Real Options and Game Theory in Incomplete Markets

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Strategic Decision Making

Suppose we want to assign *monetary values* to the strategic decision to:

- create a new firm;
- invest in a new project;
- start a real estate development;
- finance R&D;
- abandon a non-profitable project;
- temporarily suspend operations under adverse conditions and reactive them when conditions improve.

Valuation Elements

In all of the previous problems, we can identify the following common elements:

- uncertainty about the future;
- some degree of irreversibility;
- timing and managerial flexibility;
- interaction with other people's decisions.

To account for these elements, we are going to base our decisions on *values* obtained using the following theoretical tools:

- Net Present Value
- Real Options
- Game Theory

Net Present Value

- Net Present Values takes into account the *intrinsic* advantages of a given investment when compared to capital markets.
- This are essentially due to market imperfections, such as entry barriers, product differentiation, economy of scale, etc...
- For instance, denoting the expected present value of *future cash flows* of a given project by $\widetilde{V}$ and the corresponding *sunk cost* by I, then its NPV is

$$\text{NPV} = \widetilde{V} - I$$

- Therefore, the decision rule according to this NPV is to invest whenever $\widetilde{V} > I$.

The Real Options Approach

- If we view the project value V as an underlying asset, then an investment opportunity with a sunk cost I is the *formal* analogue of an American call option on $\widetilde{V}$ with strike price I.
- The Real Options Approach then applies techniques used for *financial* options to determined the value $\widetilde{C}$ for the option to invest.
- Therefore, an investors possessing an opportunity with value $\widetilde{C}$ will invest only when

$$\widetilde{V} - I > \widetilde{C}.$$

- This then results in *higher* exercise thresholds, taking into account the *value of waiting*.

Successes and Limitations

- According to a recent survey, 26% of CFOs in North America "always or almost always" consider the value of real options in projects.
- This is due to familiarity with the option valuation paradigm in financial markets and its lessons.
- *But* most of the literature in Real Options is based on one or both of the following assumptions: (1) *infinite time horizon* and (2) perfectly correlated *spanning asset* .
- Though some problems have long time horizons (30 years or more), most strategic decisions involve much shorter times.
- The vast majority of underlying projects are *not* perfectly correlated to any asset traded in financial markets.

Alternatives

- The use of well–known numerical methods (e.g binomial trees) allows for finite time horizons.
- As for the spanning asset assumption, the absence of perfect correlation with a financial asset leads to an *incomplete market*.
- Replication arguments can no longer be applied to value managerial opportunities.
- Instead, one needs to rely on *risk preferences*.
- The most widespread way to do this in the strategic decision making literature is to introduce an *internal rate of return*, which replaces the risk–free rate, and use dynamic programming.
- This approach lacks the intuitive understanding of opportunities as *options*.
- We prefer to stick with the options paradigm and use utility–based methods to calculate their values.

A one–period investment model

- Consider the two–factor market where the *discounted* project value V and the *discounted* a correlated traded asset S follow:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \tag{1}$$

  where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, V_0 and historical probabilities p_1, p_2, p_3, p_4.
- Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$.
- An investment opportunity is model as an option with *discounted* payoff $C_t = (V - e^{-rt} I)^+$, for $t = 0, T$.

European Indifference Price

- Without the opportunity to invest in the project V, a rational agent with initial wealth x will try to solve the optimization problem

$$u^0(x) = \max_H E[U(X_T^x)], \tag{2}$$

where

$$X_T^x = \xi + HS_T = x + H(S_T - S_0). \tag{3}$$

is the wealth obtained by keeping ξ dollars in a risk–free cash account and holding H units of the traded asset S.

- An agent with initial wealth x who pays a price π for the opportunity to invest in the project will try to solve the modified optimization problem

$$u^C(x - \pi) = \max_H E[U(X_T^{x-\pi} + C_T)] \tag{4}$$

- The *indifference price* for the option to invest in the final period as the amount π^C that solves the equation

$$u^0(x) = u^C(x - \pi). \tag{5}$$

Explicit solution

Denoting the two possible pay-offs at the terminal time by C_h and C_ℓ, the European indifference price defined in (5) is given by

$$\pi^C = g(C_h, C_\ell) \tag{6}$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} g(x_1, x_2) \quad &= \quad \frac{q}{\gamma} \log \left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}} \right) \\ &\quad + \frac{1-q}{\gamma} \log \left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}} \right), \end{aligned} \tag{7}$$

with

$$q = \frac{1-d}{u-d}.$$

Early exercise

- When investment at time $t = 0$ is allowed, it is clear that immediate exercise of this option will occur whenever its *exercise value* $(V_0 - I)^+$ is larger than its *continuation value* given by π^C.
- That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT} I)^+, (\ell V_0 - e^{-rT} I)^+)\}.$$

