Inquiring the Crisis: relationships between mathematical models and the financial world

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October 29, 2012

A brief history of financial math: infancy

- Bachelier's 1900 thesis *Théorie de la Spéculation*.
  - The examining committee consisted of Appell, Boussinesq and Poincaré (supervisor).
  - Obtain the distribution properties of what would be called a Wiener process W_t.
  - Introduced (arithmetic) Brownian motion to model prices in the *Bourse de Paris*.
  - Derived the price for a barrier option, using the *wrong* probability measure.
- Portfolio selection: Harry Markowitz (1952)
- CAPM: William Sharpe (1964)
- Brownian motion-driven models: Paul Samuelson (1965)
- Option pricing: Black–Scholes–Merton (1973)

Everyone's favourite example

- Consider a stock with initial price S_0 at time $t = 0$ and a bank account with interest rate r.
- At time $T = 1$ we assume that the price changes to

$$S_T = \begin{cases} uS_0 & \text{with probability } p \\ dS_0 & \text{with probability } 1 - p, \end{cases}$$

  for some $0 < d < e^r < u$.
- What is the fair price at time $t = 0$ of a call option with pay-off $(S_T - K)^+$?
- Bachelier's answer would be

$$c_0 = e^{-r}[p(uS_0 - K)^+ + (1 - p)(dS_0 - K)^+]$$

Replication

- Suppose that, at time $t = 0$ we buy a shares and invest b dollars in the bank.
- This portfolio *replicates* the call option at time $T = 1$ if and only if

$$\begin{aligned} a(uS_0) + be^r &= c^u := (uS_0 - K)^+ \\ a(dS_0) + be^r &= c^d := (dS^0 - K)^+ \end{aligned}$$

- The *Law of one price* then dictates that

$$c_0 = aS_0 + b = e^{-r}\left[qc^u + (1-q)c^d\right],$$

where $q = \frac{e^r - d}{u - d}$ defines the so called *risk-neutral probability*.

The Black–Scholes analysis

- Assume that the stock price follows a geometric Brownian motion
$$dS = \mu S dt + \sigma S dW$$
- Use Ito calculus to write down the differential for the value of a *derivative* on the stock, that is $f(S_t)$:
$$df = (\cdots)dt + (\cdots)dW$$
- Find Δ such that the combination $f + \Delta S$ has a differential with no dW term.
- Equate the drift term in the differential for $f + \Delta S$ to that of a risk-free bank account.
- This leads to the Black–Scholes partial differential equation.
- By Feynman-Kac, its solutions has the representation
$$f(t, s) = E^Q[e^{-r(T-t)} f(S_T)],$$
where, under Q, we have $dS = rSdt + \sigma S dW$.

A brief history of financial math: adolescence

- Risk–neutral pricing: Harrison–Pliska (1981)
- Utility maximization: Karatzas–Lehoczky–Shreve–Xu (1990)
- Modeling with jump processes: Madan–Seneta (1990)
- Interest rate modeling: Heath–Jarrow–Morton (1992)
- Stochastic volatility modeling: Heston (1993), Dupire (1994)
- FTAP: Delbaen–Schachermayer (1994)

The Fundamental Theorems of Asset Pricing

- An *arbitrage* is a portfolio with zero initial cost an strictly positive expected pay-off.
- A *martingale measure* is a measure Q such that $E_s^Q[e^{-rt}S_t] = e^{-rs}S_s$ for all $s \leq t$.
- A market is said to be *complete* if every contingent claim can be replicated.
- First FTAP: a market is arbitrage-free if and only if there exists an equivalent martingale measure.
- Second FTAP: an arbitrage-free market is complete if and only if the equivalent martingale measure is unique.
- Therefore, complete markets lead to a "no-economics principle".

Ressucitating economics

- Incompleteness is everywhere (jumps, defaults, market imperfections, information asymmetry).
- Absence of arbitrage alone leads to an arbitrage-free *interval* for the price of a derivative, often too wide to be of any practical significance.
- An additional criterion is needed to select a martingale measure (or equivalently, a price).
- One such criterion is given by *utility theory*.
- An agent's attitude towards risk is codified in terms of an increasing concave function U. To price a derivative B the agent tries to solve the stochastic control problem

$$\sup_H U(X_T - B),$$

where $X_T = \int_0^T H dS$ is the wealth produced by the portfolio H.

