Applications of utility-based pricing to stochastic volatility and real options models

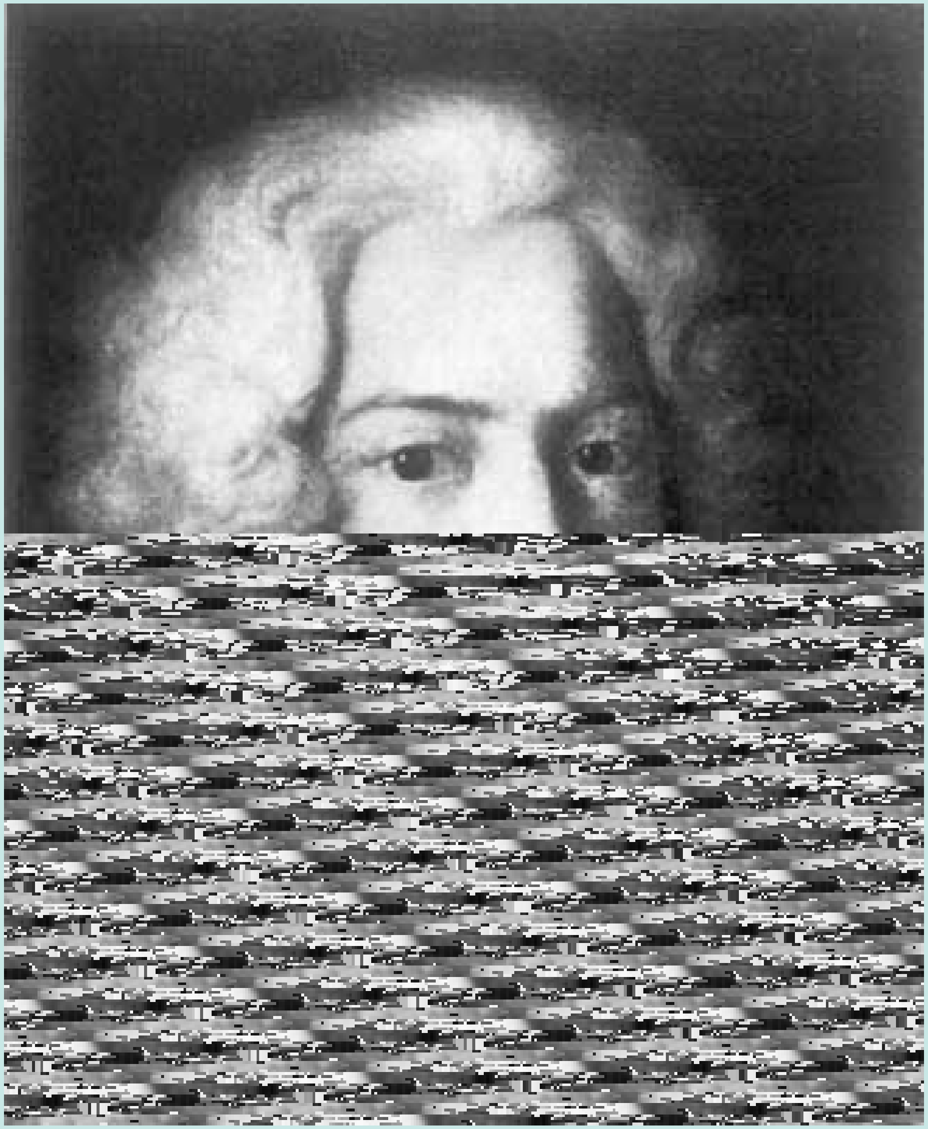
M. R. Grasselli
Dept. of Mathematics and Statistics
McMaster University

Heriot-Watt University
June 10, 2005

"Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats.

If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble (...) the determination of the value of an item must not be based on its price, but rather on the utility it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man (...)"

"Another rule which may prove useful can be derived from our theory. This is the rule that it is advisable to divide goods which are exposed to some danger into several portions than to risk them all together."



Daniel Bernoulli (1738)

1. Introduction

Market Model: We consider two-factor models of the form

$$\begin{aligned} dS_t &= S_t(\mu - r)dt + S_t \sigma dW_t \\ dY_t &= a dt + b[\rho dW_t + \sqrt{1-\rho^2} dZ_t] \\ dV_t &= Y_t dt \end{aligned} \qquad (1)$$

for a deterministic functions μ, σ, a, b and independent one dimensional P–Brownian motions (W_t, Z_t) and a constant correlation coefficient ρ.

If $\sigma = \sigma(t, Y_t)$ the model is interpreted as a stochastic volatility one. When μ and σ are independent of Y_t we interpret it as a model for a non-traded asset Y_t correlated with the (discounted) trade asset S_t.

Optimal hedging portfolio: the strategy followed by an investor who, when faced with a (discounted) financial liability B maturing at a future time T, tries to solve the stochastic control problem

$$u(x) = \sup_{H \in \mathcal{A}} E\left[U\left(X_T - B\right) | X_0 = x\right], \tag{2}$$

where X_T is the (discounted) terminal wealth obtained by holding H_t units of the risky asset S_t.

Utility function: $U(x) = -e^{-\gamma x}$, where $\gamma > 0$ is the risk aversion parameter.

Admissible strategies: In addition to the self–financing condition, we restrict the class $\mathcal{A}$ of admissible portfolios to

$$\mathcal{A} = \{H \in L(S) : (H \cdot S)_t \text{ is a Q–martingale for all Q } \in \mathcal{M}_f\}.$$

Claims: Finally, the (discounted) liability B is assumed to be a random variable of the form $B = B(S_T, Y_T, V_T)$, satisfying

$$E[e^{(\gamma+\iota)B}] < \infty \qquad \text{and} \qquad E[e^{-\iota B}] < \infty \qquad \text{for some } \iota > 0$$

2. Utility based pricing

Under these conditions, it follows from convex duality (Becherer 2004; Delbaen *et al* 2002; Kabanov and Stricker 2002; Owen 2002) that the optimal hedging problem (2) has a unique solution $H^B \in \mathcal{A}$ satisfying

$$U'(X_T^B - B) = \xi \frac{dQ^B}{dP}, \tag{3}$$

where $\xi = u'(x)$ and $Q^B \in \mathcal{M}^f \cap \mathcal{M}^e$ is the unique maximizer of the corresponding dual problem

$$\sup_{Q \in \mathcal{M}^f} E^Q \left[\gamma B - \log \left(\frac{dQ}{dP} \right) \right]. \tag{4}$$

Let H^B be the unique solution to (2) and define the exponential certainty equivalent for the claim B at time t as the unique semimartingale c_t^B satisfying

$$U(X_t - c_t^B) = E\left[U\left(X_t + \int_t^T H^B dS - B\right)\middle|\mathcal{F}_t\right]. \tag{5}$$

In other words,

$$c_t^B = \frac{1}{\gamma}\log E\left[\exp\left(-\gamma\int_t^T H^B dS + \gamma B\right)\middle|\mathcal{F}_t\right]. \tag{6}$$

The (selling) indifference price for the claim B is defined to be the premium that makes the agent indifferent between selling it or not, that is, the unique solution π_t^B to the equation

$$\sup_{H\in\mathcal{A}} E\left[U\left(X_t + \int_0^T H dS\right)\right] = \sup_{H\in\mathcal{A}} E\left[U\left(X_t + \pi_t^B + \int_0^T H dS - B\right)\right].$$

We see that this equation is equivalent to

$$U(X_t - c_t^0) = U(X_t + \pi_t^B - c_t^B),$$

so that the indifference price process is given by

$$\pi_t^B = c_t^B - c_t^0. \tag{7}$$

Pricing by marginal utility: Consider the indifference price $\pi^{\varepsilon B}$ for the claim εB. By differentiating the identity

$$U(x - c_0^{\varepsilon B}) = E\left[U\left(x + \int_0^T H^{\varepsilon B} dS - \varepsilon B\right)\right]$$

at $\varepsilon = 0$ we obtain Davis's formula

$$\left.\frac{dc_0^{\varepsilon B}}{d\varepsilon}\right|_{\varepsilon=0} = E_t^{Q^0}[B].$$

Therefore, to first order in ε, the exponential indifference price for εB can be obtained by taking the expectation of the claim B with respect to the optimal measure Q^0 which solves the dual to Merton's problem. From (4), it is clear that this is the (local) martingale measure with minimal relative entropy with respect to P.

