The Wiener chaos expansion for the Cox–Ingersoll–Ross model

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Joint work with T.R Hurd
Supported by NSERC and MITACS
Special thanks to Chemistry Department - KCL

July 1, 2003

1. **Introduction**

- We take the CIR model as given by

$$dr_t = a(b - r_t)dt + c\sqrt{r_t}d\tilde{W}_t, \qquad \|\lambda_t\|^2 = \bar{\lambda}^2 r_t \tag{1}$$

for some positive constants $a, b, c, \bar{\lambda}$ with $4ab > c^2$, where $\tilde{W}_t$ is a standard one dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ equipped with a Brownian filtration $(\mathcal{F}_t)_{0\le t\le\infty}$.

- We seek for the chaotic representation of the underlying random variable X_∞ in the Hughston/Rafailidis framework.

2. **Positive Interest Rates**

Let $P_{tT}, 0 \leq t \leq T$ denote the price at time t for a zero coupon bond which pays one unit of currency at its maturity T. Clearly $P_{tt} = 1$ for all $0 \leq t < \infty$ and furthermore, positivity of the interest rate is equivalent to having

$$P_{ts} \leq P_{tu}, \tag{2}$$

for all $0 \leq t \leq u \leq s$.

2.1 **The Flesaker–Hughston approach**

FH introduced zero coupon bond prices in the form

$$P_{tT} = \frac{\int_T^\infty h_s M_{ts} ds}{\int_t^\infty h_s M_{ts} ds}, \quad \text{for } 0 \le t \le T < \infty, \tag{3}$$

where M_{ts} is a family of strictly positive continuous martingales satisfying $M_{0s} = 1$ and $h_T = -\frac{\partial P_{0T}}{\partial T}$ is a positive deterministic function obtained from the initial term structure. Then the positivity condition (2), as well as $P_{tt} = 1$, holds for all $0 \le t \le T < \infty$.

For concrete examples, they introduce the process

$$V_t = \int_t^\infty h_s M_{ts} ds, \tag{4}$$

which is easily seen to be a strictly positive supermartingale.

Due to the martingale property for M_{ts}, the bond prices can be rewritten as

$$P_{tT} = \frac{E_t[V_T]}{V_t}. \tag{5}$$

2.2 **State price density and the potential approach**

One can elevate equation (5) to the starting point of the modelling and concentrate on the positive adapted continuous process V_t, called the *state price density*.

Positivity of the interest rates is then equivalent to V_t being a supermartingale. In order to match the initial term structure, it needs to be chosen so that $E[V_T] = P_{0T}$. If we further impose that $P_{0T} \to 0$ as $T \to \infty$, then V_t satisfies all the properties of what is known in probability theory as a *potential* (namely, a positive supermartingale with expected value going to zero at infinity).

It follows from the Doob–Meyer decomposition that any continuous potential satisfying

$$E\left(\sup_{0\leq t\leq\infty} V_t^2\right) < \infty \tag{6}$$

can be written as

$$V_t = E_t[A_\infty] - A_t, \tag{7}$$

for a unique (up to indistinguishibility) adapted continuous increasing process A_t with $E(A_\infty^2) < \infty$, satisfying the constraint that

$$E\left[\frac{\partial A_T}{\partial T}\right] = -\frac{\partial P_{0T}}{\partial T}. \tag{8}$$

2.3 **Related quantities**

Given a strictly positive supermartingale V_t, there exist a unique strictly positive (local) martingale Λ_t such that the process

$$B_t = \Lambda_t / V_t \tag{9}$$

is strictly increasing. We identify B_t with a riskless money market account initialized at $B_0 = 1$ and write it as

$$B_t = \exp(\int_0^t r_s ds), \tag{10}$$

for an adapted process $r_s > 0$, the short rate process.

The market price of risk then arises as the adapted vector valued process λ_t such that

$$d\Lambda_t = -\lambda_t \Lambda_t dW_t. \tag{11}$$

It is immediate to see that

$$dV_t = -r_t V_t dt - \lambda_t V_t dW_t. \tag{12}$$

so that the specification of the process V_t is enough to produce both the short rate r_t and the market price of risk λ_t.

