Chaotic Interest Rate Model Calibration

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Axiomatic Interest Rate Theory

We follow the axiomatic framework proposed by Hughston and Rafailidis (2005). For this, we need:

- a probability space $(\Omega, \mathcal{F}, P)$ (physical measure)
- the augmented filtration $\mathcal{F}_t$ generated by a k–dimensional Brownian motion W_t
- asset prices S_t given by continuous semimartingales
- a non–dividend–paying asset with adapted price process $\xi_t > 0$ (natural numeraire).

Axiomatic Interest Rate Theory (continued)

The following axioms define an arbitrage–free interest rate model:

1. There exists a strictly increasing asset with absolutely continuous price process B_t (bank account).
2. If S_t is the price of any asset with an adapted dividend rate D_t then
$$\frac{S_t}{\xi_t} + \int_0^t \frac{D_s}{\xi_s} ds \qquad \text{is a martingale} \tag{1}$$
3. There exists an asset that offers a dividend rate sufficient to ensure that the value of the asset remains constant (floating rate note).
4. There exists a system of discount bond price processes P_{tT} satisfying
$$\lim_{T \to \infty} P_{tT} = 0.$$

The state price density

- Define $V_t = 1/\xi_t$ (state price density).
- Since $B_t V_t$ is a martingale (A2) and B_t is strictly increasing (A1), we have

$$E_t[V_T] = E_t\left[\frac{B_T V_T}{B_T}\right] < E_t\left[\frac{B_T V_T}{B_t}\right] = \frac{B_t V_t}{B_t} = V_t,$$

  which means that V_t is a positive supermartingale.
- Writing $B_t = B_0 \exp\left(\int_0^t r_s ds\right)$ for an adapted process $r_t > 0$ and

$$d(B_t V_t) = -(B_t V_t)\lambda_t dW_t,$$

  for an adapted vector process λ_t, we have that the dynamics for V_t is

$$dV_t = -r_t V_t dt - V_t \lambda_t dW_t. \tag{2}$$

Conditional variance representation

- Integrating (2), taking conditional expectations and the limit $T \to \infty$ (all well–defined thanks to (A3) and (A4)) leads to

$$V_t = E_t\left[\int_t^{\infty} r_s V_s ds\right].$$

- Now let σ_t be a vector process satisfying $\sigma_t^2 = r_t V_t$ and define the square integrable random variable

$$X_\infty := \int_0^{\infty} \sigma_s dW_s.$$

- It then follows from the Ito isometry that

$$V_t = E_t\left[(X_\infty - X_t)^2\right], \tag{3}$$

where $X_t := E_t[X_\infty] = \int_0^t \sigma_s dW_s$.

Wiener chaos

- Define the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{4}$$

- For $h \in L^2(\mathbb{R}_+^k)$, define the Gaussian random variable

$$W(h) := \int_0^\infty h(s) dW_s.$$

- Then the Wiener chaos of order n,

$$\begin{aligned} \mathcal{H}_n &:= \mathrm{span}\{H_n(W(h)) | h \in L^2(\Delta)\}, \quad n \geq 1, \\ \mathcal{H}_0 &:= \mathbb{C}, \end{aligned}$$

  provide an orthogonal decomposition of square integrable random variables:

$$L^2(\Omega, \mathcal{F}_\infty, P) = \bigoplus_{n=0}^\infty \mathcal{H}_n.$$

Wiener chaos expansion

- Let $\Delta_n := \{(s_1, \dots, s_n) \in \mathbb{R}_+^n | 0 \le s_n \le \dots \le s_2 < s_1 \le \infty\}$.
- Each $\mathcal{H}_n$ can be identified with $L^2(\Delta_n)$ via the isometries

$$J_n : L^2(\Delta_n) \to \mathcal{H}_n$$

given by

$$\phi_n \mapsto J_n(\phi_n) = \int_{\Delta_n} \phi_n(s_1, \dots, s_n) dW_{s_1} \dots dW_{s_n}, \tag{5}$$

- With these ingredients, one is then led to the result that any $X \in L^2(\Omega, \mathcal{F}_\infty, P)$ can be represented as a Wiener chaos expansion

$$X = \sum_{n=0}^{\infty} J_n(\phi_n), \tag{6}$$

where the deterministic functions $\phi_n \in L^2(\Delta_n)$ are uniquely determined by the random variable X.