3. Two-factor Markovian Markets

The market price of volatility risk

Consider the density

$$\Lambda_t^B := E\left[\left.\frac{dQ^B}{dP}\right| \mathcal{F}_t\right], \tag{8}$$

which, being an exponential martingale, can be expressed as

$$\frac{d\Lambda_t^B}{\Lambda_t^B} = -[\lambda_t dW_t + \nu_t^B dZ_t], \tag{9}$$

with $\lambda_t := \frac{\mu - r}{\sigma}$.

The process ν_t^B is then defined as the utility based market price of risk associated with the claim B.

Proposition 1 *The utility based market price of risk for B is given by*

$$\nu_t^B = -\gamma b \partial_y c^B \sqrt{1-\rho^2}, \tag{10}$$

and the unique optimizer $H_t^B = h^B(t, S_t, Y_t)$ for the hedging problem (2) can be expressed as

$$h^B(t,s,y) = \partial_s c_t^B + \frac{b\rho}{s\sigma}\partial_y c_t^B + \frac{(\mu - r)}{\gamma s \sigma^2}. \tag{11}$$

In particular

$$\nu_t^0 = -\gamma b \partial_y c^0 \sqrt{1-\rho^2}. \tag{12}$$

Corollary 2 *The certainty equivalent process* $c_t^B = c^B(t, S_t, Y_t, V_t)$ *satisfies*

$$
\begin{aligned}
c_t^B \; + \; & \left[a - \frac{b\rho(\mu - r)}{\sigma}\right] c_y^B + y c_v^B + \frac{1}{2}\left(s^2\sigma^2 c_{ss}^B + 2s\sigma\rho c_{sy}^B + b^2 c_{yy}^B\right) \\
& - \frac{(\mu - r)^2}{2\gamma\sigma^2} + \frac{\gamma(1-\rho^2)}{2} b^2 (c_y^B)^2 = 0 \qquad (13)
\end{aligned}
$$

with terminal condition $c^B(T, s, y, v) = B(s, y, v)$.

Residual risk

Musiela and Zariphopoulou (2004) define the difference

$$(X_t^B - X_t^0) - \pi_t^B$$

as the residual risk associated with the claim B.

Proposition 3 *The process* $e^{-\gamma(X_t^B - X_t^0 - \pi_t^B)}$ *is an exponential martingale under the optimal measure* Q^0 *obtained from the solutions of Merton's problem.*

Pay-off decomposition

For the case $B = B(Y_T)$ (μ and σ constant), MZ (2004) also show that the final payoff can be written as the sum of three terms: the indifference price, the wealth obtained by trading according to the optimal hedging portfolio and a term corresponding to the unhedgeable risk associated with the process Y_t. Here is a generalized result:

Proposition 4 *The payoff $B = B(S_T, Y_T, V_T)$ admits the decomposition*

$$\begin{aligned} B(S_T, Y_T, V_T) &= \pi_t^B + \int_t^T \left(S_u \partial_s \pi_u^B + \frac{b\rho}{\sigma} \partial_y \pi_u^B \right) \frac{dS_u}{S_u} \\ &\quad + \sqrt{1-\rho^2} \int_t^T b\, \partial_y \pi_u^B dZ_u^0 - \frac{\gamma}{2}(1-\rho^2) \int_t^T b^2 (\partial_y \pi_u^B)^2 du \end{aligned} \tag{14}$$

where $dZ_t^0 = dZ_t + \nu_t^0 dt$ defines a Brownian motion under Q^0.

3. Volatility claims

Proposition 5 *Consider a claim of the form $B = B(Y_T, V_T)$ and assume that λ_t is an adapted process for which the minimal martingale measure*

$$\frac{d\widetilde{Q}}{dP} = \exp\left(-\int_0^T \frac{\lambda_s^2}{2} ds - \int_0^T \lambda_s dW_s\right) \tag{15}$$

is well defined. Then the process

$$\Xi_t = e^{\gamma(1-\rho^2)c_t^B} e^{-\int_0^t \frac{(1-\rho^2)}{2}\lambda_s^2 ds} \tag{16}$$

is a local $\widetilde{Q}$–martingale. If moreover ν^B is the weak solution of an SDE, then Ξ_t is a true martingale.

Corollary 6 *The indifference price for volatility claims* $B = B(Y_T, V_T)$ *can be written as*

$$\pi_t^B = \frac{1}{\gamma(1-\rho^2)} \log \left(\frac{E_t^{\widetilde{Q}} \left[e^{\gamma(1-\rho^2) B(Y_T, V_T)} e^{-\int_t^T \frac{(1-\rho^2)}{2} \lambda_s^2 ds} \right]}{E^{\widetilde{Q}} \left[e^{-\int_t^T \frac{(1-\rho^2)}{2} \lambda_s^2 ds} \right]} \right) \quad (17)$$

4. Reciprocal affine models

Let us first consider the case where $B = B(Y_T)$. We take

$$\sigma(t, Y_t) = \sqrt{\frac{(1-\rho^2)(\mu - r)^2}{2}} \frac{1}{Y_t + \varepsilon}, \tag{18}$$

with $\varepsilon > 0$. Then the denominator and the numerator in (17) are, respectively, the formal equivalent of the price of a zero coupon bond and an interest rate derivative under the measure $\tilde{Q}$ with a "risk-free rate"

$$\frac{(1-\rho^2)}{2} \lambda_t^2 = Y_t + \varepsilon. \tag{19}$$

We write the dynamics for Y_t under the measure $\tilde{Q}$ as

$$dY_t = \tilde{\alpha}(\tilde{\kappa} - Y_t)dt + \tilde{\beta}\sqrt{Y_t}\left[\rho d\widetilde{W}_t + \sqrt{1-\rho^2}dZ_t\right], \qquad (20)$$

for constants $\tilde{\alpha}, \tilde{\kappa}, \tilde{\beta} > 0$ satisfying $4\tilde{\alpha}\tilde{\kappa} > \tilde{\beta}^2$. We obtain that the coefficients for the dynamics of Y_t under the economic measure P are

$$a(t, Y_t) = \tilde{\alpha}(\tilde{\kappa} - Y_t) + \tilde{\beta}\rho\sqrt{\frac{2(\varepsilon + Y_t)Y_t}{1-\rho^2}} \qquad (21)$$

$$b(t, Y_t) = \tilde{\beta}\sqrt{Y_t} \qquad (22)$$

Pricing and hedging formulas

Proposition 7 *In the reciprocal CIR model, the indifference price of $B = B(Y_T)$ is*

$$\frac{1}{\gamma(1-\rho^2)} \log \left\{ \frac{\int_{-\infty}^{\infty} \exp\left[(M(u,t,T) + N(u,t,T)y\right] \widehat{g}(u) du}{\exp(M(0,t,T) + N(0,t,T)y)} \right\}, \quad (23)$$

where $\widehat{g}$ denotes the Fourier transform of $g(y) = e^{\gamma(1-\rho^2)B(y)}$ and

$$N(u,t,T) = \frac{(b_2+iu)b_1 - (b_1+iu)b_2 e^{\Delta(t-T)}}{(b_2+iu) - (b_1+iu)e^{\Delta(t-T)}}, \quad (24)$$

$$M(u,t,T) = \frac{-2\alpha\kappa}{\beta^2} \log\left(\frac{b_2+iu}{b_2-N}\right) + \alpha\kappa b_1(t-T), \quad (25)$$

with $b_2 > b_1$ being the two roots of $x^2 - \frac{2\widetilde{\alpha}}{\tilde{\beta}^2}x - \frac{2}{\tilde{\beta}^2}$ and $\Delta = \sqrt{\tilde{\alpha}^2 + 2\tilde{\beta}^2}$.