A brief history of financial math: adulthood

- *Financial Mathematics* at Isaac Newton Institute (1995)
- Bachelier Finance Society (1996)
- Explosion of Master's programs
- Creation of research groups in mathematics departments
- Creation of dedicated journals
- *Developments in Quantitative Finance* at Isaac Newton Institute (2005)
- *Quantitative Finance: foundations and applications* at the Fields Institute (2010)

Prediction is very hard, especially about the future...

From our 2005 Letter of Intent:

- "In the past decade, Mathematical Finance (MF) has outstripped its infancy...
- "In its current phase, MF has developed a vast range of applications, and has brought in many new mathematical, statistical and computational methods...
- "MF abounds with fresh new applications and new mathematics is constantly being introduced and developed. This trend will continue- over future decades we expect MF will impact many areas as yet undeveloped..."
- "...the general trend of securitization, of which collateralised debt obligations (CDO's) and general asset backed securities are particularly important examples, offers a new level of mathematical complexity guaranteed to keep MF vigorous, useful and fascinating well into its twilight years."

Chronology of the Crisis

- Default of subprime mortgages causes freezing of interbank lending (August 2007)
- Rescue of Bear Stearns (March 2008)
- Bailout of Fannie and Freddie (July 2008)
- Failure of Lehman Brothers causes credit crunch and stock crash (September 2008)
- Bailout of AIG (September 2008)
- Crisis spreads to Europe and emerging countries (October 2008)
- Beginning of the Great Contraction !

Causes of the Crisis

- Monetary Excess (abnormally low interest rates)
- Change in regulation (repeal of Glass-Steagal Act)
- Lax oversight (growth of shadow banking)
- Housing bubble
- **Financial Innovation**

Monetary policy

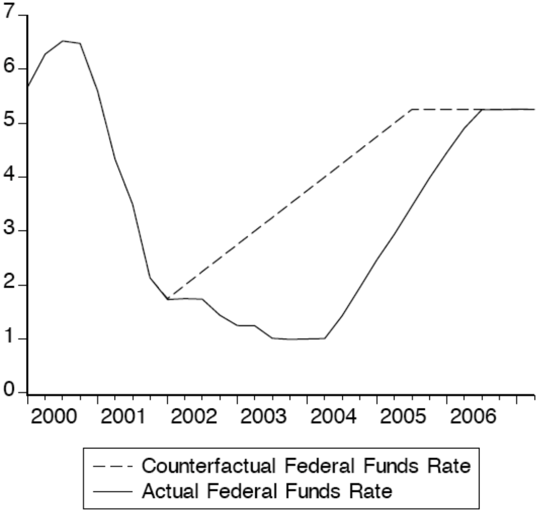


Figure: John Taylor, 2007

Shadow banking

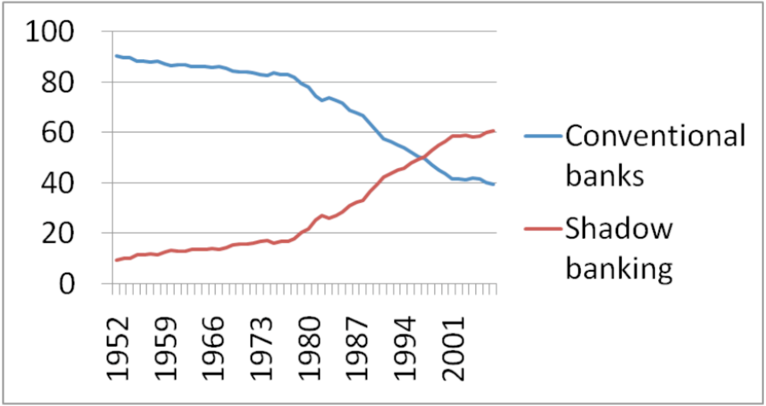
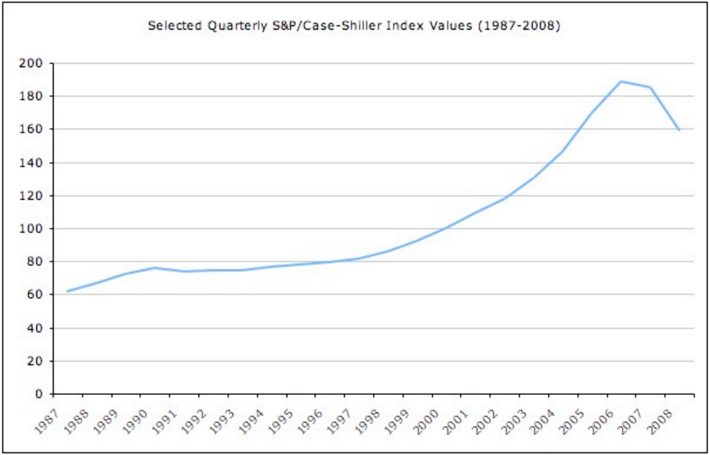


