Nonlinearity, correlation and the valuation of employee stock options

M. R. Grasselli

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1. **Introduction**

We consider an employee who has been awarded a compensation package consisting of $A$ identical call options on the company’s stock with the following features:

- strike price $K$, maturity date $T$ and vesting period $T_v < T$;
- options are non-transferible;
- hedge using the underlying stock $Y_t$ is not allowed;
- hedge using a correlated asset $S_t$ is allowed.
2. Accounting recommendations

- The Financial Accounting Standard Board instructed in 1972 (Opinion 25) that stock options should be accounted according to their **intrinsic value**, that is \((Y_t - K)^+\) on the date they are granted.

- In 1995, the FASB 123 recommended using a **fair value** approach instead: estimate the expected life of the option and insert this into either Black–Scholes or a Cox–Rubenstein-Ross tree. It still accepted Opinion 25 as a valid method.

- In 2004, it revised FASB 123, eliminating the possibility of using intrinsic value methods.
3. Previous literature

- Detemple and Sudaresan (1999) and Hall and Murphy (2002) propose to use utility methods to deal with the market incompleteness created by trading and hedging restrictions, but without using a correlated asset.

- Musiela and Zariphopoulou (2004) developed a multiperiod model to price European style contracts based on a non-traded underlying asset in the presence of a correlated traded asset using indifference pricing techniques.

- Henderson (2005) applied indifference pricing to value a single American call options on a non-traded asset.
- Rogers and Scheinkman (2003) and Jain and Subramanian (2004) investigate the effect of partial exercise, but with no correlated asset.

- Hull and White (2004) use a binomial model with no correlated asset, no partial exercise and no risk preferences. The incompleteness is accounted for by a parameter $M$ - the effective stock-to-strike exercise threshold.
4. The one-period model

Consider a one-period market model

\[
(S_T, Y_T) = \begin{cases} 
(uS_0, hY_0) & \text{with probability } p_1, \\
(uS_0, \ell Y_0) & \text{with probability } p_2, \\
dS_0, hY_0 & \text{with probability } p_3, \\
dS_0, \ell Y_0 & \text{with probability } p_4,
\end{cases} \tag{1}
\]

where \(0 < d < 1 < u\) and \(0 < \ell < 1 < h\), for positive initial values \(S_0, Y_0\) and historical probabilities \(p_1, p_2, p_3, p_4\)
Let $C_T = C(Y_T)$ be a $T$–claim and consider a utility function $U(x) = -e^{-\gamma x}$. An investor who buys this claim for a price $\pi$ will then try to solve the optimal portfolio problem

$$u^C(x - \pi) = \sup_H E[U(X_T + C_T)].$$

(2)

The **indifference price** for this claim is defined to be a solution to the equation

$$u^0(x) = u^C(x - \pi),$$

where $u^0$ is defined by (2) for the degenerate case $C \equiv 0$. 
An explicit calculation then leads to

$$\pi = g(C_h, C_\ell)$$  \hspace{1cm} (3)

where, for fixed parameters \((u, d, p_1, p_2, p_3, p_4)\) the function \(g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) is given by

$$g(x_1, x_2) = \frac{q}{\gamma} \log \left( \frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}} \right) + \frac{1 - q}{\gamma} \log \left( \frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}} \right),$$

with

$$q = \frac{1 - d}{u - d}.$$
Now suppose $C$ is an American claim. It is clear that early exercise will occur whenever

$$C(Y_0) \geq \pi,$$

where $\pi^B$ is the (European) indifference price. For example, an American call option with strike price $K$ will be exercised if $Y_0$ exceeds the solution to

$$(Y^* - K)^+ = g((hY_0 - K)^+, (\ell Y_0 - K)^+)$$
5. **Multiple claims**

As a result of risk aversion, the early exercise threshold for an American call option obtained above is different (and higher) than the exercise threshold for a contract consisting of $A$ units of identical American calls. Explicitly, it is the solution to

\[
A(Y^* - K)^+ = g(A(hY_0 - K)^+, A(\ell Y_0 - K)^+) 
\]

(4)
The diagram shows the relationship between the number of options and the exercise threshold for different values of \( \rho \). The exercise threshold decreases as the number of options increases. The curves are labeled for \( \rho = 1 \), \( \rho = 0.9554 \), and \( \rho = 0 \).
If partial exercise is allowed, then the optimal number of options to be exercised is the solution $a^*$ to

$$\max_a \left[ a(Y_0 - K)^+ + \pi^{(A-a)B} \right].$$

(5)

The value of $A$ units of the option is therefore

$$C_0^{(A)} = a_0(Y_0 - K)^+ + \pi^{(A-a_0)}$$
6. **Two-period model: inter-temporal exercise**

Let us label the nodes in of a two-period binomial tree by 0 at time \( t_0 = 0 \), \((h, \ell)\) at time \( t_1 \) and \((hh, h\ell, \ell\ell)\) at time \( t_2 = T \).