General claims and the market price of risk

For claims written on the trade asset S_t, we can still use the reciprocal CIR to compute the marginal utility prices and the market price of risk ν_t^0 associated to this pricing scheme. Observe

$$c^0(t,y) = \frac{(t-T)\varepsilon}{\gamma(1-\rho^2)} + \frac{1}{\gamma(1-\rho^2)} \log \Psi(0,y,t,T), \tag{26}$$

so that,

$$\nu^0(t,y) = \frac{2\tilde{\beta}\sqrt{y}(1-e^{\Delta(t-T)})}{\Delta\sqrt{1-\rho^2}(1+e^{\Delta(t-T)})}.$$

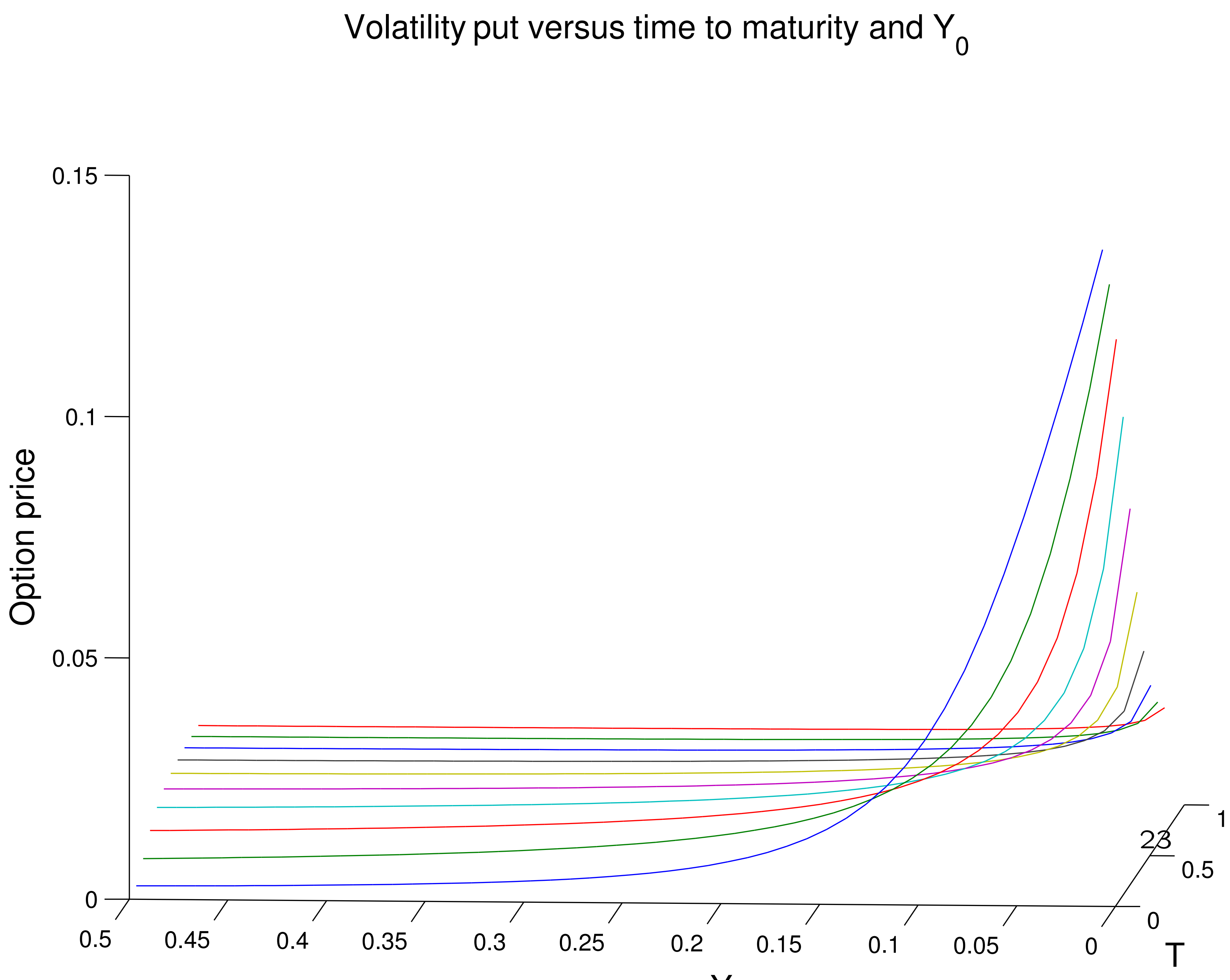
From this, we can calculate the density for the Merton measure Q^0 and use the result to obtain the marginal utility price for general claims $B = B(S_T, Y_T, V_T)$ as $E_t^{Q^0}[B]$.

Numerical results

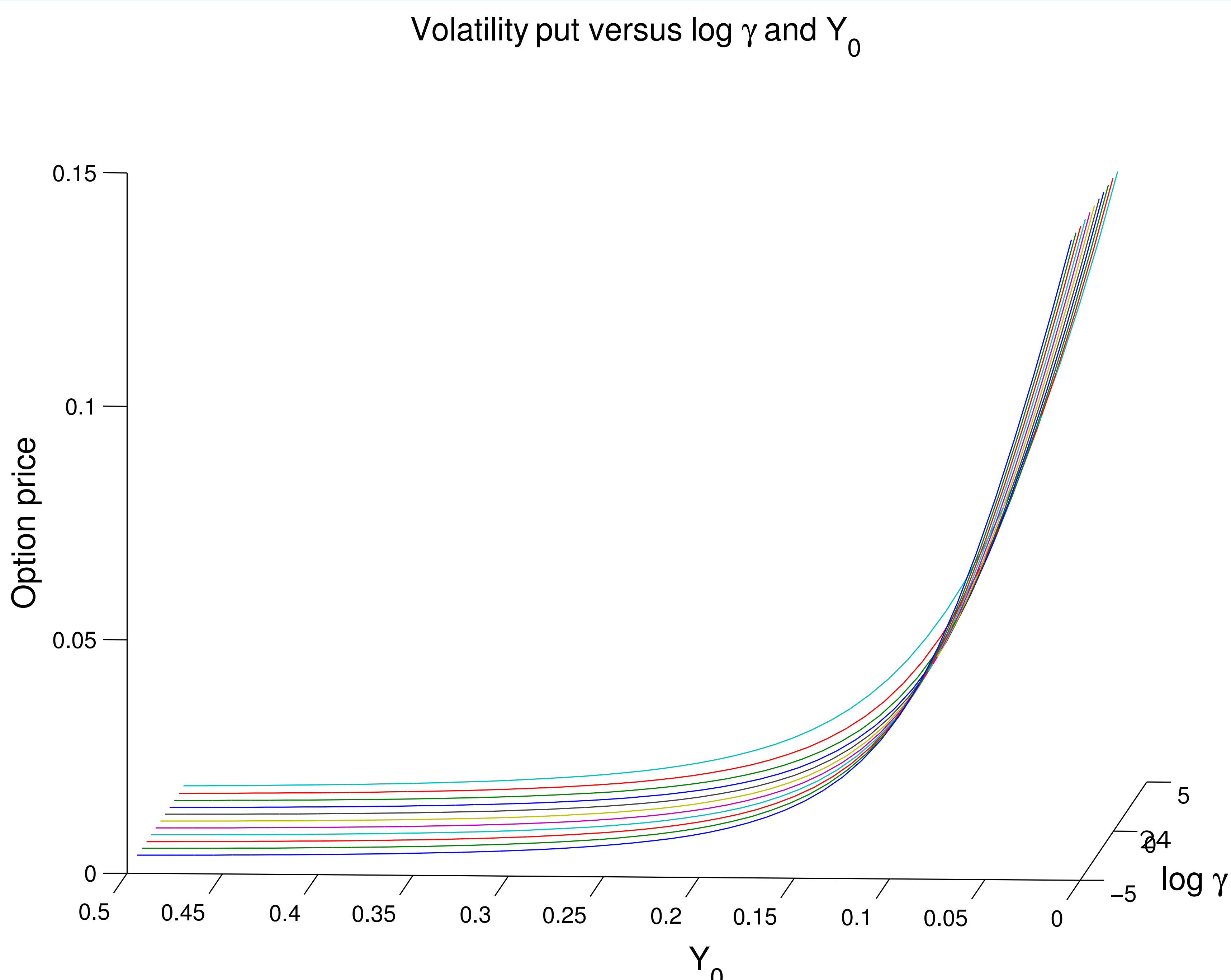
We illustrate the range of possibilities for model parameters fixed at reasonable values:

$$\begin{aligned} \alpha &= 5, \quad \beta = 0.04, \quad \kappa = 0.001, \\ \mu &= 0.04, \quad r = 0.02, \quad \rho = 0.5 \end{aligned}$$