Figure: Paul Krugman, 2009

Housing Bubble



Financial Innovation

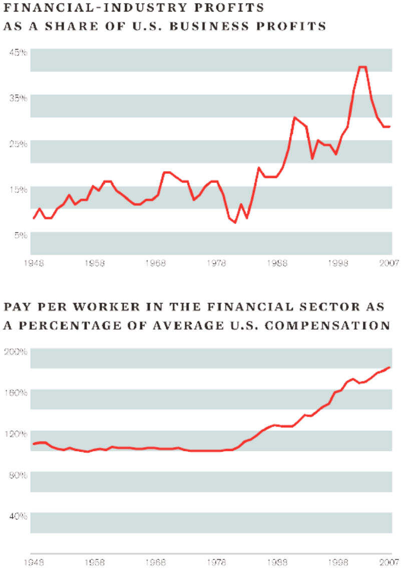


Figure: Johnson and Kwak , 2009

Theory of Crisis

- Mitchell (1913), Fisher (1933), Minsky (1977), Kindleberger (1978)
- Upswing driven by a displacement (new profitable opportunities to invest)
- Investment boom financed by bank money
- **Financial innovation** and leverage
- State of euphoria, bubble, over–indebtedness
- Dangerous timing game, bust
- Fire sale os assets, bankruptcies, bank failures
- Recession !

Modern twists

- Shifting risk from banks to shadow institutions increases the downside of a tail event.
- Over–reliance on superstructures (exchanges, rating agencies, mathematical models) makes the market less informationally diverse.
- Compensation for new intermediaries (fund managers, etc) is *convex* in returns, leading to riskier behavior.
- Relative performance evaluation induces herding, puts limits to arbitrage, leads to bubbles.
- Current risk management techniques (VaR) led to *pro-cyclicality* of leverage.
- The financial innovation at the core of the crisis (securitization) was overused without sufficient theoretical understanding.

WIRED MAGAZINE: ISSUE 17.03

THE SECRET FORMULA *That Destroyed Wall Street*

$$P=\phi(A,B,\gamma)$$

Targeting quants

- Warren Buffet: derivatives are financial weapons of mass destruction, carrying dangers that, while now latent, are potentially lethal.
- Lord Turner (chaiman of FSA): ...misplaced reliance on sophisticated maths to manage the risks
- Felix Salmon (Wired Magazine): And Li's Gaussian copula formula will go down in history as instrumental in causing the unfathomable losses that brought the world financial system to its knees.
- Paul Volcker: I will not accept the Nuremberg excuse.
- Nassim Taleb: we have to unmask the charlatans of risk like Myron Scholes. This guy should be in a retirement home doing Sudoku. His funds have blown up twice. He shouldn't be allowed in Washington to lecture anyone on risk.

Defending quants

- Steven Shreve: When a bridge collapses, no one demands the abolition of civil engineering. One first determines if faulty engineering or shoddy construction caused the collapse. If engineering is to blame, the solution is better–not less–engineering. Furthermore, it would be preposterous to replace the bridge with a slower, less efficient ferry rather than to rebuild the bridge and overcome the obstacle.
- Sir David Wallace (chair of CMS): Mathematics is surely the only medium capable of describing quantitatively the complex nature of the products that traders, risk managers, etc are handling, and the economic environment which they are operating in and influencing
- Carmona and Sircar: In fact, these geeks should have been listened to instead of being ignored or quarantined.

