Hedging Insurance Contracts in Incomplete Markets

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- By combining the two, one can use the financial market to:
  - hegde independent insurance risks;
  - hedge equity-linked insurance products;
  - design new insurance insurance contracts (e.g: flood insurance).
- The presence of insurance risks render the markets generally incomplete.
- Our approach to pricing and hedging is to use risk preferences induced by an exponential utility $U(x) = -e^{-\gamma x}$.

Optimal Hedging and Indifference Pricing

- We assume that, after selling an insurance contract B_T maturing at a future time T, the insurance company tries to solve the stochastic control problem

$$u^B(x) = \sup_{H \in \mathcal{A}} E\left[U\left(X_T^{H,x,B}\right)\right],$$

where $X_t^{H,x}$ is value of a self–financing portfolio with initial wealth x, a short position in the contract B_T, and H_t units of the stock, with the remaining value invested in the bank account.

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- The sellers indifference price for the claim B is the solution π^s to the equation

$$u^0(x) = u^B(x + \pi^s)$$

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- In this case, we have

$$\begin{aligned} u^s(x + \pi^s) &= \sup_{H \in \mathcal{A}} E\left[-e^{-\gamma(x + \pi^s + \int_0^T H_s dS_s - B_T)}\right] \\ &= e^{-\gamma \pi^s} E\left[e^{\gamma B_T}\right] \sup_{H \in \mathcal{A}} E\left[-e^{-\gamma(x + \int_0^T H_s dS_s)}\right] \\ &= e^{-\gamma \pi^s} E\left[e^{\gamma B_T}\right] u^0(x). \end{aligned}$$

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- Therefore, the indifference price in this case is given by

$$\pi^s = \frac{1}{\gamma}\log E\left[e^{\gamma B_T}\right].$$

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- Variable Annuities in US: around $100bn per year in recently, mostly GMDB.
- Unit-linked Insurance in UK: over $3bn in 2010, moved from GMMB to GMDB.

Stochastic Volatility Markets

- We consider two factor stochastic volatility models where the financial asset satisfies:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma(Y_t) S_t dW_t^1 \\
dY_t &= a(Y_t) dt + b(Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2]
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- Here μ and $-1 < \rho < 1$ are constants, $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are deterministic functions, and W_t^1 and W_t^2 are independent one dimensional P–Brownian motions.

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- Here μ and $-1 < \rho < 1$ are constants, $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are deterministic functions, and W_t^1 and W_t^2 are independent one dimensional P–Brownian motions.
- In addition, we assume the existence of a risk-free bank account paying a constant interest rate $r = 0$.

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- To obtain the equation satisfied by u^s in this case, consider the interval $[t, t+h)$ and observe that,

$$\begin{aligned} u^B(x, s, y, t) \ \geq \ & E[u^B(X_{t+h}, S_{t+h}, Y_{t+h}, t+h)]p(h) \\ & + E[u^0(X_{t+h} - B(S_{t+h}), Y_{t+h}, t+h)]q(h) \end{aligned}$$

where $p(h) = P(\tau > t + h | \tau > t)$ and $q(h) = 1 - p(h)$.

The HJB equation

- Using a function of the form $u^B(x,S,y,t) = -e^{-\gamma x} e^{\phi(S,y,t)}$ leads to

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 S^2 \phi_{SS} + \rho\sigma b S \phi_{yS} + \frac{1}{2} b^2 \phi_{yy} + \left(a - \frac{\mu b \rho}{\sigma}\right)\phi_y \\ + \frac{1}{2} b^2 (1-\rho^2)\phi_y^2 + \lambda(t)\left[e^{\gamma B + \psi - \phi} - 1\right] = \frac{\mu^2}{2\sigma^2} \\ \phi(y,S,T) = 0 \end{cases}, \tag{1}$$

where, as it is well-known,

$$\psi(y,t) = \frac{1}{1-\rho^2} \log \widetilde{E}^{y,t}\left[e^{-\int_0^T \frac{(1-\rho^2)\mu^2}{2\sigma^2(Y_s)} ds}\right],$$

with $\widetilde{E}[\cdot]$ denoting an expectation with respect to the *minimal martingale measure* for this market.