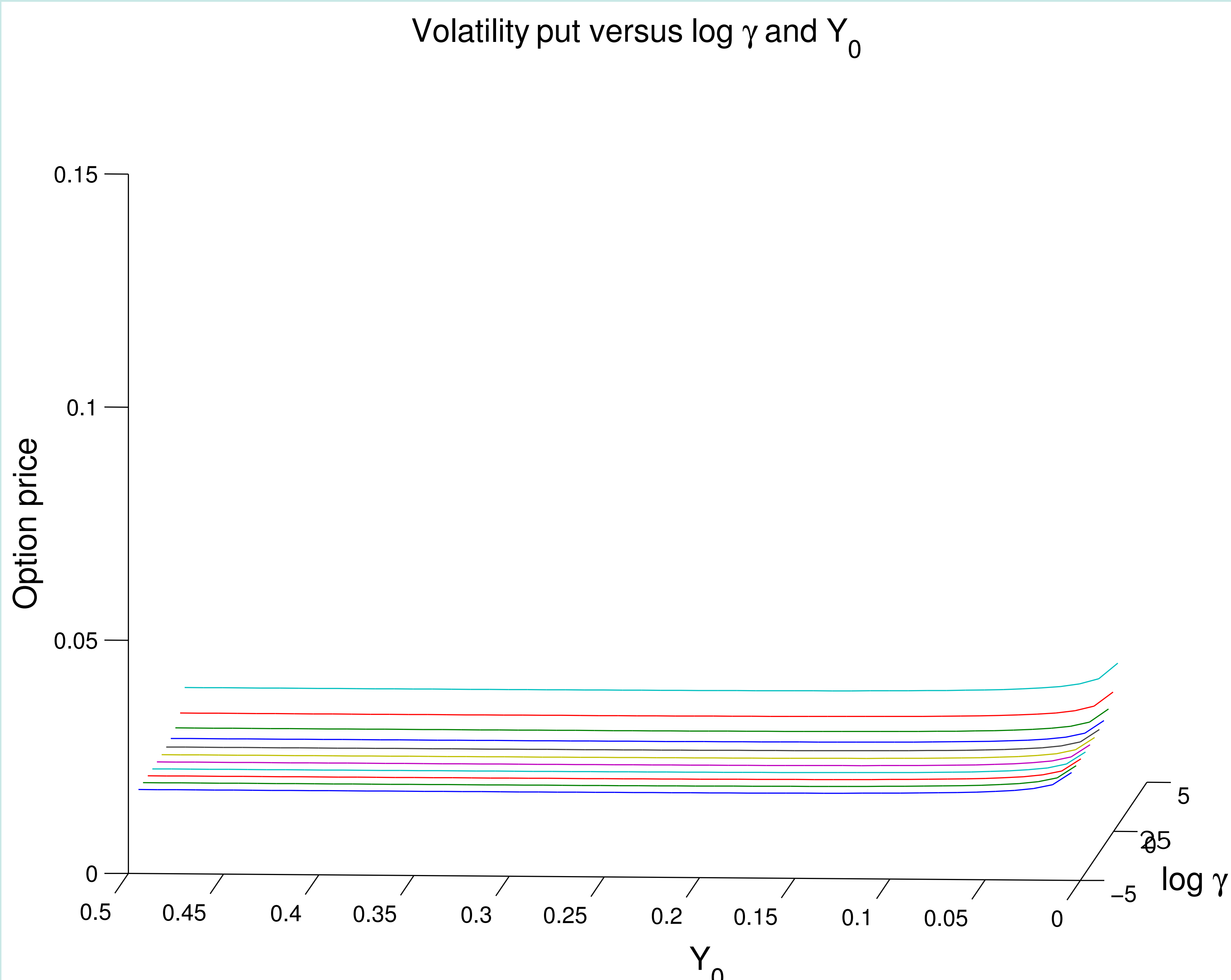
and initial squared volatility ranging in the interval $[0, 0.5]$. With these parameters the volatility process has a mean reversion time of approximately two months and an equilibrium distribution with expected value approximately 40%. We calculate the price of a put option on volatility with payoff $(0.15 - \sigma_T^2)^+$. When not mentioned the risk aversion parameter is set to $\gamma = 1$.

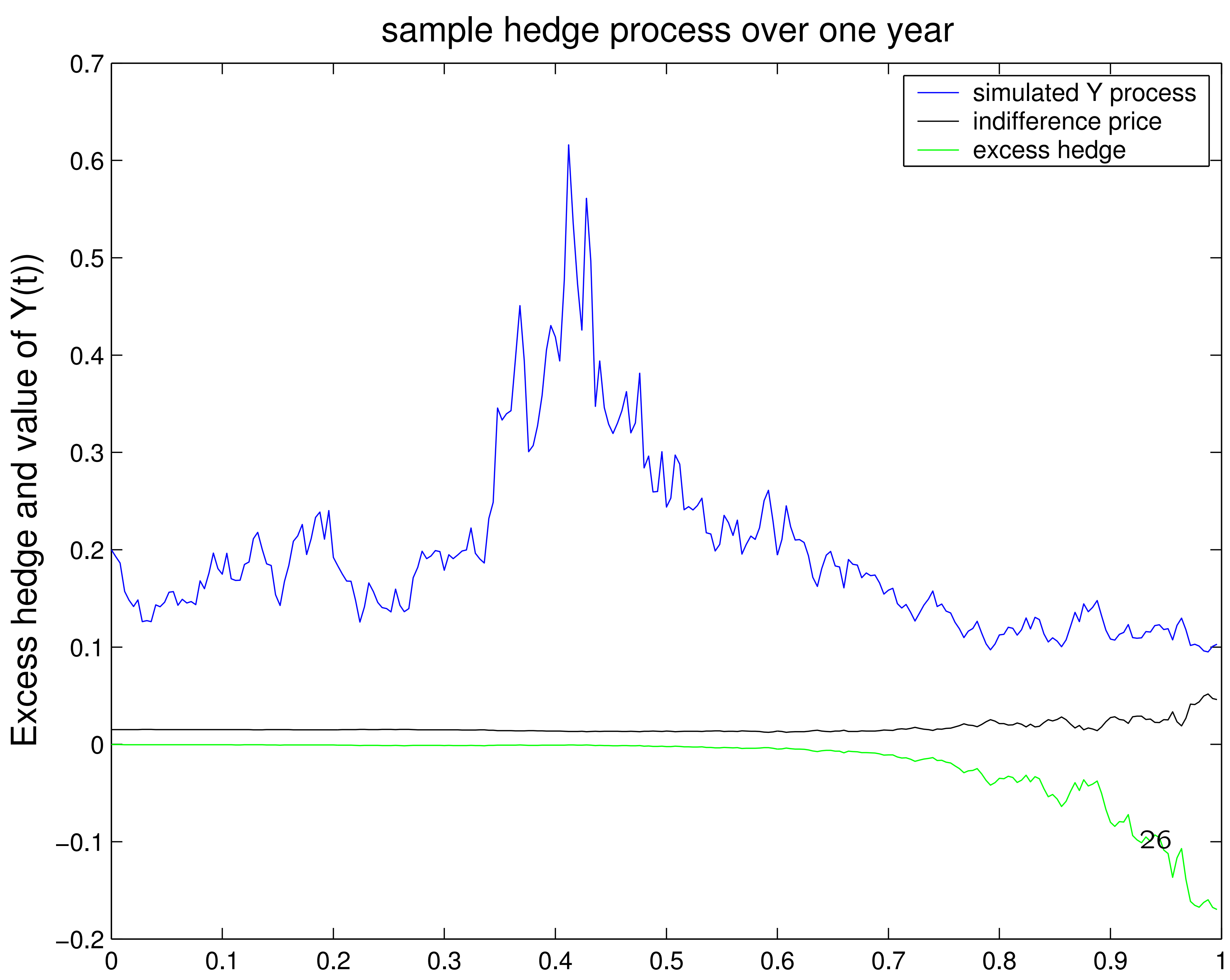


Volatility put versus time to maturity and Y_0
Option price
0.15
0.1
0.05
0
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
T
1
0.5
0
23