Let no crisis go to waste: new areas for financial math

- Limits of arbitrage
- Liquidity and leverage
- Accounting rules
- Incentive structures
- Improved risk management
- Systemic stability
- Market microstructure
- **Macroeconomics**

Dynamic Stochastic General Equilibrium

- Overwhelmingly dominant school in macroeconomics.
- Seeks to explain the aggregate economy using theories based on strong microeconomic foundations.
- All variables are assumed to be simultaneously in equilibrium.
- The only way the economy can be in disequilibrium at any point in time is through decisions based on wrong information.
- Money is neutral in its effect on real variables and only affects price levels.
- Largely ignores the role of irreducible uncertainty.

Hardcore (freshwater) DSGE

- The strand of DSGE economists affiliated with RBC theory made the following predictions after 2008:
  1. Increases government borrowing would lead to higher interest rates on government debt because of "crowding out".
  2. Increases in the money supply would lead to inflation.
  3. Fiscal stimulus has zero effect in an ideal world and negative effect in practice (because of decreased confidence).

Wrong prediction number 1

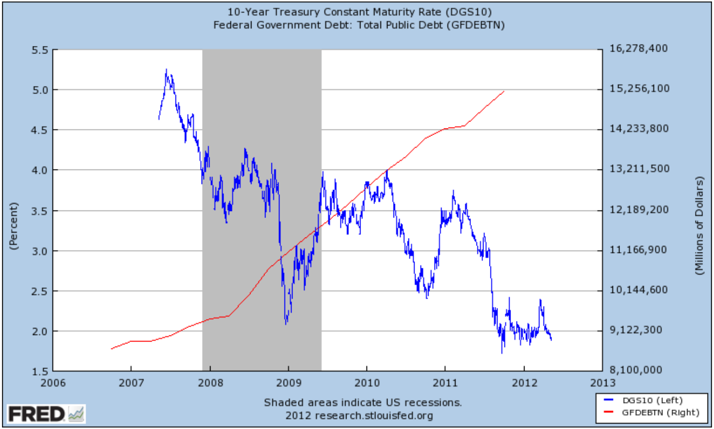


Figure: Government borrowing and interest rates.

Wrong prediction number 2

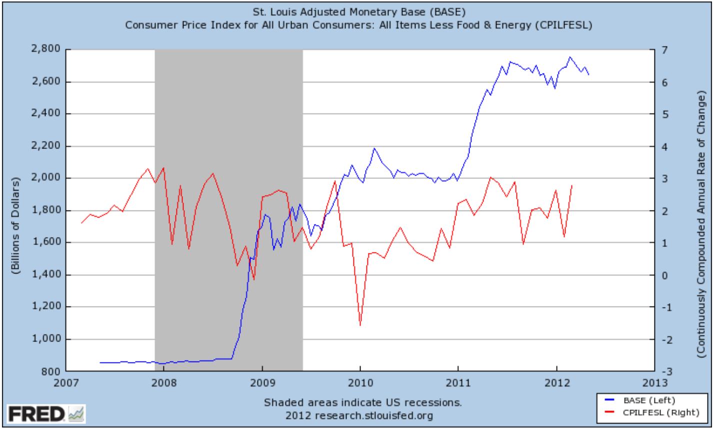


Figure: Monetary base and inflation.

Wrong prediction number 3

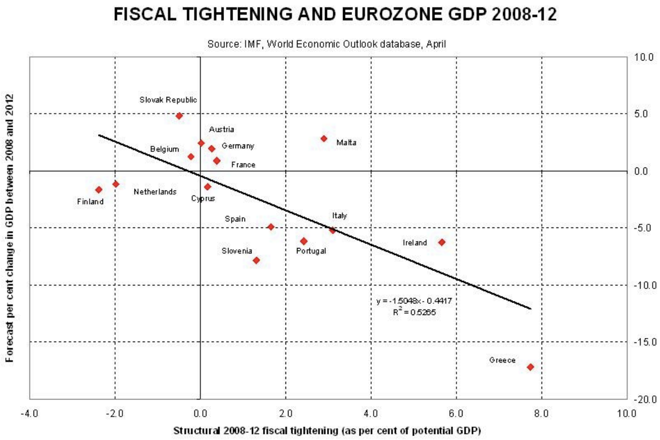


Figure: Fiscal tightening and GDP.