A multi–period model

- Consider now a continuous-time two–factor market of the form

$$
\begin{aligned}
dS_t &= (\mu_1 - r)S_t dt + \sigma_1 S_t dW \\
dV_t &= (\mu_2 - r)V_t dt + \sigma_2 V_t(\rho dW + \sqrt{1-\rho^2} dZ_t).
\end{aligned}
$$

- We want to approximate this market by a discrete–time processes (S_n, V_n) following the one–period dynamics (1).
- This leads to the following choice of parameters:

$$
\begin{aligned}
u &= e^{\sigma_1\sqrt{\Delta t}}, \qquad h = e^{\sigma_2\sqrt{\Delta t}}, \\
d &= e^{-\sigma_1\sqrt{\Delta t}}, \qquad \ell = e^{-\sigma_2\sqrt{\Delta t}}, \\
p_1 + p_2 &= \frac{e^{(\mu_1 - r)\Delta t} - d}{u - d}, \quad p_1 + p_3 = \frac{e^{(\mu_2 - r)\Delta t} - \ell}{h - \ell} \\
\rho\sigma_1\sigma_2\Delta t &= (u-d)(h-\ell)[p_1p_4 - p_2p_3],
\end{aligned}
$$

supplemented by the condition $p_1 + p_2 + p_3 + p_4 = 1$.

Grid Values

- Instead a triangular tree for project values, we consider a $(2M+1) \times N$ rectangular grid whose repeated columns are given by

$$V^{(i)} = h^{M+1-i} V_0, \qquad i = 1, \ldots, 2M+1. \tag{8}$$

  This range from $(h^M V_0)$ to $(\ell^M V_0)$, respectively the highest and lowest achievable discounted project values starting from the middle point V_0 with the multiplicative parameter $h = \ell^{-1} > 1$.
- The parameter M should be chosen so that such highest and lowest values are comfortably beyond the range of project values that can be reached during the time interval $[0, T]$ with reasonable probabilities (say four standard deviations)
- Then each realization for the discrete-time process V_n following the dynamics (1) can then be thought of as a path over this grid.

Option pricing on the grid

- We determine the *discounted* value of the option to invest on the project can is a function C_{in} on this grid.
- We start by with the boundary conditions:

$$\begin{array}{rcll} C_{iN} & = & (V^{(i)} - e^{-rT} I)^+, & i = 1, \ldots, 2M+1, \\ C_{1n} & = & V^{(1)} - e^{-rn\Delta_t}, & n = 0, \ldots, N, \\ C_{2M+1,n} & = & 0, & n = 0, \ldots, N. \end{array}$$

- Values in the interior of the grid are then obtained by backward induction as follows:

$$C_{in} = \max\left\{(V^{(i)} - e^{-rn\Delta t} I)^+, g(C_{i+1,n+1}, C_{i-1,n+1})\right\}. \quad (9)$$

- For each time t_n, the *exercise threshold* V_n^* is defined as the project value for which the exercise value becomes higher than its continuation value.

Numerical Experiments

- We now investigate how the exercise threshold varies with the different model parameters.
- The fixed parameters are

$$
\begin{aligned}
I &= 1, \quad r = 0.04, \quad T = 10 \\
\mu_1 &= 0.115, \quad \sigma_1 = 0.25, \quad S_0 = 1 \\
\sigma_2 &= 0.2, \quad V_0 = 1
\end{aligned}
$$

- Given these parameters, the CAPM equilibrium expected rate of return on the project for a given correlation ρ is

$$
\bar{\mu}_2 = r + \rho \left(\frac{\mu_1 - r}{\sigma_1} \right) \sigma_2. \tag{10}
$$

- The difference $\delta = \bar{\mu}_2 - \mu_2$ is the *below–equilibrium rate–of–return shortfall* and plays the role of a dividend rate paid by the project, which we fix at $\delta = 0.04$.