Volatility put versus log γ and Y_0
Option price
0.15
0.1
0.05
0
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Y_0
5
4
−5
log γ


Volatility put versus log γ and Y0
Option price
0.15
0.1
0.05
0
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
Y0
5
0
−5
log γ


sample hedge process over one year
simulated Y process
indifference price
excess hedge
Excess hedge and value of Y(t))
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

5. Claims on Integrated Volatility

Suppose now that $B = B(V_T)$. We now take $\sigma = \sqrt{Y_t}$, so that $\lambda_t = \frac{\mu - r}{\sqrt{Y_t}}$. As we have seen, the indifference price is given by

$$\pi_t^B = \frac{1}{\gamma(1-\rho^2)} \log \left(\frac{E_t^{\widetilde{Q}}\left[e^{\gamma(1-\rho^2)B(V_T)} e^{-\int_t^T \frac{(1-\rho^2)(\mu-r)^2}{2Y_s} ds} \right]}{E^{\widetilde{Q}}\left[e^{-\int_t^T \frac{(1-\rho^2)(\mu-r)^2}{2Y_s} ds} \right]} \right),$$

which we can calculate by simulating the processes Y_t and V_t under the minimal martingale measure $\widetilde{Q}$.

Let us adopt the Heston model for stochastic volatility, that is, under the physical measure P we take

$$\begin{aligned} dS_t &= S_t(\mu - r)dt + S_t\sqrt{Y_t}dW_t \\ dY_t &= \alpha(\kappa - Y_t) + \beta\sqrt{Y_t}[\rho dW + \sqrt{1-\rho^2}dZ] \\ dV_t &= Y_t dt \end{aligned}$$

Therefore, under the minimal martingale measure $\tilde{Q}$ we obtain

$$\begin{aligned} dS_t &= S_t\sqrt{Y_t}d\widetilde{W}_t \\ dY_t &= \alpha\left[\left(\kappa - \frac{\beta\rho(\mu - r)}{\alpha}\right) - Y_t\right] + \beta\sqrt{Y_t}[\rho d\widetilde{W} + \sqrt{1-\rho^2}dZ] \\ dV_t &= Y_t dt \end{aligned}$$

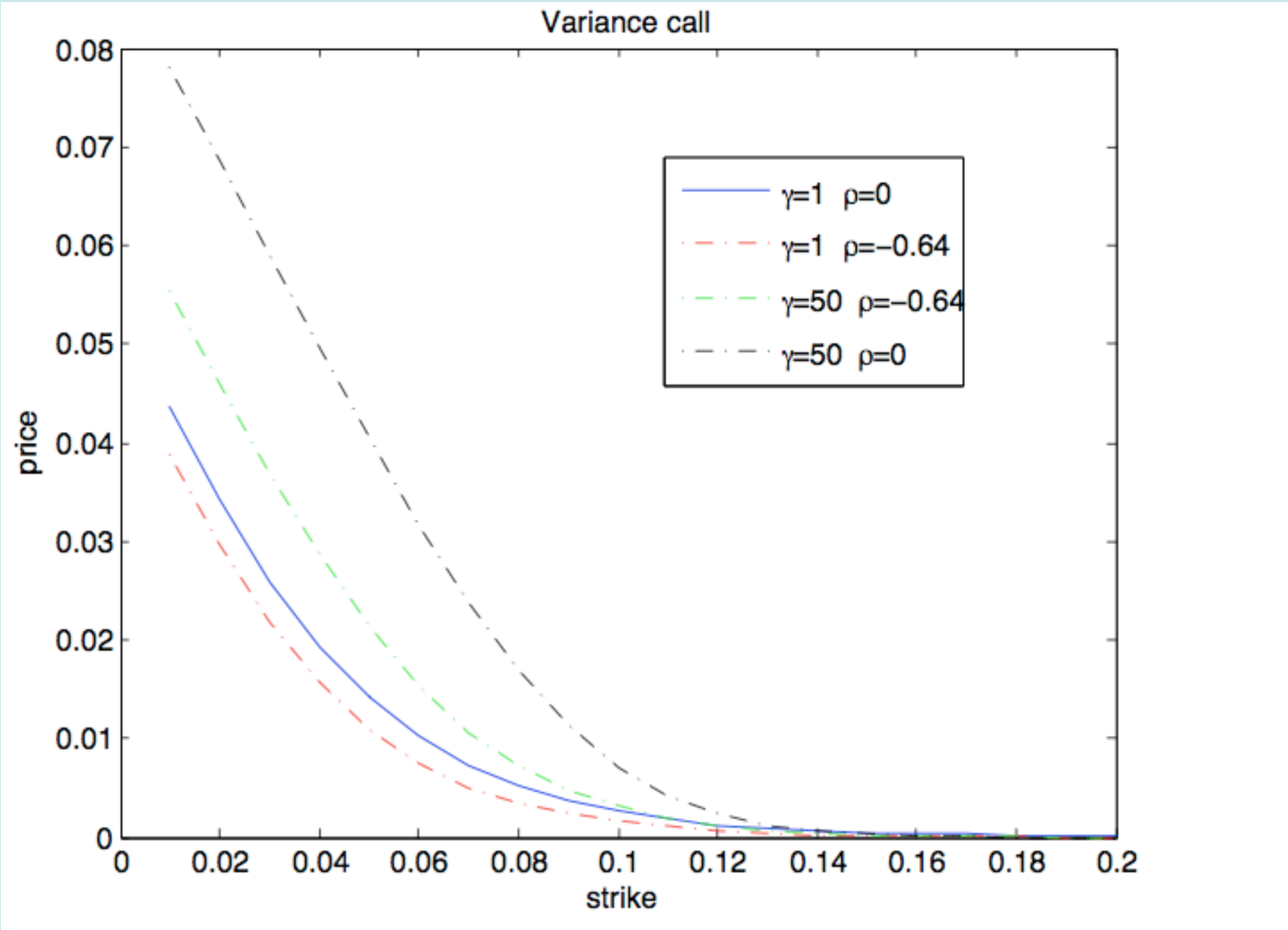
Numerical results

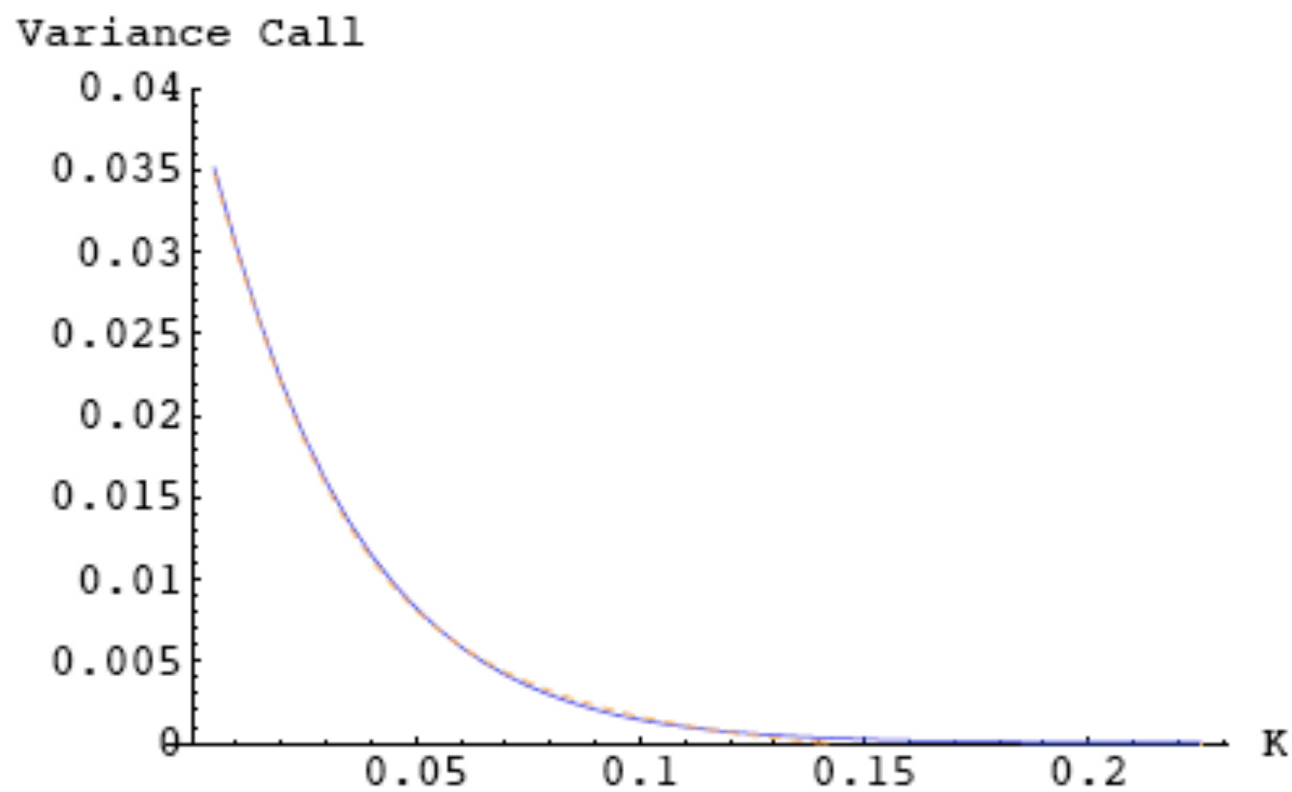
We compute the indifference price for the following claims:

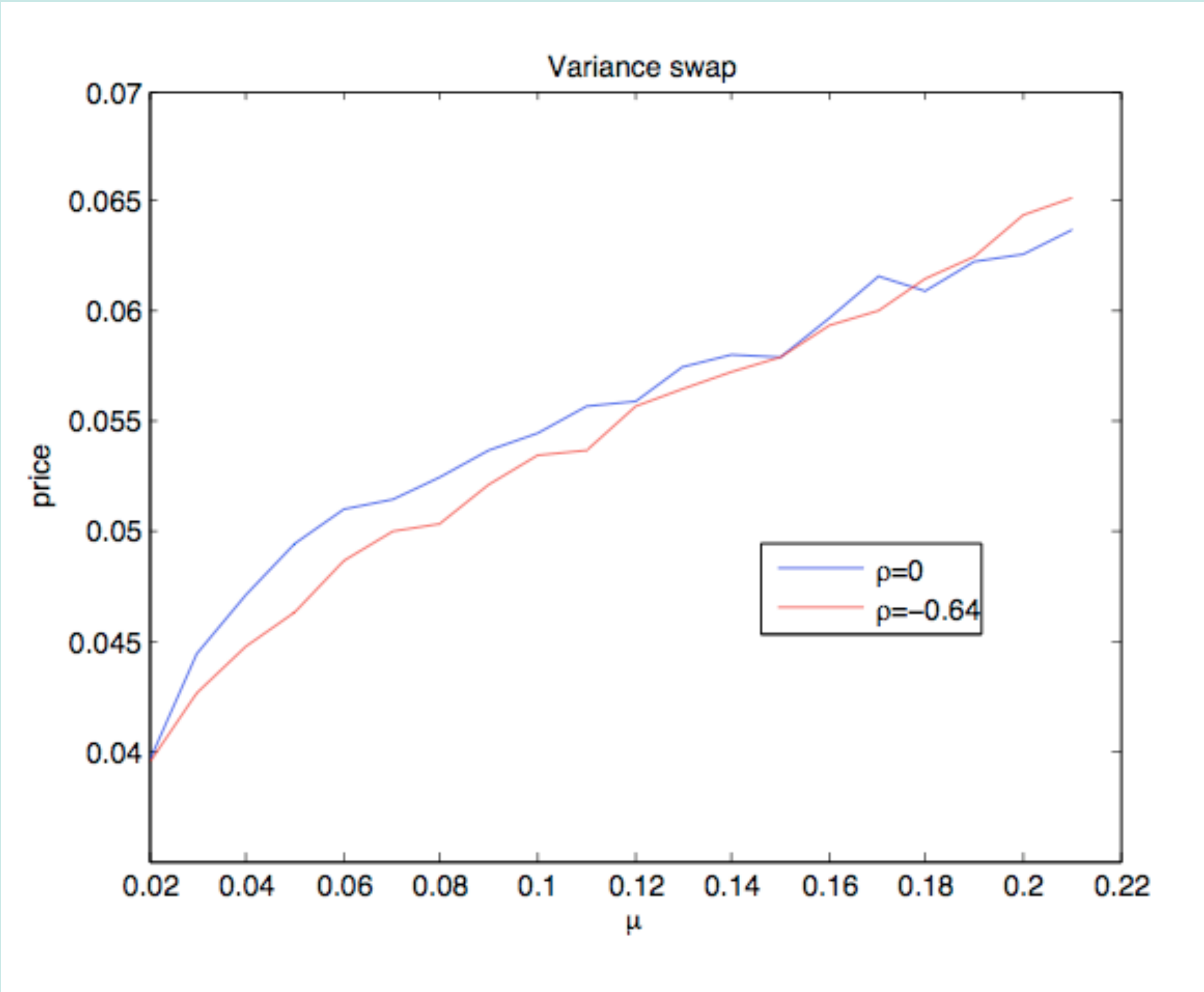
- Variance call: $B(V_T) = (V_T - K)^+$
- Variance swap: $B(V_T) = V_T$
- Volatility swap: $B(V_T) = \sqrt{V_T}$

We take the following parameters:

$$\begin{aligned} \alpha &= 1.15, \quad \beta = 0.39, \quad \kappa = 0.04, \\ \mu &= 0.10, \quad r = 0.04. \end{aligned}$$


Variance call
price
strike
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
γ=1 ρ=0
γ=1 ρ=−0.64
γ=50 ρ=−0.64
γ=50 ρ=0


Variance Call
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.05
0.1
0.15
0.2
K


Variance swap
price
0.07
0.065
0.06
0.055
0.05
0.045
0.04
0.02
0.04
0.06
0.08
0.1
0.12
0.14
0.16
0.18
0.2
0.22
μ
ρ=0
ρ=−0.64

6. A binomial model for real options

Consider a one-period model with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, and historical probabilities $P\{\omega_i\} = p_i > 0$ such that

$$\begin{aligned} S_T(\omega_1) &= uS_0, \qquad & Y_T(\omega_1) = hY_0, \\ S_T(\omega_2) &= uS_0, \qquad & Y_T(\omega_2) = \ell Y_0, \\ S_T(\omega_3) &= dS_0, \qquad & Y_T(\omega_3) = hY_0, \\ S_T(\omega_4) &= dS_0, \qquad & Y_T(\omega_4) = \ell Y_0, \end{aligned}$$

where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, Y_0.