Soft core (saltwater) DSGE

- The strand of DSGE economists affiliated with New Keynesian theory got all these predictions right.
- They did so by augmented DSGE with 'imperfections' (wage stickiness, asymmetric information, imperfect competition, etc).
- Still DSGE at core - analogous to adding epicycles to Ptolemaic planetary system.
- For example: "Ignoring the foreign component, or looking at the world as a whole, the overall level of debt makes no difference to aggregate net worth – one person's liability is another person's asset." (Paul Krugman and Gauti B. Eggertsson, 2010, pp. 2-3)

Then we can safely ignore this...

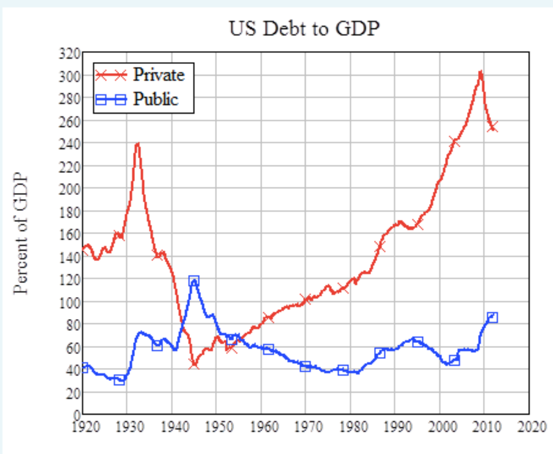


Figure: Private and public debt ratios.

Really?

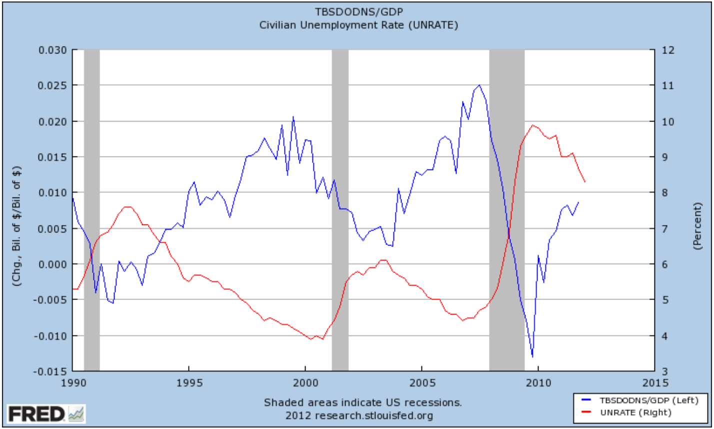


Figure: Change in debt and unemployment.

A different approach: the Goodwin model (1967)

- Assume that

$$\begin{aligned} N(t) &= N_0 e^{\beta t} && \text{(total labour force)} \\ a(t) &= a_0 e^{\alpha t} && \text{(productivity per worker)} \\ Y(t) &= \nu K(t) = a(t)L(t) && \text{(total yearly output)} \end{aligned}$$

  where K is the total stock of capital and L is the employed population.

- Assume further that

$$\begin{aligned} \dot{\mathrm{w}} &= \Phi(\lambda)\mathrm{w} && \text{(Phillips curve)} \\ \dot{K} &= (Y - \mathrm{w}L) - \delta K && \text{(Say's Law)} \end{aligned}$$

Differential Equations

- Define

$$\omega = \frac{\mathrm{w}L}{Y} = \frac{\mathrm{w}}{a} \quad \text{(wage share)}$$
$$\lambda = \frac{L}{N} = \frac{Y}{aN} \quad \text{(employment rate)}$$