Known Thresholds

- In the limit $\rho \to \pm 1$ (complete market), the closed–form expression for the investment threshold obtained in the case $T = \infty$ gives $V^*_{DP} = 2$.
- This should be contrasted with the NPV criterion (that is, invest whenever the net present value for the project is positive) which in this case gives $V^*_{NPV} = 1$.
- The limit $\gamma \to 0$ in our model corresponds to the McDonald and Siegel (1986) threshold, obtained by assuming that investors are averse to market risk but neutral towards idiosyncratic risk.
- For our parameters, the adjustment to market risks is accounted by CAPM and this threshold coincides with market risk is threshold is $V^*_{DP} = 2$

Dependence with Correlation and Risk Aversion

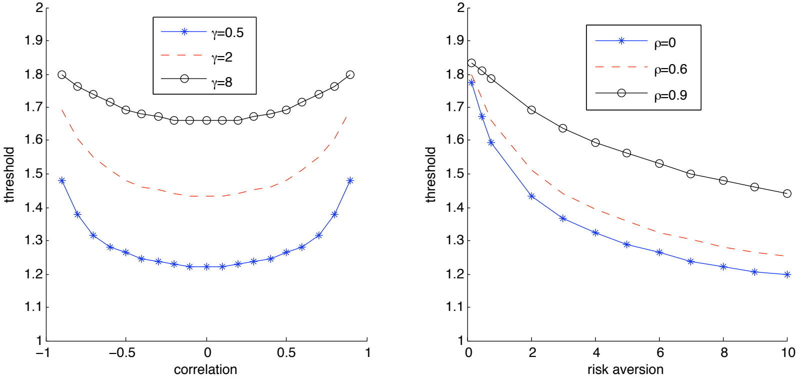

Figure: Exercise threshold as a function of correlation and risk aversion.

Dependence with Correlation and Risk Aversion

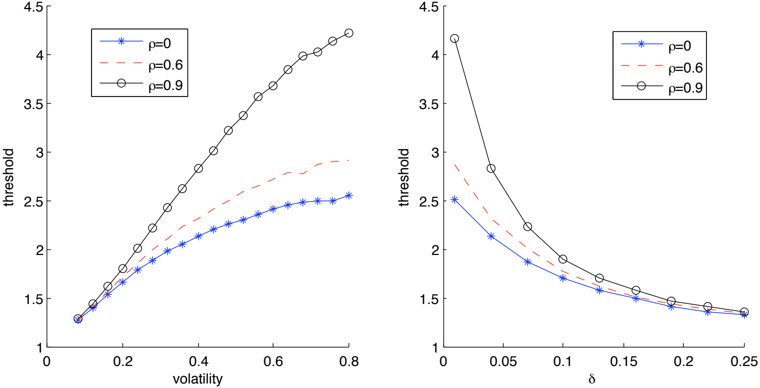

Figure: Exercise threshold as a function of volatility and dividend rate.

Dependence with Time to Maturity

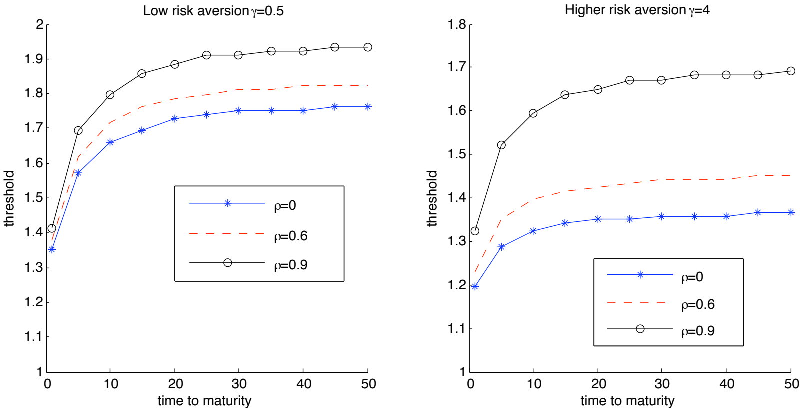

Figure: Exercise threshold as a function of time to maturity.

Values for the option to invest

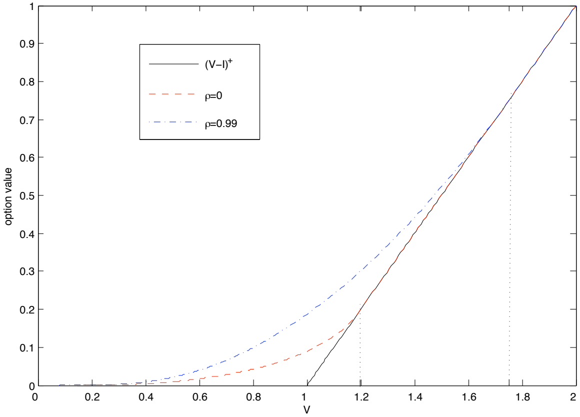

Figure: Option value as a function of underlying project value. The threshold for $\rho = 0$ is 1.1972 and the one for $\rho = 0.99$ is 1.7507.