2.4 **The Chaotic Approach**

Assume that the state price density V_t is a potential satisfying

$$E\left[\int_0^\infty r_s V_s ds\right] < \infty \tag{13}$$

By integrating (12) on the interval (t, T), taking conditional expectations at time t and the limit $T \to \infty$, one finds that

$$V_t = E_t\left[\int_t^\infty r_s V_s ds\right] \tag{14}$$

Now let σ_t be a vector valued process such that

$$\|\sigma_t\|^2 = r_t V_t, \tag{15}$$

and define the square integrable random variable

$$X_\infty = \int_0^\infty \sigma_s dW_s. \tag{16}$$

It follows from the Ito isometry that

$$V_t = E_t[X_\infty^2] - E_t[X_\infty]^2, \tag{17}$$

which is called the conditional variance representation of the state price density V_t. A direct comparison between (14) and (4) gives that

$$h_s M_{ts} = E_t\left[\|\sigma_s\|^2\right]. \tag{18}$$

Similarly, by comparing the conditional variance representation (17) with the decomposition (7), we see that

$$E_t[X_\infty^2] - X_t^2 = E_t[A_\infty] - A_t,$$

where $X_t = E_t[X_\infty]$. It follows from the uniqueness of the Doob-Meyer decomposition that

$$A_t = [X, X]_t,$$

that is, the quadratic variation of the process X_t.

2.5 **Wiener chaos**

Let W_t be an N–dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}_+}, P)$. To streamline the handling of vector indices by a compact notation

$$\tau = (s, \mu) \in \Delta \doteq \mathbb{R}_+ \times \{1, \ldots, N\}$$

and express integrals as

$$\begin{aligned} \int_\Delta f(\tau) d\tau &\doteq \sum_\mu \int_0^\infty f(s, \mu) ds \\ \int_\Delta f(\tau) dW_\tau &\doteq \sum_\mu \int_0^\infty f(s, \mu) dW_s^\mu \end{aligned} \tag{19}$$

For each $n \geq 0$, let

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{20}$$

be the nth Hermite polynomial. For $h \in L^2(\Delta)$, let $W(h)$ denote the Gaussian random variable $\int h(\tau) dW_\tau$. The spaces

$$\begin{aligned} \mathcal{H}_n &\doteq \text{span}\{H_n(W(h)) | h \in L^2(\Delta)\}, \quad n \geq 1, \\ \mathcal{H}_0 &\doteq \mathbb{C} \end{aligned}$$

form an orthogonal decomposition of the space $L^2(\Omega, \mathcal{F}_\infty, P)$ of square integrable random variables:

$$L^2(\Omega, \mathcal{F}_\infty, P) = \oplus_{n=0}^{\infty} \mathcal{H}_n$$

Each $\mathcal{H}^{(n)}$ can be understood completely via the *isometries*

$$f_n \mapsto J_n(f_n) = \int_{\Delta_n} f_n(\tau_1, \dots, \tau_n) dW_{\tau_1} \dots dW_{\tau_n} \tag{21}$$

where $\Delta_n \doteq \{(\tau_1, \dots, \tau_n) | \tau_i = (s_i, \mu_i) \in \Delta, 0 \leq s_1 \leq s_2 \leq \dots \leq s_n < \infty\}$.

With these ingredients, one is then led to the result that any $X \in L^2(\Omega, \mathcal{F}_\infty, P)$ can be represented as a *Wiener chaos expansion*

$$X = \sum_{n=0}^{\infty} J_n(f_n) \tag{22}$$

where the deterministic functions $f_n \in L^2(\Delta_n)$ are uniquely determined by the random variable X.