First order chaos

- In a first order chaos model we have
$$X_\infty = \int_0^\infty \phi(s) dW_s.$$
- In this case $\sigma_s = \phi(s)$, so that $M_{ts} := E_t[\sigma_s^2] = \phi^2(s)$ and
$$V_t = \int_t^\infty M_{ts} ds = \int_t^\infty \phi^2(s) ds$$
- This corresponds to a deterministic interest rate theory, since
$$P_{tT} = \frac{\int_T^\infty \phi^2(s) ds}{\int_t^\infty \phi^2(s) ds}, \quad f_{tT} = \frac{\phi^2(T)}{\int_T^\infty \phi^2(s) ds} = r_T.$$
- The remaining asset prices can be stochastic, however. Indeed, for a derivative with payoff H_T we have
$$H_t = \frac{E_t[V_T H_T]}{V_t} = \frac{V_T}{V_t} E_t[H_T] = P_{tT} E_t[H_T]$$

Second order chaos: definition

- In a second order chaos model we have

$$X_\infty = \int_0^\infty \phi_1(s)dW_s + \int_0^\infty \int_0^s \phi_2(s,u)dW_u dW_s$$

- In this case $M_{ts} = E_t[\sigma_s^2]$ where

$$\sigma_s = \phi_1(s) + \int_0^s \phi_2(s,u)dW_u.$$

- Using the Ito isometry we find that

$$M_{ts} = \left(\phi_1(s) + \int_0^t \phi_2(s,u)dW_u\right)^2 + \int_t^s \phi_2^2(s,u)du,$$

which, for each t, is a parametric family of squared Gaussian r.v. plus a constant.

Second order chaos: bond and option prices

- Defining $Z_{tT} = \int_T^\infty M_{ts} ds$, we see that bond prices are given by
$$P_{tT} = \frac{Z_{tT}}{Z_{tt}}.$$
- In particular, since $M_{0s} = \phi_1^2(s) + \int_0^s \phi_2^2(s,u)du$, it follows that
$$P_{0T} = \frac{\int_T^\infty \left(\phi_1^2(s) + \int_0^s \phi_2^2(s,u)du\right) ds}{\int_0^\infty \left(\phi_1^2(s) + \int_0^s \phi_2^2(s,u)du\right) ds}.$$
- Moreover, the price at time zero of an option with payoff $(P_{tT} - K)^+$ is
$$ZBC(0,t,T,K) = \frac{1}{V_0} E\left[V_t\left(P_{tT} - K\right)^+\right] = \frac{1}{V_0} E\left[(Z_{tT} - KZ_{tt})^+\right],$$
which can be calculated in terms of the joint distribution of Z_{tT_1} and Z_{tT_2}.

Factorizable second order chaos: definition

- Consider $\phi_1(s) = \alpha(s)$ and $\phi_2(s, u) = \beta(s)\gamma(u)$.
- Then $\sigma_s = \phi(s) + \beta(s)R_s$ where

$$R_t = \int_0^t \gamma(s)dW_s$$

  is a martingale with quadratic variation $Q(t) = \int_0^t \gamma^2(s)ds$.
- Therefore

$$\begin{aligned} M_{ts} &= (\alpha^2(s) + \beta(s)R_t)^2 + \beta^2(s)[Q(s) - Q(t)] \\ &= \alpha^2(s) + \beta^2(s)Q(s) + 2\alpha(s)\beta(s)R_t + \beta^2(s)(R_t^2 - Q(t)) \end{aligned}$$

- Notice that the scalar random variable R_t is the sole state variable for the interest rate model at time t, even in the case of a multidimensional Brownian motion W_t.