Suspension, Reactivation and Scrapping

- The previous framework ignores the possibility of negative cash flows arising from the active project, for instance, when operating costs exceed the revenue.
- We then have to consider the option to abandon the project when such cash flows become too negative.
- Instead of completely abandoning the project, we might have the option to "mothball" it by paying a sunk cost E_M and a maintenance rate $m < C$.
- Once prices for the output become favorable again, we have the option to reactive the project by paying a sunk cost $R < I$.
- Finally, if prices drop too much, we have the option to completely abandon the project by paying a sunk cost S (which could be negative, corresponding to a "scrap value").
- As before, the decisions to invest, mothball, reactivate and scrap are triggered by the price thresholds $P_S < P_M < P_R < P_H$.

Project values and options

- Let us denote the value of an idle project by F^0, an active project by F^1 and a mothballed project by F^M.
- Then

$$
\begin{aligned}
F^0 &= \text{option to invest at cost } I \\
F^1 &= \text{cash flow } + \text{option to mothball at cost } E_M \\
F^M &= \text{cash flow } + \text{option to reactivate at cost } R \\
&\quad + \text{option to scrap at cost } S
\end{aligned}
$$

- We obtain its value on the grid using the recursion formula

$$F^k(i,j) = \max\{\text{continuation value, possible exercise values}\}.$$

Introducing Competition

- We use game theoretical tools to introduce the effect of competition.
- The goal is to assign a strategic value $\widetilde{G}$ to both conditional and unconditional moves toward investment that can create a competitive advantaged in the market.
- This is then added to the NPV of a project.
- Therefore, the decision rule is to invest whenever

$$(\widetilde{V} - I) + \widetilde{G} > \widetilde{C}.$$

Combining options and games

- For a systematic application of both *real options* and *game theory* in strategic decisions, we consider the following rules:
    1. Outcomes of a given game that involve a "wait–and–see" strategy should be calculated by option value arguments.
    2. Once the Nahs equilibrium (NE) for a given game is found on a decision node, its value becomes the pay-off for an option at that node.
- In this way, option valuation and game theoretical equilibrium become *dynamically related* in a decision tree.
- In what follows, we denote the NE solution for a given game in bold face within the matrix of outcomes (rounded to nearest integer).

One Stage Strategic Investment

- As a first example, consider two *symmetric* firms contemplating a total investment $I = 80$ on a project with $V_0 = 100$ and equal probabilities to move up to $V^u = 200$ and down to $V^d = 50$.
- We take $u = 3/2$, $h = 2$, $p_1 = p_4 = 255/256$, $p_2 = p_3 = 1/256$, $\gamma = 0.1$, $r = 0$.
- Therefore, using expression (7) to calculate the *option value* for the "wait–and–see" strategy, we have the following matrix of outcomes for this game:

		B: Invest	B: Wait
A	Invest	**(10,10)**	(20,0)
A	Wait	(0,20)	(11,11)

- For comparison, the complete market gives an option value of 48 to be shared by the firms.

Two–stage competitive R&D

- Suppose now that firm A is the only firm facing an R&D investment at cost $I_0 = 25$ at time t_0, whereas at time t_1 the firms can equally share the follow–on cost $I_1 = 80$.
- We will assume that the technology resulting from the R&D investment is either *proprietary*, so that the market share of firm A after the R&D phase is $s = 3/5$.
- Moreover, we assume that the market value continues to evolve from time t_1 to time t_2 following the same dynamics, that is, at time t_2 the possible market values in these two–period tree are

$$V^{uu} = 400, \quad V^{ud} = 100, \quad V^{dd} = 25.$$

Analyzing the game

- If demand is high at time t_1 ($V^u = 200$), we have:

		B (follower)	
		Invest	Wait
A (leader)	Invest	**(80,40)**	(120,0)
	Wait	(0,120)	(42,22)

- If demands is low at time t_1 ($V^d = 60$), we have:

		B (follower)	
		Invest	Wait
A (leader)	Invest	(-10,-20)	(-30,0)
	Wait	(0,-30)	**(8,0)**

- Then $C_A = -I_0 + g(80, 8) = -25 + 30 = 5 > 0$,
- whereas $C_B = g(40, 0) = 15$
- Therefore the R&D investment is recommended for A.
- For comparison, the complete market results are $C_A = 10$ and $C_B = 7$.