The number of option that the holder of \( A \) calls at the node \( h \) should immediately exercise is given by

\[
a_h = \arg \max_{0 \leq a \leq A} \left[ a(hY_0 - K)^+ + \pi_h^{(A-a)} \right],
\]

(6)

where \( \pi_h^{(A-a)} \) denotes the indifference of an European claim to starting at the node \( h \) and maturing at time \( T \).
The pay-offs for such claim are with

\[ C_{hh}^{(A-a)} = (A - a)(hhY_0 - K)^+ \]

with probability \((p_1 + p_3)\) and

\[ C_{hl}^{(A-a)} = (A - a)(hlY_0 - K)^+ \]

with probability \((p_2 + p_4)\), where we have used \(hh\) and \(hl\) to denoted, respectively, the nodes where the non-traded asset has values \(hhY_0\) and \(hlY_0\). Its indifference price is explicitly given by

\[ \pi_h^{(A-a)} = g(C_{hh}^{(A-a)}, C_{hl}^{(A-a)}) \] (7)
In the same vein, the optimal number of options to be exercised at the node $\ell$, where the non-trade asset has value $\ell Y_0$, is

$$a_\ell = \arg \max_{0 \leq a \leq A} \left[ a(\ell Y_0 - K)^+ + \pi_\ell^{(A-a)} \right],$$

where

$$\pi_\ell^{(A-a)} = g(C_{h\ell}^{(A-a)}, C_{\ell\ell}^{(A-a)})$$

(9)

Therefore, at the intermediate time $t_1$, the total value of $A$ options at the node $h$ is

$$C_h^{(A)} := \left[ a_h(hY_0 - K)^+ + \pi_h^{(A-a_h)} \right],$$

(10)

while the total value of $A$ options at the node $\ell$ is

$$C_\ell^{(A)} := \frac{1}{A} \left[ a_\ell(\ell Y_0 - K)^+ + \pi_\ell^{(A-a_\ell)} \right].$$

(11)
Finally, starting with $A$ units of the option, the number of options that should be exercised at the initial time $t_0$ is

$$a_0 = \arg \max_{0 \leq a \leq A} \left[ a(Y_0 - K)^+ + \pi_0^{(A-a)} \right],$$

(12)

where

$$\pi_h^{(A-a)} = g(C_h^{(A-a)}, C_\ell^{(A-a)}).$$

(13)

Therefore the value at time zero of $A$ units of an American call option on the non-traded asset is

$$C_0^{(A)} := \left[ a_0(Y_0 - K)^+ + \pi_0^{(A-a_0)} \right].$$

(14)
6. The multi-period model: inter-temporal exercise

We first have to choose discrete time parameters $(u, d, h, \ell, p_1, p_2, p_3, p_4)$ that match the distributional properties of the continuous time diffusion

\begin{align*}
\frac{dS}{S} &= (\mu - r)dt + \sigma dW \\
\frac{dY}{Y} &= (a - r - \delta) dt + bY (\rho dW + \sqrt{1 - \rho^2}) dZ,
\end{align*}

(15) (16)
These are given by the system

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}}, \\
    h &= e^{b \sqrt{\Delta t}}, \\
    d &= e^{-\sigma \sqrt{\Delta t}}, \\
    \ell &= e^{-b \sqrt{\Delta t}}, \\
    p_1 + p_2 &= e^{(\mu - r) \Delta t - d} \\
    u - d \\
    p_1 + p_3 &= e^{(a - r - \delta) \Delta t - \ell} \\
    h - \ell \\
    \rho b \sigma \Delta t &= (u - d)(h - \ell)[p_1 p_4 - p_2 p_3] \\
    1 &= p_1 + p_2 + p_3 + p_4
\end{align*}
\]
The valuation algorithm is then:

- Begin at the final period.

- At each node of the tree, compute the (European) indifference prices for different values of \((A - a)\).

- Determining the maximum of (5).

- Use this as the value for the entire position at that node.

- Iterate backwards.
7. Numerical Results

We first determine the optimal exercise surface for the holder of $A = 10$ options with strike price $K = 1$ and

\[
\begin{align*}
\mu &= 0.12, & \sigma &= 0.2, & S_0 &= 1 \\
a &= 0.15, & b &= 0.3, & Y_0 &= 1 \\
r &= 0.06, & T &= 5, & N &= 500
\end{align*}
\]

(17) (18) (19)

For our base case, $\delta = 0.075$, $\gamma = 0.125$ and $\rho = -0.5$. We then modify it by having $\delta = 0$, $\gamma = 10$ and $\rho = 0.95$. 
Next we consider the impact that time-to-maturity, risk aversion, correlation and volatility have on the option price. When not indicated in the graphs, the parameter values are

\[
\begin{align*}
\mu &= 0.09, & \sigma &= 0.4, & S_0 &= 1 \\
\alpha &= 0.08, & b &= 0.45, & Y_0 &= 1 \\
r &= 0.06, & \delta &= 0, & N &= 100
\end{align*}
\]
Low risk aversion $\gamma=0.1$

- $\rho=0$
- $\rho=0.6$
- $\rho=0.9$
- $\rho=0.9$ constrained
- Black-Scholes

Higher risk aversion $\gamma=2$

- $\rho=0$
- $\rho=0.6$
- $\rho=0.9$
- $\rho=0.9$ constrained
- Black-Scholes
Short maturity $T=1$

Long maturity $T=10$
Short maturity $T=1$, low risk aversion $\gamma=0.5$

Long maturity $T=10$, low risk aversion $\gamma=0.5$

Short maturity $T=1$, high risk aversion $\gamma=4$

Long maturity $T=10$, high risk aversion $\gamma=4$
Short maturity T=1

Long maturity T=10

- gamma=0.5
- gamma=2
- gamma=8
- constrained
- black-scholes