Factorizable second order chaos: bond prices

- Integrating the previous expression gives

$$Z_{tT} = \int_T^\infty M_{ts} ds = A(T) + B(T)R_t + C(T)(R_t^2 - Q(t)),$$

where

$$\begin{aligned} A(T) &= \int_t^\infty (\alpha^2(s) + \beta^2(s)Q(s))ds \\ B(T) &= 2\int_T^\infty \alpha(s)\beta(s)ds, \quad C(T) = \int_T^\infty \beta^2(s)ds \end{aligned}$$

- Therefore

$$V_t = A(t) + B(t)R_t + C(t)(R_t^2 - Q(t))$$

and

$$P_{tT} = \frac{A(T) + B(T)R_t + C(T)(R_t^2 - Q(t))}{A(t) + B(t)R_t + C(t)(R_t^2 - Q(t))}$$

Factorizable second order chaos: option prices

- Fixing t, T and K, it follows that

$$Z_{tT} - KZ_{tt} = A + BY + CY^2,$$

where $Y = R(t)/\sqrt{Q(t)} \sim N(0,1)$ and

$$\begin{aligned} A &= [A(T) - KA(t)] - [C(T) - KC(t)]Q(t) \\ B &= [B(T) - KB(t)]\sqrt{Q(t)}, \quad C = [C(T) - KC(t)]Q(t) \end{aligned}$$

- Therefore, defining $p(y) = A + By + Cy^2$, we have

$$ZBC(0,t,T,K) = \frac{1}{A(0)\sqrt{2\pi}} \int_{p(y)\geq 0} p(y) e^{-\frac{1}{2}y^2} dy,$$

which can be calculated explicitly in terms of the roots of the polynomial $p(y)$.

- Analogous expressions can be derived for puts, swaptions, caps, floors, etc...

One–variable second order chaos

- Consider now

$$
\begin{aligned}
X_\infty &= \int_0^\infty \alpha(s) dW_s + \int_0^\infty \int_0^s \beta(s) dW_u dW_s \\
&= \int_0^\infty [\alpha(s) + \beta(s) W_s] dW_s
\end{aligned}
$$

- For fitting the initial term structure, this behaves like a first order chaos model with $\phi^2(s) = \alpha^2(s) + \beta^2(s)s$
- However, the stochastic evolution of bond prices is now

$$
P_{tT} = \frac{A(T) + B(T)W_t + C(T)(W_t^2 - t)}{A(t) + B(t)W_t + C(t)(W_t^2 - t)}
$$

- Option prices are determined by the same expression as before by setting $Q(t) = t$.

One–variable third order chaos

- Motivated by the previous example, we consider

$$X_\infty = \int_0^\infty \alpha(s)dW_s + \int_0^\infty\int_0^s \beta(s)dW_u dW_s + \int_0^\infty\int_0^s\int_0^u \delta(s)dW_v dW_u dW_s$$
$$= \int_0^\infty \left[\alpha(s) + \beta(s)W_s + \frac{1}{2}\delta(s)(W_s^2 - s)\right] dW_s$$

- Again, for fitting P_{0T} this behaves like a first order chaos model with $\phi(s) = \alpha^2(s) + \beta^2(s)s + \delta^2(s)s^2/2$.
- Moreover, since

$$Z_{tT} = a(T) + b(T)W_t + c(T)W_t^2 + d(T)W_t^3 + e(T)W_t^4,$$

general bond prices are expressed as the ratio of 4th–order polynomials in W_t.
- Similarly, option prices can be found explicitly by integrating a 4th–order polynomial of a standard normal random variable.

Data

- For P_{0T} we use clean prices of treasury coupon strips in the Gilt Market using data from the UK Debt Management Office (DMO) at 146 dates (every other business day) from January 1998 to January 1999 with 50 maturities for each date.
- We also consider yield data from money market at 53 dates (every Friday) from September 2000 to August 2001 with 23 maturities for each date.
- For interest rate options for the same period we use ATM caps (39 caplets) and swaptions (6 maturities and 7 tenors).