A special example arises by noting that for $h \in L^2(\Delta)$

$$n!J_n(h^{\otimes n}) = \|h\|^n H_n\left(\frac{W(h)}{\|h\|}\right) \tag{23}$$

and furthermore

$$\exp\left[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau\right] = \sum_{n=0}^{\infty} \frac{\|h\|^n}{n!} H_n\left(\frac{W(h)}{\|h\|}\right) \tag{24}$$

In the notation of quantum field theory, this example defines the Wick ordered exponential and Wick powers

$$\begin{aligned} :\exp[W(h)]: \quad &\doteq \exp\left[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau\right] \\ :W(h)^n: \quad &\doteq n!J_n(h^{\otimes n}) \end{aligned} \tag{25}$$

Theorem 1 *For any random variable* $X \in L^2(\Omega, \mathcal{F}_\infty, P)$*, the generating functional* $Z_X(h) : L^2(\Delta) \to \mathbb{C}$ *defined by*

$$Z_X(h) \doteq E\left[X \exp\left[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau\right]\right] \tag{26}$$

is an entire analytic functional of $h \in L^2(\Delta)$ *and hence has an absolutely convergent expansion*

$$Z_X(h) = \sum_{n \geq 0} F_X^{(n)}(h) \tag{27}$$

where

$$F_X^{(n)}(h) = \int_{\Delta_n} f_X^{(n)}(\tau_1, \ldots, \tau_n) h(\tau_1) \ldots h(\tau_n) d\tau_1 \ldots d\tau_n \tag{28}$$

Here, $f_X^{(n)}(\tau_1, \ldots, \tau_n)$ *is the* n*th Frechet derivative of* Z_X *at* $h = 0$*. Finally, the Wiener–Itô chaos expansion of* X *is*

$$X = \sum_{n \geq 0} \int_{\Delta_n} f_X^{(n)}(\tau_1, \ldots, \tau_n) dW_{\tau_1} \ldots dW_{\tau_n} \tag{29}$$

3. **Squared Gaussian models**

The CIR model with an integer constraint $N \doteq \frac{4ab}{c^2} \in \mathbb{N}_+ \setminus \{0,1\}$ lies in the class of so–called squared Gaussian models. By introducing an $\mathbb{R}^N$–valued Ornstein–Uhlenbeck process R_t, governed by the stochastic differential equation

$$dR_t = -\frac{a}{2} R_t \; dt + \frac{c}{2} dW_t \tag{30}$$

where W_t is N–dimensional Brownian motion, one verifies that the square $r_t = \|R_t\|^2$ satisfies (1) where

$$\tilde{W}_t = \int_0^t \|R_t\|^{-1} R_t \cdot dW_t$$

is itself a one–dimensional Brownian motion.

Definition 2 *A pair (r_t, λ_t) of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ processes is called an $N-$dimensional squared Gaussian model of interest rates ($N \geq 2$) if there is an $\mathbb{R}^N-$valued Ornstein–Uhlenbeck process such that $r_t = \|R_t\|^2$ and $\lambda_t = \bar{\lambda} R_t$. The process R_t satisfies*

$$dR_t = \alpha(t)(\bar{R}(t) - R_t)dt + \gamma(t)dW_t, \quad R|_{t=0} = R_0 \tag{31}$$

where $\alpha, \gamma, \bar{\lambda}$ are symmetric matrix valued and $\bar{R}$ vector valued deterministic measurable functions on $\mathbb{R}_+$. W is standard $N-$dimensional Brownian motion. In addition we impose the boundedness condition:

- *there is some constant $M > 0$ such that $\alpha(t) \geq M$ and $|\bar{\lambda}(t)|^{-2} \geq M$ for all t*

The exact solution of (31) is easily seen to be

$$R_t = \tilde{R}(t) + \int K(t, t_1)(\gamma dW)_{t_1} \tag{32}$$

where

$$\tilde{R}(t) = K(t, 0)R_0 + \int K(t, t_1)\alpha(t_1)\bar{R}(t_1)dt_1 \tag{33}$$

and $K(t, s), t \geq s$ is the matrix valued solution of

$$\begin{cases} dK(t, s)/dt = -\alpha(t)K(t, s) & 0 \leq s \leq t \\ K(t, t) = I & 0 \leq t \end{cases} \tag{34}$$

which generates the Ornstein–Uhlenbeck semigroup.