Let B be a contingent claim on Y. If we denote

$$\begin{aligned} B_h &= B_T(\omega_1) = B_T(\omega_3) = B(hY_0) \\ B_\ell &= B_T(\omega_2) = B_T(\omega_4) = B(\ell Y_0), \end{aligned}$$

then its indifference price is

$$\begin{aligned} \pi^B &= -\frac{1}{\gamma}\left(q \log\left[\frac{e^{-\gamma B_h} p_1 + e^{-\gamma B_\ell} p_2}{p_1 + p_2}\right] + \right. \\ & \left. (1-q)\log\left[\frac{e^{-\gamma B_h} p_3 + e^{-\gamma B_\ell} p_4}{p_3 + p_4}\right]\right), \end{aligned} \tag{27}$$

where

$$q = \frac{1-d}{u-d}.$$

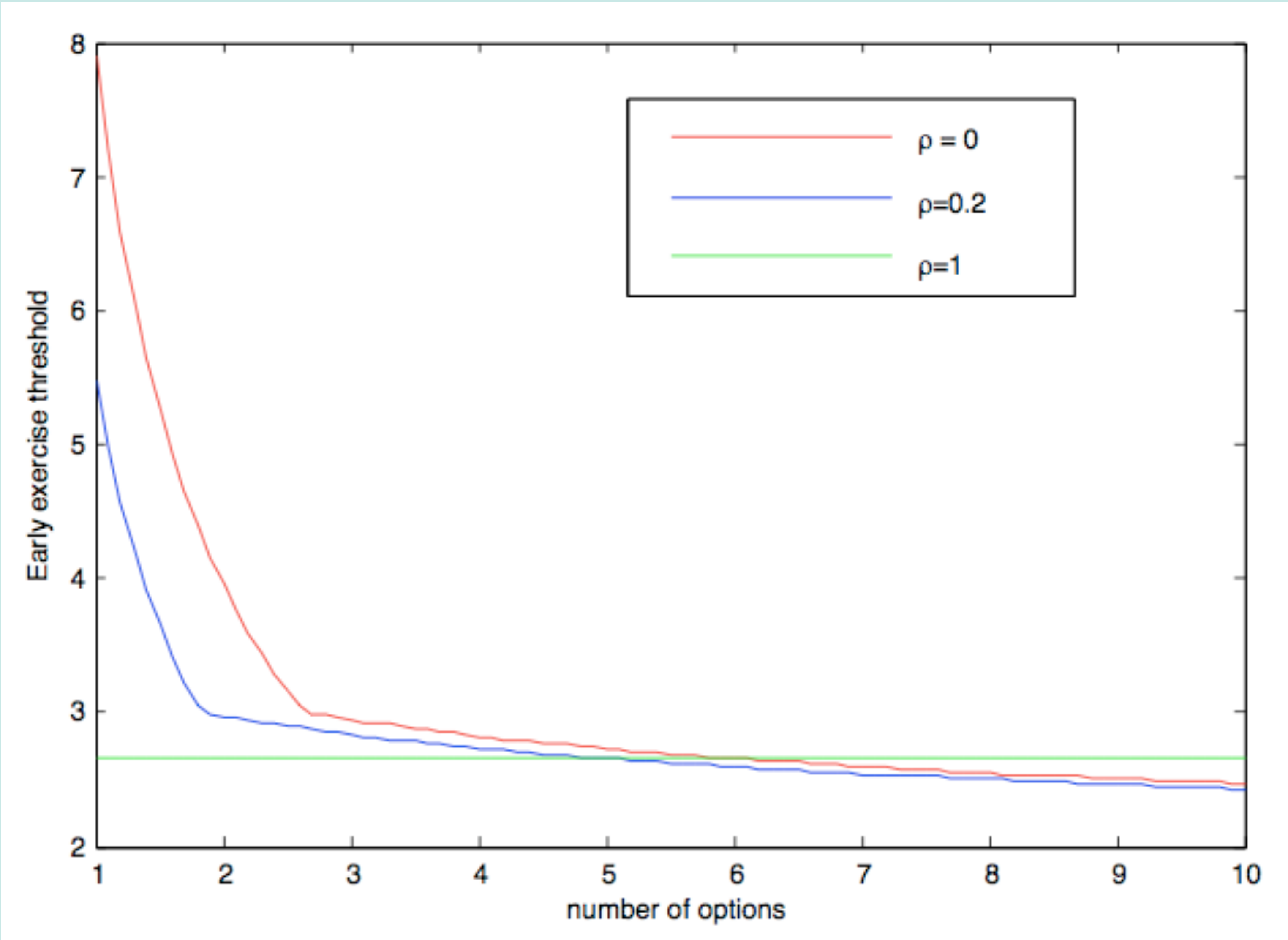
Now suppose B is an American claim. It is clear that early exercise will occur whenever

$$B(Y_0) \geq \pi^B,$$

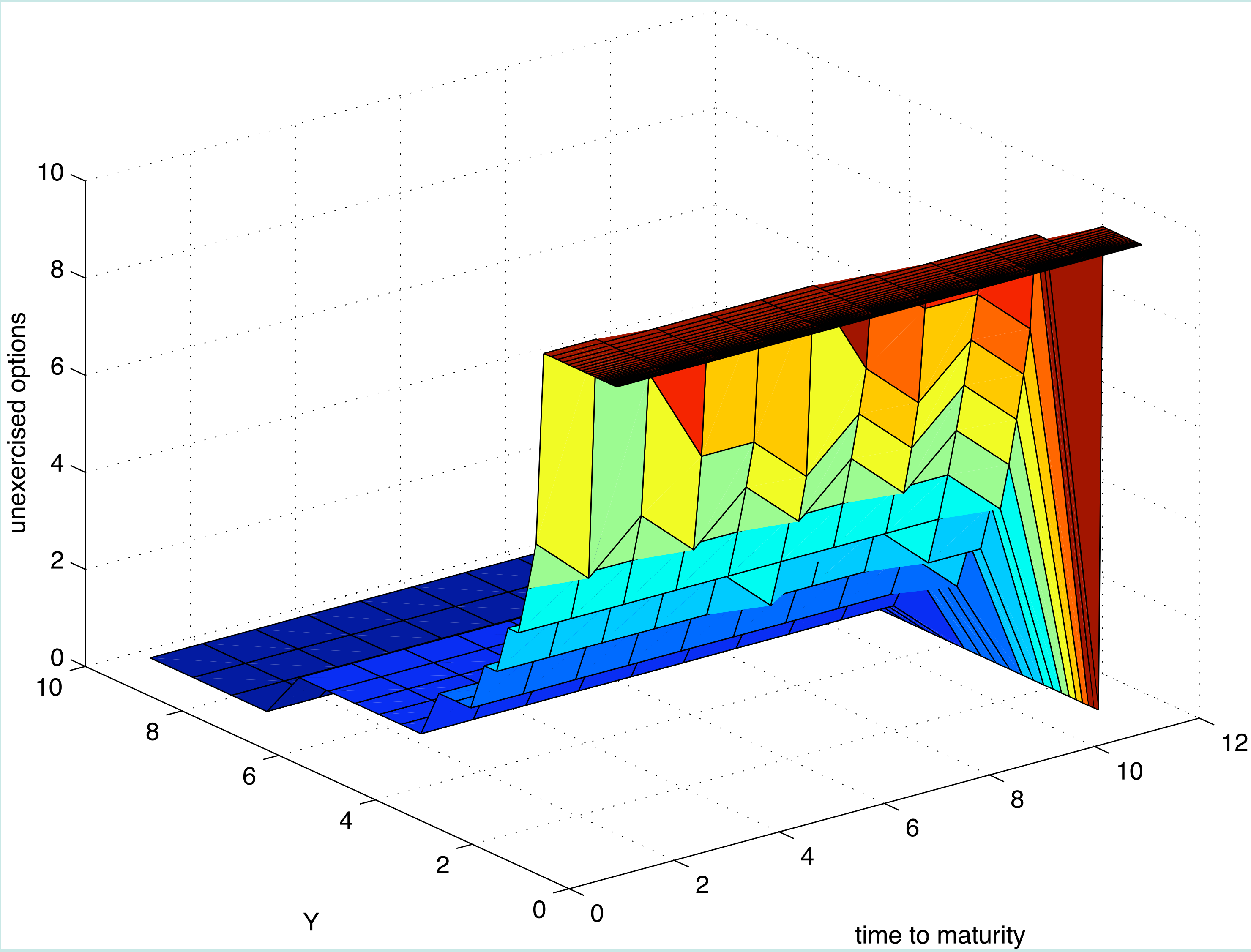
where π^B is the (European) indifference price. For example, an American call option with strike price K will be exercised if Y_0 exceeds the solution to

$$Y^* - K = \log\left[\left(\frac{p_1 + p_2}{e^{-\gamma B_h} p_1 + e^{-\gamma B_\ell} p_2}\right)^{\frac{q}{\gamma}} \left(\frac{p_3 + p_4}{e^{-\gamma B_h} p_3 + e^{-\gamma B_\ell} p_4}\right)^{\frac{1-q}{\gamma}}\right].$$