Empirical yield curves

Empirical yields and caplets

Empirical swaption data

Parametric specification

- Motivated by the vast literature on forward rate curve fitting (so-called descriptive–form interest rate models), we consider the exponential–polynomial family (Bjork and Christensen 99):

$$\phi(s) = \sum_{i=1}^{n} \left(\sum_{j=1}^{\mu_i} b_{ij} s^j \right) e^{-c_i s}$$

- Special cases in this family are the Nelson–Sigel (87), Svensson (94) and Cairns (98) models:

$$\begin{aligned} \phi_{NS}(s) &= b_0 + (b_1 + b_2 s) e^{-c_1 s} \\ \phi_{Sv}(s) &= b_0 + (b_1 + b_2 s) e^{-c_1 s} + b_3 s e^{c_2 s} \\ \phi_C(s) &= \sum_{i=1}^{4} b_1 e^{c_i s} \end{aligned}$$

Descriptive fit for yield curves

Chaos fit for yield curves

Calibration results: bonds from Jan/98 to Feb/99

	Chaos Order	N	-L	RMSE	RMSPE
1	1st chaos	3	4435.6428	0.0073	0.0452
2	Descriptive NS	4	2116.9623	0.0049	0.0275
3	1st chaos	5	269.7759	0.0019	0.0096
4	factorizable 2nd chaos	6	254.1958	0.0020	0.0072
5	one-var 2nd chaos	6	180.0528	0.0015	0.0094
6	one-var 3rd chaos	6	179.8550	0.0015	0.0087
7	Descriptive Sv	6	170.7942	0.0015	0.0074
8	factorizable 2nd chaos	7	172.9058	0.0015	0.0082
9	one-var 2nd chaos	7	163.5599	0.0014	0.0086
10	one-var 3rd chaos	7	63.5103	0.0014	0.0083

Stability of parameters

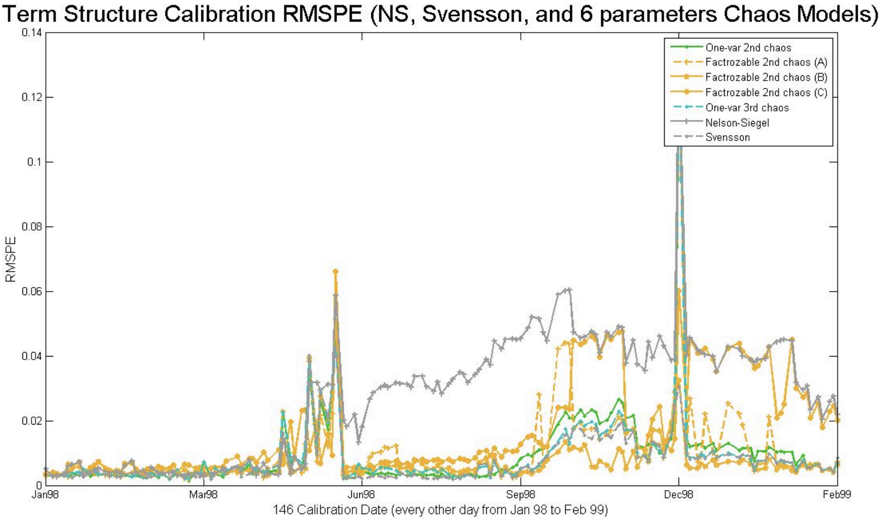

Figure: RMSE as a function of time.