By (12), the state price density process is

$$V_t = \exp\left[-\int_0^t \left(R_s^\dagger \left(1 + \frac{\bar{\lambda}^2}{2}\right) R_s ds + R_s^\dagger \bar{\lambda} dW_s\right)\right] \tag{35}$$

We thus have a natural candidate for the random variable X_∞:

$$X_\infty = \int_0^\infty \sigma_t^\dagger \; dW_t \tag{36}$$

where σ_t is the $\mathbb{R}^N$–valued process

$$\sigma_t \doteq \exp\left[-\int_0^t \left(R_s^\dagger \left(\frac{1}{2} + \frac{\bar{\lambda}^2}{4}\right) R_s ds + \frac{1}{2} R_s^\dagger \bar{\lambda} dW_s\right)\right] R_t \tag{37}$$

is the natural solution of $\sigma_t^\dagger \sigma_t = r_t V_t$.

Proposition 3 $\lim_{T\to\infty} E[X_T^2] = 1$

4. **Exponentiated second chaos**

The chaos expansion we seek for the CIR model will be derived from a closed formula for expectations of e^{-Y} for elements

$$Y = A + \int_{\Delta} B^{\dagger}(\tau_1) dW_{\tau_1} + \int_{\Delta_2} C(\tau_1, \tau_2) dW_{\tau_1} dW_{\tau_2} \tag{38}$$

in a certain subset $\mathcal{C}^{+} \subset \mathcal{H}_{\leq 2} \doteq \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

If in (38) we define $C(\tau_1, \tau_2) = C(\tau_2, \tau_1)$ when $\tau_1 > \tau_2$, then C is the kernel of a symmetric integral operator on $L^2(\Delta)$:

$$[Cf](\tau) = \int_0^\infty C(\tau, \tau_1) f(\tau_1) d\tau_1 \tag{39}$$

Recall that integral Hilbert-Schmidt operators are finite norm operators under the norm:

$$\|C\|_{HS}^2 = \int_{\Delta^2} |C(\tau_1, \tau_2)|^2 d\tau_1 d\tau_2$$

We say that $Y \in \mathcal{H}_{\leq 2}$ is in $\mathcal{C}^+$ if C is the kernel of a symmetric Hilbert–Schmidt operator on $L^2(\Delta)$ such that $(1 + C)$ has non–negative spectrum.

Proposition 4 *Let $Y \in \mathcal{C}^+$. Then*

$$E[e^{-Y}] = [\mathrm{det}_2(1+C)]^{-1/2} \exp\left[-A + \frac{1}{2}\int_{\Delta_2} B^\dagger(\tau_1)(1+C)^{-1}(\tau_1,\tau_2)B(\tau_2)d\tau_1 d\tau_2\right]$$

Remark 5 *The Carleman–Fredholm determinant is defined as the extension of the formula*

$$\mathrm{det}_2(1+C) = \det(1+C)\exp[-\mathrm{Tr}(C)] \tag{40}$$

from finite rank operators to bounded Hilbert–Schmidt operators; the kernel $(1+C)^{-1}(\tau_1,\tau_2)$ is also the natural extension from the finite rank case.

Corollary 6 (Wick's theorem) *The random variable* $X = e^{-Y}$*, for* $Y = \int_{\Delta_2} C(\tau_1, \tau_2) dW_{\tau_1} dW_{\tau_2} \in \mathcal{C}^+$ *has Wiener chaos coefficient functions*

$$f_n(\tau_1, \ldots, \tau_n) = \begin{cases} K \sum_{G \in \mathcal{G}_n} \prod_{g \in G} [C(1+C)^{-1}](\tau_{g_1}, \tau_{g_2}) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

where $K = [\det_2(1 + \mathrm{C})]^{-1/2}$ *and for* n *even,* $\mathcal{G}_n$ *is the set of* Feynman graphs *on the* n *marked points* $\{\tau_1, \ldots, \tau_n\}$*. Each Feynman graph* G *is a disjoint union of unordered pairs* $g = (\tau_{g_1}, \tau_{g_2})$ *with* $\cup_{g \in G}\, g = \{\tau_1, \ldots, \tau_n\}$*.*