Optimal hedge

- In terms of ϕ, the optimal portfolio is

$$\pi_t^B = \frac{1}{\gamma}\left[\frac{\mu}{\sigma^2(y)} + \phi_S S + \frac{b(y,t)\rho}{\sigma(y)}\phi_y\right]$$

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- Subtracting one from the other we obtain the *excess hedge*

$$\pi_t^B - \pi_t^0 = P_S(S,y,t)S_t + \frac{b(y,t)\rho}{\gamma\sigma(y)}P_y(S,y,t),$$

  which has the form of a *delta* hedge plus a volatility correction.

Fast-mean reversion asymptotics

- Let us now take

$$dY_t = \alpha(m - Y_t)dt + \beta(\rho dW_t + \sqrt{1-\rho^2} dZ_t)$$

and consider the regime $\frac{1}{\alpha} = \varepsilon << 1$, with $\beta = \sqrt{2\nu}/\sqrt{\varepsilon}$ where ν^2 is a fixed variance for the invariant distribution of Y_t.

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- We then look for expansion of the form

$$\phi^\varepsilon = \phi^{(0)}(y, S, t) + \sqrt{\varepsilon}\phi^{(1)}(y, S, t) + \varepsilon\phi^{(2)}(y, S, t) + \ldots$$

Main result

- The insurer's indifference price satisfy:

$$|P(y,S,t) - P^{(0)}(S,t) - \widetilde{P^1}(y,S,t)| = \mathcal{O}(\varepsilon) \qquad (2)$$

where

$$\widetilde{P^1}(y,S,t) = -(T-t)(V_3 S^3 P^{(0)}_{SSS} + V_2 S^2 P^{(0)}_{SS})$$

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- Here $P^{(0)}$ satisfies

$$\begin{cases} P^{(0)}_t + \frac{1}{2}\sigma_\star^2 P^{(0)}_{SS} + \frac{\lambda(t)}{\gamma}\left[e^{\gamma(g-P^{(0)})} - 1\right] = 0 \\ P^{(0)}(S,T) = 0 \end{cases} \qquad (3)$$

  where $\sigma_\star^2 = \langle \sigma^2 \rangle$.

Example

- Consider the contract

$$B(S) = \begin{cases} 4, & \text{if } 0 \leq S \leq 50 \\ 0.8S, & \text{if } 5 \leq S \leq 20 \\ 16, & \text{if } 20 \leq S \leq 100 \end{cases} \tag{4}$$

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- The mortality rate by Gompertz mortality,

$$\lambda_x(t) = \frac{1}{b} e^{\frac{x+t-m}{\beta}}$$

with $\beta = 8.75$ and $m = 92.63$.

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- The other model parameters are:

$$\alpha = 200, \quad m = \log 0.1, \quad \nu = \frac{1}{\sqrt{2}}, \quad \rho = -0.2, \quad \mu = 0.2.$$

Price correction

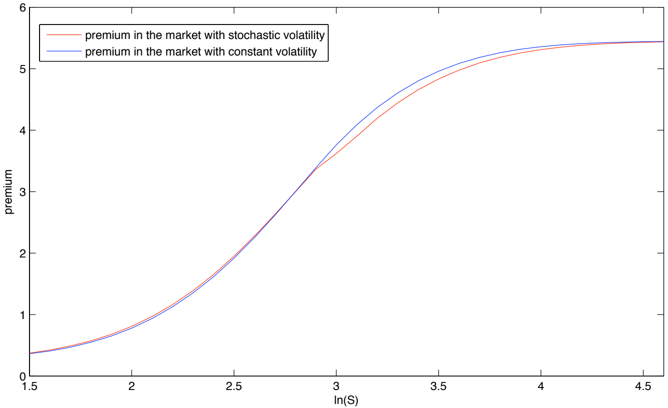

Figure: Premium for the equity linked contract in a market with constant volatility $\sigma = 0.165$ and in the market with stochastic volatility for $T - t = 15$ years and $\gamma = 0.3$.