Forward rates

Short rate models

- For interest rate options, we need to have at least a short rate model, such as the CIR process

$$dr_t = \kappa(\theta - r_t) + \sigma\sqrt{r_t}dW_t. \tag{7}$$

- To fit the initial term structure we also use the Hull–White model

$$dr_t = \kappa(\Theta(t) - r_t) + \sigma\sqrt{r_t}dW_t. \tag{8}$$

CIR fit for yields and caplets

CIR fit for yields and swaptions

Hull–White fit for yields and caplets

Hull–White fit for yields and swaptions

The rational lognormal model (Flesaker and Hughston 96)

- For comparison, we also consider the following model:

$$M_{ts} = g_1(s)M_t + g_2(s),$$

where

$$M_t = \exp\left[\int_0^t \theta(s)dW_s - \frac{1}{2}\int_0^t \theta(s)^2 ds\right].$$

- Bond prices in this model are given by

$$P_{tT} = \frac{G_1(T)M_t + G_2(T)}{G_1(t)M_t + G_2(t)}$$

where

$$G_1(t) = \int_t^\infty g_1(s)ds, \qquad G_2(t) = \int_t^\infty g_2(s)ds.$$

- Because M_t is an exponentiated first–chaos, this interest rate model has chaos terms of all orders.

Rational lognormal fit for yields and caplets

Chaos fit for yields and caplets (I)

Chaos fit for yields and caplets (II)

Chaos fit for yields and caplets (III)

Chaos fit for yields and swaptions

Joint calibration results (I): yields and caplets from Sept/00 to Aug/01

	Model	N	Yield Error	Caplet Error	RMSPE
1	CIR	3	0.0010	0.0181	0.1323
2	one-var 2nd chaos	6	0.0019	0.0031	0.0685
3	fac. 2nd chaos	6	0.0004	0.0020	0.0483
4	one-var 3rd chaos	6	0.0005	0.0023	0.0516
5	Hull-White	8	0.0003	0.0168	0.1244
6	one-var 3rd chaos	7	0.0002	0.0021	0.0454
7	rational lognormal	9	0.0021	0.0023	0.0625
8	one-var 3rd chaos	9	0.0002	0.0012	0.0359

Joint calibration results (II): yields and swaptions from Sept/00 to Aug/01

	Model	N	Yield Error	Swaption Error	RMSPE
1	CIR	3	0.0030	0.0045	0.0852
2	one-var 2nd chaos	6	0.0004	0.0048	0.0707
3	fac. 2nd chaos	6	0.0005	0.0046	0.0714
4	one-var 3rd chaos	6	0.0010	0.0018	0.0526
5	one-var 3rd chaos	7	0.0003	0.0012	0.0376
6	Hull-White	8	0.0001	0.0104	0.0971
7	rational lognormal	9	0.0002	0.0030	0.0557
8	one-var 3rd chaos	9	0.0003	0.0011	0.0376

Joint calibration results (III): yields, caplets and swaptions from Sept/00 to Aug/01

	Model	N	YE	SE	CE	RMSPE
1	CIR	3	0.0032	0.0089	0.0203	0.1778
2	one-var 2nd chaos	6	0.0013	0.0087	0.0069	0.1274
3	Factorizable 2nd chaos	6	0.0010	0.0078	0.0076	0.1268
4	one-var 3rd chaos	6	0.0021	0.0027	0.0035	0.0894
5	one-var 3rd chaos	7	0.0020	0.0027	0.0035	0.0891
6	Hull-White	8	0.0020	0.0142	0.0250	0.1936
7	Rational Log-normal	9	0.0024	0.0170	0.0105	0.1719

Performance of Chaos Models

Figure: Squared-errors for different dates (green for yields, yellow for options)

Conclusions

1. We propose a systematic way to calibrate interest rate model in the chaotic approach.
2. For term structure calibration, the performance of 1st–order chaos is comparable to their deterministic descriptive form analogues (Nelson–Sigel and Svensson).
3. One–variable higher–order chaos slightly improves the performance, with the advantage of being fully stochastic and consistent with non-arbitrage and positivity conditions.
4. Option calibration requires at least a factorizable 2nd–order chaos.
5. One–variable 3rd–order chaos outperforms rational lognormal.
6. Further work will compare factorizable 2nd–order and two–variable 3rd–order chaos models for smile calibration.
7. Higher–order chaos models are likely to be unnecessary (and possibly made illegal anyway...)