Proof: The generating functional for $X = e^{-Y}$ is

$$
\begin{aligned}
Z_X(h) &= E\left[X \exp\left(\int h(\tau)dW_\tau - \frac{1}{2}\int h(\tau)^2 d\tau\right)\right] \\
&= E\left[\exp\left(\int h(\tau)dW_\tau - \frac{1}{2}\int h(\tau)^2 d\tau - \int_{\Delta_2} C(\tau_1,\tau_2)dW_{\tau_1}dW_{\tau_2}\right)\right],
\end{aligned}
$$

so we can use Proposition 4

$$
\begin{aligned}
Z_X(h) &= \det{}_2(1+C)^{-1/2} \\
&\quad \exp\left[-\frac{1}{2}\int_{\Delta^2} h^\dagger(\tau_1)[\delta(\tau_1,\tau_2) - (1+C)^{-1}(\tau_1,\tau_2)]h(\tau_2)d\tau_1 d\tau_2\right]
\end{aligned}
$$

using the last part of Theorem 1, the result comes by evaluating the nth Fréchet derivative at $h = 0$, or equivalently by expanding the exponential and symmetrizing over the points $\tau_1, \ldots, \tau_n$ in the $n/2$th term. $\sqcap\!\!\!\!\sqcup$

5. **The chaotic expansion for squared Gaussian models**

We now derive the chaos expansion for the squared Gaussian model defined by (31). For simplicity, we assume from now on that $\bar{\lambda} = 0$. In view of (36) it will be enough to find the chaos expansion for σ_T^{μ}, $T < \infty$. We start by finding its the generating functional $Z_{\sigma_T^{\mu}}$. Define the functional $Z(h,k) = E\left[e^{-Y_T}\right]$ with

$$\begin{aligned} Y_T &= \frac{1}{2}\int_0^T R_t^{\dagger} R_t dt + -\int_0^T h^{\dagger}(t) dW_t \\ &\quad + \frac{1}{2}\int_0^T \|h(t)\|^2 dt - \int_0^T k^{\dagger}(t) R_t dt. \end{aligned} \tag{41}$$

Proposition 7 *$Z(h,k)$ is an entire analytic functional of $(h,k) \in L^2([0;T]) \otimes \mathbb{R}^{2N}$. Moreover*

$$\lim_{t \to T^-} \frac{\delta Z(h,k)}{\delta k^\mu(t)}\bigg|_{k=0} = Z_{\sigma_T^\mu}(h) \tag{42}$$

where $Z_{\sigma_T^\mu}(h)$ is defined by (26) with $X = \sigma_T^\mu, \mu = 1, \ldots, N$.

We want to use Proposition 4 in order to comnpute $Z(h,k)$. Substitution of (32) into the first and last terms of (41) leads to

$$\begin{aligned}
\frac{1}{2}\int_0^T R_t^\dagger\, R_t dt &= \frac{1}{2}\int_0^T \tilde{R}^\dagger(t)\tilde{R}(t)dt + \int_0^T \left(\int_0^T \tilde{R}_s^\dagger K_T(s,t)ds\right)\gamma(t)dW_t \\
&\quad + \int_{\Delta_2}\gamma(t_1)\left(\int_0^T K_T^\dagger(s,t_1)K_T(s,t_2)ds\right)\gamma(t_2)dW_{t_1}dW_{t_2} \\
&\quad + \frac{1}{2}\int_0^T \mathrm{tr}\left[\gamma(t)\left(\int_0^T K_T^\dagger(s,t)K_T(s,t)ds\right)\gamma(t)\right]dt, \\
\int_0^T k_t^\dagger R_t dt &= \int_0^T k_t^\dagger \tilde{R}_t dt + \int_0^T \left(\int_0^T k_s^\dagger K_T(s,t)ds\right)\gamma(t)dW_t,
\end{aligned}$$

where we define $K_T(t_1,t_2) = \mathbb{1}(t_1 \le T)K(t_1,t_2)$ and use an operator multiplication notation.