Risk aversion

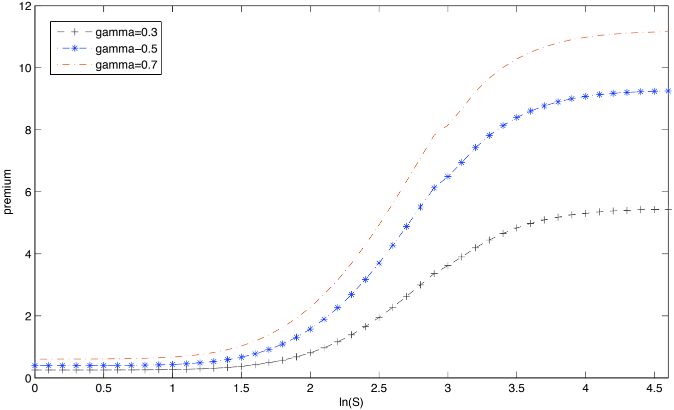

Figure: Dependence with risk aversion

Hedge

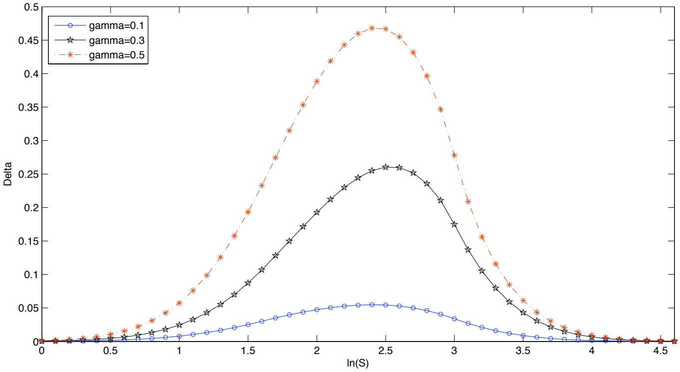

Figure: Hedge ratio for different risk aversion parameters

Stochastic Interest Rates

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- We model the short rate as

$$\begin{cases} dr_s = (a_0(s)r_s + b_0(s))ds + \sqrt{c(s)r_s + d(s)}dZ_s \\ r_t = r \end{cases},$$

where $Z_t = \rho W_t^1 + \sqrt{1-\rho^2}dW_t^2$.

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  where $Z_t = \rho W_t^1 + \sqrt{1-\rho^2}dW_t^2$.
- It then follows that the price of a zero–coupon bond with maturity T_1 is given by

$$F_{tT_1} = e^{A(t,T_1) - C(t,T_1)r_t},$$

  for deterministic functions $A(\cdot,\cdot)$ and $C(\cdot,\cdot)$.

Portfolio choice

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- We assume the market for bonds of different maturities has a market price of risk of the form

$$q(r_s, s) = \frac{(a_0(s) - a(s))r_s + (b_0(s) - b(s))}{\sqrt{c(s)r_s + d(s)}} \tag{5}$$

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- Under this assumption, one can show that the dynamics of the discounted bond price is

$$\begin{aligned} \frac{d(e^{-\int_0^s r_u du} F_{sT_1})}{e^{-\int_0^s r_u du} F_{sT_1}} &= -C(s, T_1)\big[(\Delta a(s) r_s + \Delta b(s))dt \\ &\quad + \sqrt{c(s)r_s + d(s)}\, dZ_s\big] \end{aligned}$$

Path–dependent claims

- We consider path–dependent claims of the form $B_t = B(S_t, r_t, v_t)$, where

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$$\begin{cases} dX_s = \pi_s \frac{dS_s}{S_s} + \eta_s \frac{d(e^{-\int_0^s r_u du} F_{sT_1})}{e^{-\int_0^s r_u du} F_{sT_1}} \\ dX_s = [\pi_s(\mu - r) - \eta_s C(s, T_1)(\Delta a(s) r_s + \Delta b(s))] ds \\ +\pi_s \sigma dW^1 - \eta_s C(s, T_1)\sqrt{c(s) r_s + d(s)} dZ_s \\ X_\tau = X_{\tau -} - B(S_\tau, r_\tau, V_\tau), \quad \tau < T \\ X_t = x \end{cases}$$

The solution to Merton's Problem

- The Merton problem for the insurance company is now

$$u^0(x, r, t) = \sup_{\pi, \eta \in \mathcal{A}} E^{x,r,t} \left[U \left(X_T \right) \right].$$

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- Using the same reasoning as before for the function $u^0(x,r,t) = -e^{-\gamma x}e^{\psi(r,t)}$ we arrive at the following PDE:

$$\psi_t+(ar+b)\psi_r+\frac{1}{2}\psi_{rr}(cr+d)-\left[\frac{1}{2}\left(\frac{\mu-r-\sigma\rho q}{\sqrt{1-\rho^2}\sigma}\right)^2+\frac{q^2}{2}\right]=0,$$

subject to $\psi(r,T)=0$.