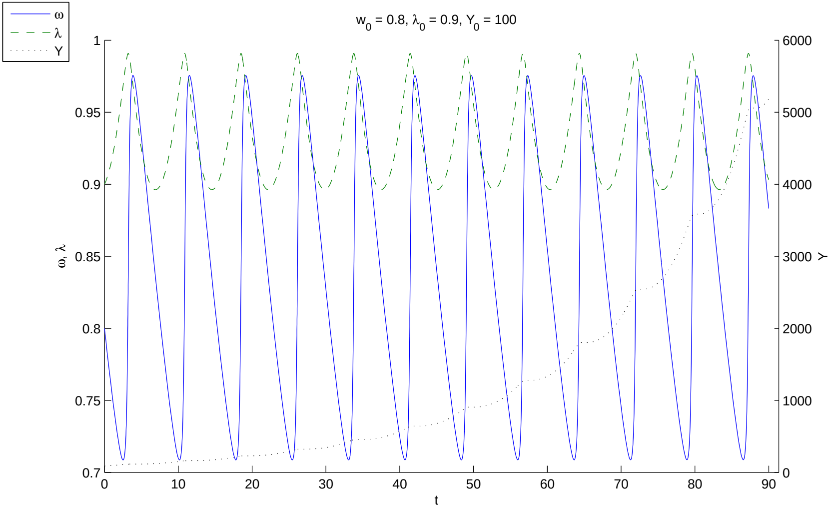
- It then follows that

$$\dot{\omega} = \omega(\Phi(\lambda) - \alpha)$$
$$\dot{\lambda} = \lambda\left(\frac{1-\omega}{\nu} - \alpha - \beta - \delta\right)$$

- This system is globally stable and leads to endogenous cycles of employment.

Example 1: basic Goodwin model

Example 1 (continued): basic Goodwin model



Introducing a financial sector (Keen 1995)

- Assume now that new investment is given by

$$\dot{K} = \kappa(1 - \omega - rd)Y - \delta K$$

where $\kappa(\cdot)$ is $C^1(-\infty, \infty)$ increasing function satisfying certain technical conditions.

- Accordingly, total output evolves as

$$\frac{\dot{Y}}{Y} = \frac{\kappa(1 - \omega - rd)}{\nu} - \delta := g(\omega, d)$$

- This leads to external financing through debt evolving according to

$$\dot{D} = \kappa(1 - \omega - rd)Y - (1 - \omega - rd)Y$$

Keen model - Differential Equations

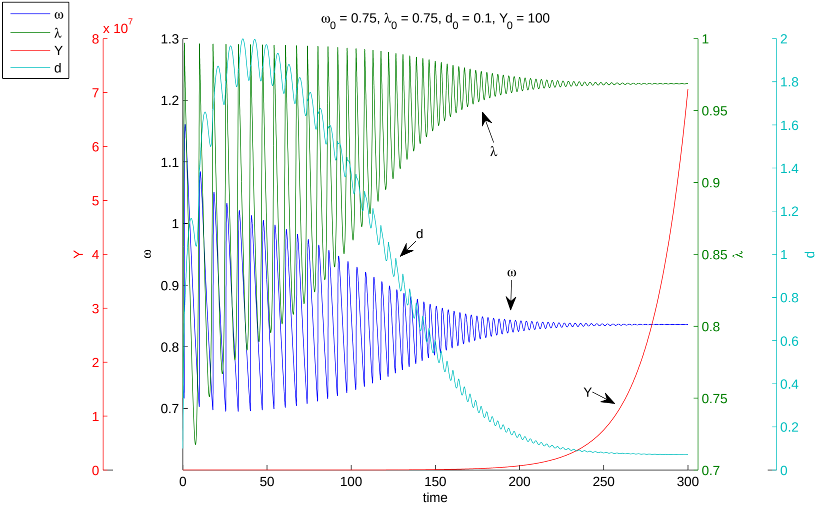
- Denote the debt ratio in the economy by $d = D/Y$, the model can now be described by the following system

$$
\begin{aligned}
\dot{\omega} &= \omega\left[\Phi(\lambda) - \alpha\right] \\
\dot{\lambda} &= \lambda\left[\frac{\kappa(1-\omega-rd)}{\nu} - \alpha - \beta - \delta\right] \\
\dot{d} &= d\left[r - \frac{\kappa(1-\omega-rd)}{\nu} + \delta\right] + \kappa(1-\omega-rd) - (1-\omega)
\end{aligned}
\tag{1}
$$

- This system has a *good* equilibrium with finite debt and a *bad* equilibrium with infinite debt.
- Both equilibria are locally stable.

