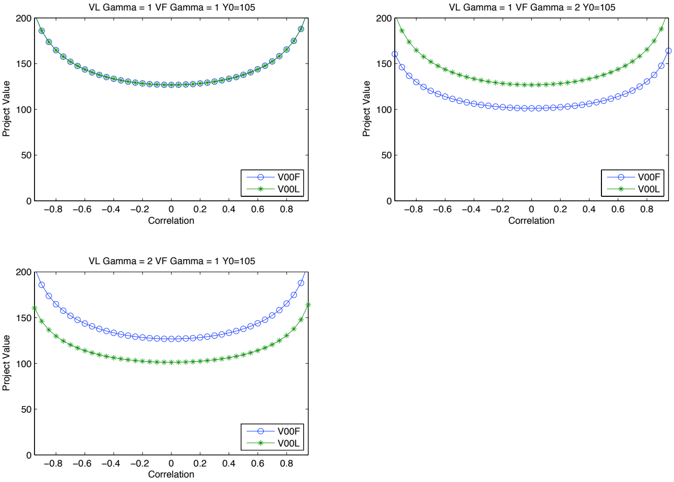



Figure: Project values in SMA case as function of correlation.

# SMA x FMA

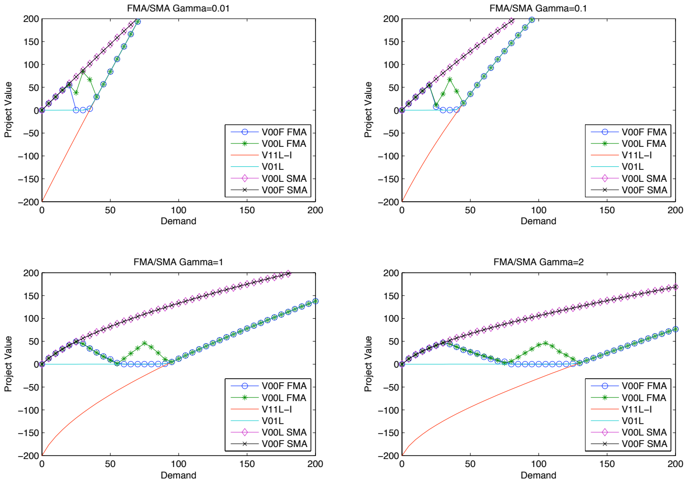



Figure: Project values for FMA and SMA.