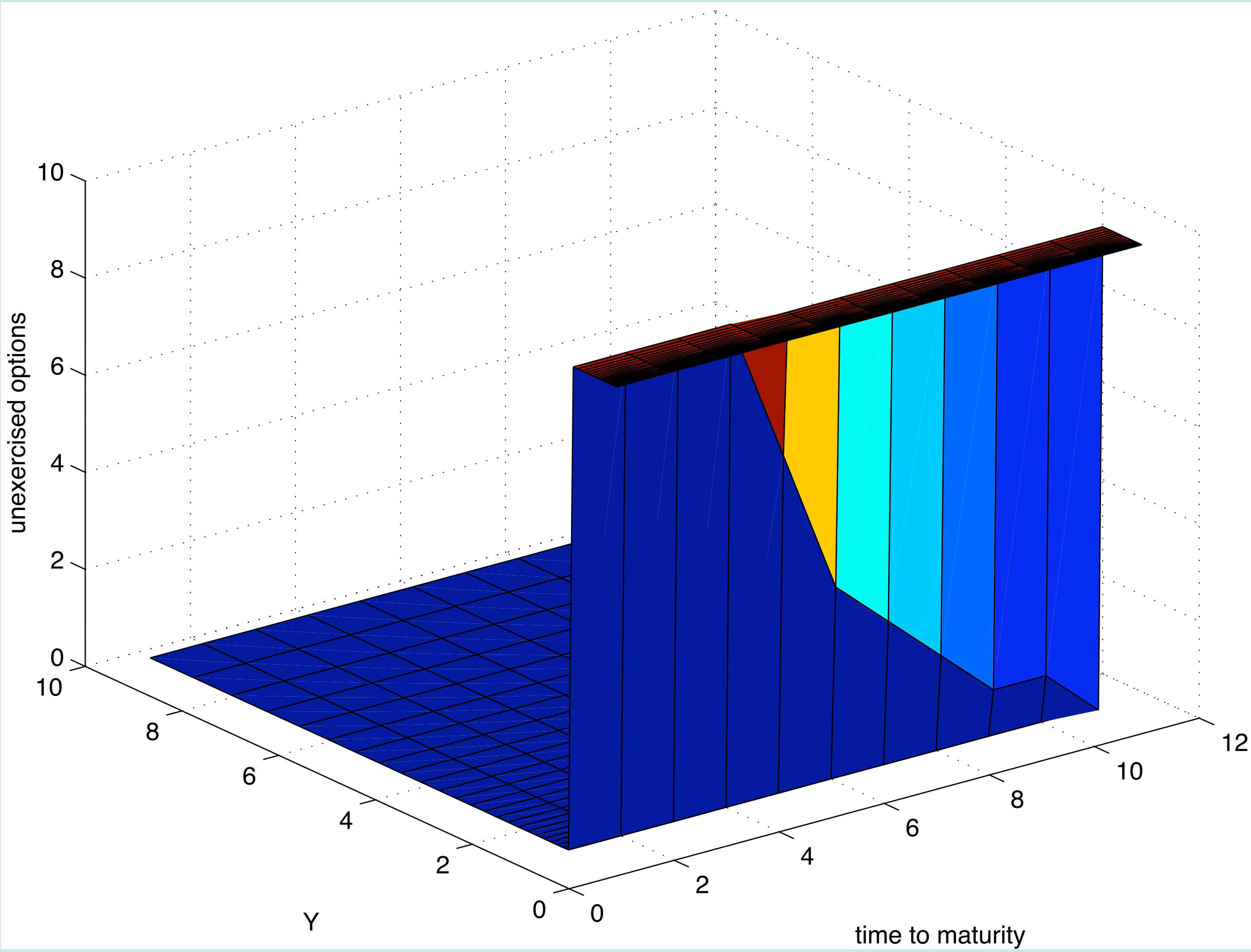
As a result, the early exercise threshold for an American call option obtained above is different (and higher) than the exercise threshold for a contract consisting of A units of identical Americal calls.



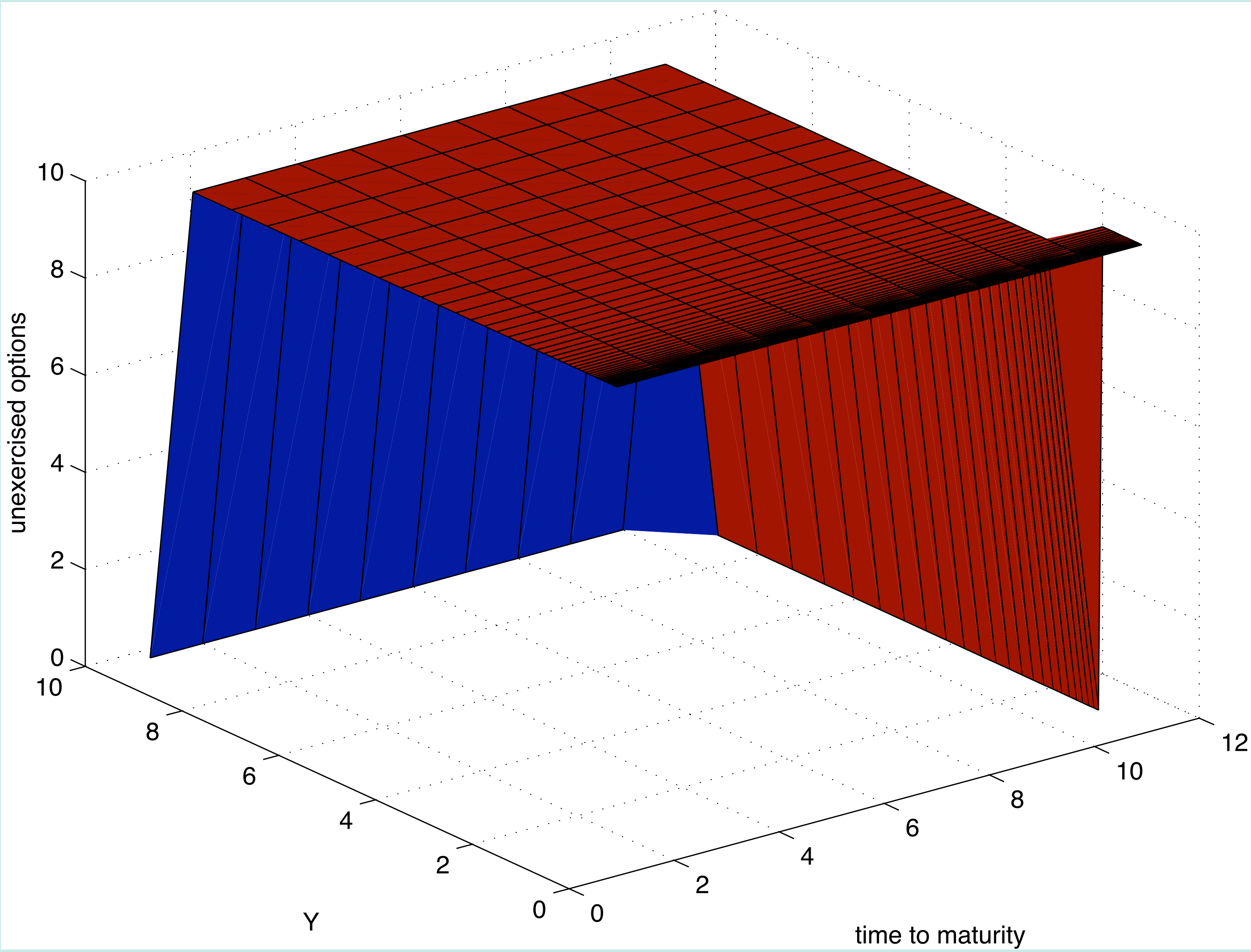
Early exercise threshold
number of options
ρ = 0
ρ=0.2
ρ=1
8
7
6
5
4
3
2
1
2
3
4
5
6
7
8
9
10



unexercised options
10
8
6
4
2
0
Y
10
8
6
4
2
0
time to maturity
0
2
4
6
8
10
12



10
8
6
4
2
0
unexercised options
10
8
6
4
2
0
Y
0
2
4
6
8
10
12
time to maturity


unexercised options
0
2
4
6
8
10
Y
0
2
4
6
8
10
time to maturity
0
2
4
6
8
10
12