Thus the exponent Y_T appearing in (41) has the form of (38) with

$$\begin{aligned}
A_T &= \frac{1}{2}\int_0^T \mathrm{tr}\left[\gamma(t)\left(\int_0^T K_T^\dagger(s,t)K_T(s,t)ds\right)\gamma(t)\right]dt \\
& \frac{1}{2}\int_0^T \left[\tilde{R}_t^\dagger \tilde{R}_t + h_t^\dagger h_t - 2k_t^\dagger \tilde{R}_t\right]dt, \\
B_T(t) &= -h_t - \gamma(t)\int_0^T K_T^\dagger(s,t)k_s ds + \gamma(t)\int_0^T K_T^\dagger(s,t)\tilde{R}_s ds, \\
C_T(t_1,t_2) &= \gamma(t_1)\left(\int_0^T K_T^\dagger(s,t_1)K_T(s,t_2)ds\right)\gamma(t_2).
\end{aligned}$$

Now we note that $C_T(\cdot,\cdot)$ is the kernel of an integral operator which is manifestly positive and has Hilbert–Schmidt norm $\|C_T\|_{HS}^2 = \mathcal{O}(T)$. Therefore, we can use Proposition 4 for $E[e^{-Y_T}]$, leading to a general formula for the generating functional $Z(h,k)$

Differentiation once with respect to k then yields

$$
\begin{aligned}
Z_{\sigma_T}(h) \;&=\; M_T \exp\left[-\frac{1}{2}\int_0^T \left(\tilde{R}_t^\dagger \tilde{R}_t + h_t^\dagger h_t\right) dt\right] \\
&\quad \left[-\tilde{R} + K_T\gamma(1+C_T)^{-1}(h-\gamma K_T^\dagger \tilde{R})\right](T) \\
&\quad \exp\left[\frac{1}{2}\int_{\Delta_2}\left(h^\dagger - \tilde{R}^\dagger K_T\gamma\right)(1+C_T)^{-1}\left(h-\gamma K_T^\dagger \tilde{R}\right) dt_1 dt_2\right]
\end{aligned}
$$

where

$$
M_T = e^{-\frac{1}{2}\mathrm{tr}C_T}(\det{}_2(1+C_T))^{-1/2} = (\det(1+C_T))^{-1/2} \qquad (43)
$$

For the CIR model with $r_0 = 0$ we have $\tilde{R} = 0$, $\alpha(t) = a/2$ and $\gamma(t) = c/2$, so that $K_T(s,t) = e^{-a(s-t)/2}\mathbb{1}(t \leq s \leq T)$ and

$$C_T(t_1, t_2) = \frac{c^2}{4}\int_0^T K_T(s, t_1) K_T(s, t_2) ds = \frac{c^2}{4a} e^{\frac{a}{2}(t_1+t_2)}(e^{-at_2} - e^{-aT}), \tag{44}$$

for $0 \leq t_1 \leq t_2 \leq T$.

Moreover, the previous expression for $Z_{\sigma_T}(h)$ reduces to

$$\begin{aligned} Z_{\sigma_T}(h) \;\; = \;\; & M_T \left[K_T \gamma (1 + C_T)^{-1} h\right](T) \\ & \exp\left[-\frac{1}{2}\int_0^T h_t^\dagger h_t dt + \frac{1}{2}\int_{\Delta_2} h_{t_1}^\dagger (1 + C_T)^{-1}(t_1, t_2) h_{t_2} dt_1 dt_2\right]. \end{aligned}$$

Theorem 8 *The nth term of the chaos expansion of σ_T for the CIR model with $\bar{\lambda} = 0$ and initial condition $r_0 = 0$ is zero for n even. For n odd, the kernel of the expansion is the function $f_T^{(n)}(\cdot) : \Delta_n \to \mathbb{R}$*

$$f_T(t_1, \ldots, t_n) = M_T \sum_{G \in \mathcal{G}_n^*} \prod_{g \in G} L(g) \tag{45}$$

where

$$L(g) = \begin{cases} [C_T(1 + C_T)^{-1}](t_{g_1}, t_{g_2}) & t \notin g \\ \\ (K_T \gamma (1 + C_T)^{-1})(T, t_{g_2}) & t \in g \end{cases} \tag{46}$$

Here, $\mathcal{G}_n^*$ is the set of Feynman graphs, *each Feynman graph G being a partition of $\{t_1, \ldots, t_n, T\}$ into pairs $g = (t_{g_1}, t_{g_2})$.*