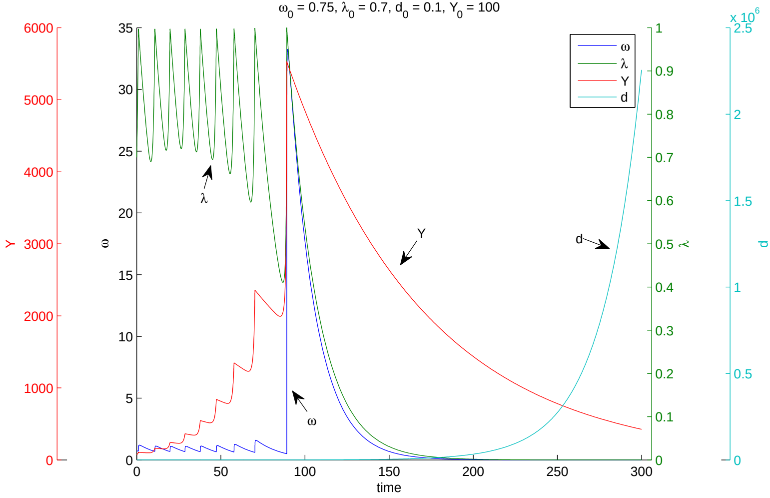
Example 2: convergent Keen model

Example 2 (continued): convergent Keen model

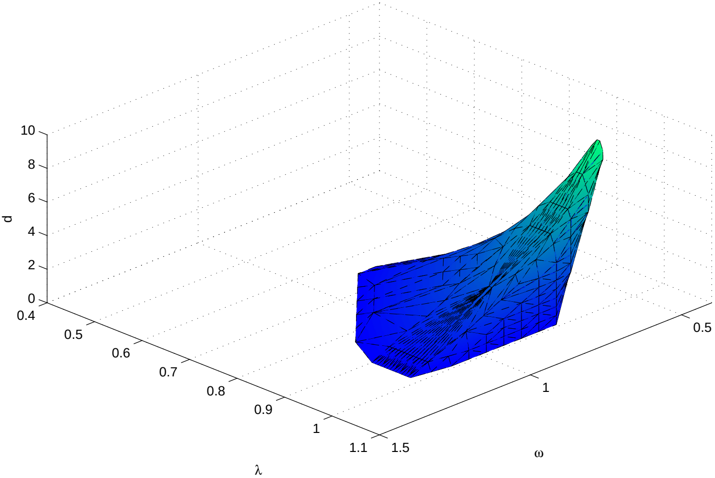


Example 3: divergent Keen model

Example 3 (continued): divergent Keen model

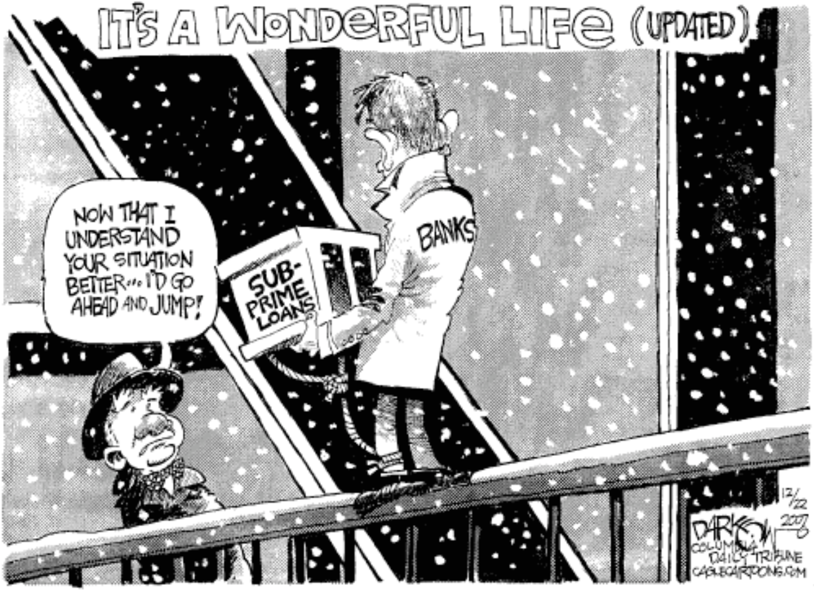


Basin of convergence for Keen model



Extensions of the Keen model

- Ponzi financing: destabilizes the good equilibrium
- Government spending: destabilizes the bad equilibrium
- Prices for capital goods, commodities, etc: different inflation regimes.
- Time–dependent interest rates: monetary policy
- Asset price models with jumps: links between real economy and finance
- These topics represent a substantial broadening of the scope of mathematical finance.
- By incorporating them, its relevance to real problems can be increased.
- Hopefully society won't pass the following judgment on us: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