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- Using Feynmann-Kac we obtain that

$$\psi(r,t) = -\int_t^T \widehat{E}^{t,r}\left[\left(\frac{\mu-r-\sigma\rho q}{2\sqrt{1-\rho^2}\sigma}\right)^2+\frac{q^2}{2}\right]dt,$$

where $\widehat{E}[\cdot]$ denotes expectation with respect to the (unique) martingale measure for bond prices defined by the market price of risk q.

The value function with the claim

- Similarly, the hedging problem for the insurance company is now

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- For a function of the form $u^B(x, S, y, t) = -e^{-\gamma x} e^{\phi(S,r,v,t)}$, we obtain that ϕ satisfies the PDE

$$\begin{cases} \phi_t + (ar+b)\phi_r + \frac{1}{2}(cr+d)\phi_{rr} + \rho\sigma\sqrt{cr+d}\,S\phi_{Sr} + \frac{1}{2}\sigma^2 S^2 \phi_{SS} \\ +f(S,r,t)\phi_v - \left[\frac{1}{2}\left(\frac{\mu - r - \sigma\rho q}{\sqrt{1-\rho^2}\sigma}\right)^2 + \frac{q^2}{2}\right] - \lambda(t)\left(1 - e^{\gamma B + \psi - \phi}\right) = \end{cases} \tag{7}$$

subject to $\phi(S, r, v, T) = 0$.

Optimal hedge

- In terms of ϕ, the optimizers for (6) are

$$
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\pi_t^B &= \frac{1}{\gamma}\left[\frac{\mu - r - q\rho\sigma}{(1 - rho^2)\sigma^2} + \phi_S S\right] \\
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- By comparison, the optimal Merton portfolio is

$$\begin{aligned}\pi_t^0 &= \frac{1}{\gamma}\left[\frac{\mu - r - q\rho\sigma}{(1 - rho^2)\sigma^2}\right] \\ \eta_t^0 &= \frac{1}{\gamma C\sqrt{cr+d}}\left[\frac{\rho\sigma(\mu - r) - q\sigma^2}{(1-\rho^2)\sigma^2} - \sqrt{cr+d}\,\Psi_r\right]\end{aligned}$$

- Subtracting one from the other we obtain the *excess hedge*

$$\begin{aligned}\pi_t^B - \pi_t^0 &= P_S(S, r, v, t)S_t \\ \eta_t^B - \eta_t^0 &= -\frac{1}{C}P_r(S, r, v, t)\end{aligned}$$

The pricing equation and integral representation

- Therefore, P satisfies the following nonlinear PDE:

$$\begin{cases} P_t + (ar+b)P_r + \dfrac{1}{2}(cr+d)P_{rr} + \rho\sigma\sqrt{cr+d}\,SP_{Sr} + \dfrac{1}{2}\sigma^2 S^2 P_{SS} \\ +f(S,r,t)P_v - \frac{\lambda(t)}{\gamma}\left(1 - e^{\gamma B - \gamma P}\right) = 0 \\ P(S,r,T) = 0 \end{cases}$$

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- This leads to an integral representation of the premium as follows:

$$P(S,r,V,t) = \frac{1}{\gamma}\sup_{y>0}\left[E^Q_{t,S,r,V}\left[\int_t^T g(S,V,r,t)e^{-\int_t^s \frac{y_s\lambda_s}{\gamma}du} y_s\lambda_s ds\right]\right.$$
$$\left. -E^Q_{t,S,r,V}\left[\int_t^T \left(\frac{1}{y_s} - \frac{1}{\gamma}\left(1 - \ln\frac{y_s}{\gamma}\right)\right) y_s\lambda_s e^{-\int_t^s \frac{y_s\lambda_s}{\gamma}du} ds\right]\right] \quad (8)$$

Future research

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