6. **Bond pricing formula**

In this section we give a derivation of the price of a zero coupon bond in the CIR model. Recall from section 2 that these are given by

$$P_{tT} = E_t[V_t^{-1} V_T]. \tag{47}$$

To mantain things as simple as possible we keep $\bar{\lambda} = 0$ so $V_t = \exp[-\int_0^T r_s ds]$, or in terms of the squared Gaussian formulation,

$$V_t = \exp\left[-\int_0^t R_s^\dagger R_s ds\right]. \tag{48}$$

But as we have seen in the previous section, for $t \leq s \leq T$

$$R_s^\mu = K_T(s,t)R_t^\mu + \frac{c}{2}\int_t^s K_T(s,s_1)dW_{s_1}^\mu. \tag{49}$$

Hence $-\log[V_t^{-1}V_T] = \sum_\mu \int_t^T (R_s^\mu)^2 ds$ can be written as

$$\sum_\mu \left[\frac{1}{2}(R_t^\mu)^2 C_T(t,t) + \frac{cR_t^\mu}{2}\int_t^T C_T(t,s)dW_s^\mu \right.$$
$$\left. + \int_t^T \int_t^{s_2} \frac{c^2}{4} C_T(s_1,s_2)dW_{s_1}^\mu dW_{s_2}^\mu \right] + N\int_t^T C_T(s,s)ds$$

where $C_T(s_1,s_2) = 2\int_0^T K_T(s,s_1)K_T(s,s_2)ds$

Taking the conditional expectation of $V_t^{-1} V_T$ by use of Proposition 4 leads to the desired formula

$$
\begin{aligned}
P_{tT} &= \left(\det(1 + \frac{c^2}{4} C_T)\right)^{-N/2} && (50) \\
& \prod_\mu \exp\left[-\frac{1}{2}(R_t^\mu)^2 \left(C_T(1 + \frac{c^2}{4} C_T)^{-1}\right)(t,t)\right] && (51)
\end{aligned}
$$

Thus P_{tT} has the exponential affine form $\exp[-\beta(t,T) r_t - \alpha(t,T)]$ with

$$
\begin{aligned}
\beta(t,T) &= \frac{1}{2}[C_T(1 + \frac{c^2}{4} C_T)^{-1}](t,t) \\
\alpha(t,T) &= \frac{N}{2} \log\left(\det(1 + \frac{c^2}{4} C_T)\right) && (52)
\end{aligned}
$$

The known formula has the same form, with

$$\begin{aligned}
\beta(t,T) &= \frac{2(e^{\rho(T-t)}-1)}{(\rho+a)(e^{\rho(T-t)}-1)+2\rho}, \qquad \rho^2 = a^2+2c^2 \\
\alpha(t,T) &= -\frac{2ab}{\rho^2}\log\left[\frac{2\rho e^{(a+\rho)(T-t)/2}}{(\rho+a)(e^{\rho(T-t)}-1)+2\rho}\right] \qquad (53)
\end{aligned}$$

which can be derived as solutions of the pair of Ricatti ordinary differential equations

$$\begin{aligned}
\frac{\partial\beta}{\partial t} &= \frac{c^2\beta^2}{2}+a\beta-1 \\
\frac{\partial\alpha}{\partial t} &= -ab\beta \qquad (54)
\end{aligned}$$

7. **Discussion**

- The CIR model, at least in integer dimensions, can be viewed within the chaos framework of Hughston and Rafailidis as arising from a random variable X_∞ derived from exponentiated second chaos random variables $e^{-Y}, Y \in \mathcal{C}^+$. Such exponentiated $\mathcal{C}^+$ variables form a rich and natural family which is likely to include many more candidates for applicable interest rate models.

- Although their analytic properties are complicated, there do exist approximation schemes which can in principle be the basis for numerical methods

- This family is invariant under conditional $\mathcal{F}_t$–expectations: $\log E_t[e^{-Y}] \in \mathcal{C}^+$ whenever $Y \in \mathcal{C}^+$.

- Illustration of the connection between methods developed for quantum field theory and the methods of Malliavin calculus.