Investment under uncertainty and competition in incomplete markets

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Séminaire CEPN-LAGA
Université Paris Nord, October 19, 2010

Real options beyond monopoly

- Traditional real option valuation assumes a monopoly right to invest in a project.
- The option value of waiting produces wider price ranges for investment/abandonment.
- This leads to a more conservative attitude than predicted by a NPV approach.
- With competition, the value of waiting should decrease because of opportunity cost.
- How can we incorporate this effect into the real options approach ?
- How does it affect the results ?

"Grief and rage, along with other outbursts of passion, were mistakes easily committed by a mind lacking in refinement. And the Count was certainly not a man who lacked refinement.

Just let matters slide. How much better to accept each sweet drop of the honey that was Time, than to stoop to the vulgarity latent in every decision. However grave the matter at hand might be, if one neglected it for long enough, the act of neglect itself would begin to affect the situation, and someone else would emerge as an ally. Such was Count Ayakura's version of political theory."

Spring Snow, Yukio Mishima

Combining options and games

- A systematic application of both real options and game theory in strategic decisions has been proposed in the literature (see Smit and Trigeorgis (2004) for a review).
- The essential idea can be summarized in two rules:
  1. whenever the outcome of a given game involves a "wait–and–see" strategy, its pay-off should be calculated as the value of a real option;
  2. whenever the pay-off of a given involves a game, its value should calculated as the equilibrium solution to the game.
- In this way, option valuation and game theoretical equilibrium become dynamically related in a decision tree.
- In what follows, we denote the NE solution for a given game in bold face within the matrix of outcomes.
- For convenience of notation we will round all number to the nearest integer.

One–stage investment: single firm

- As a first example, suppose that a single firm can make an investment of $I = 90$ either at $t = 0$ or at $t = 1$.
- Let the underlying project values be $V_0 = 100$ at time $t = 0$, then either $\overline{V}^h = 120$ or $\overline{V}^\ell = 80$ at time $t = 1$ with equal probabilities.
- If V is perfectly correlated with a traded financial asset S, then the option to invest can be valued using standard risk–neutral pricing.
- For a one–period risk–free rate $R = 0.06$, the risk–neutral probability in this case is $q = \frac{(1+R)-\overline{h}}{\overline{h}-\overline{\ell}} = 0.65$.
- If the firm postpones investment until $t = 1$ it realizes an option value $c_0 = 18.40$.
- Since $c_0 \geq V_0 - I = 10$, a firm acting in isolation should postpone the investment.

One–stage investment: two firms

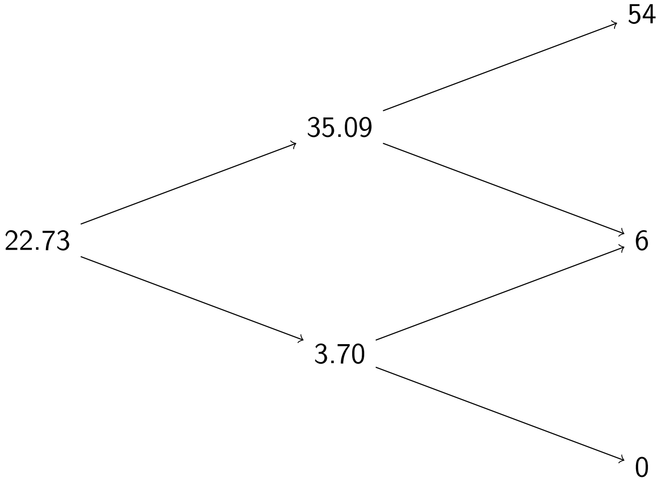
- Suppose now that two symmetric firms A and B face the same investment problem as before.
- Let us assume that if a firm invests in the project alone, then the payoff for the other firm is zero, whereas the payoff is divided equally between them if both firms reach the same decision.
- We then have the following matrix of outcomes:

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(9.20, 9.20)$

- Notice the "prisoner's dilemma" character of this game.

Two–stage investment: one firm

- Using the same setting as in the previous example, let the project value be $V_0 = 100$ at time $t = 0$, then either 120 or 80 at time $t = 1$, and finally either 144, 96, or 64 at time $t = 2$, leading to the following option values :



Two–stage investment: two firms

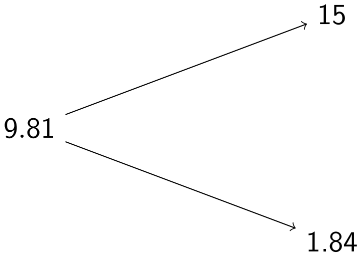
- Suppose now that two firms A and B face the same investment problem as before.
- The games played at time $t = 1$ are:

A \ B	invest	wait
invest	$(\mathbf{15}, \mathbf{15})$	$(30, 0)$
wait	$(0, 30)$	$(17.55, 17.55)$

A \ B	invest	wait
invest	$(-5, -5)$	$(-10, 0)$
wait	$(0, -10)$	$(\mathbf{1.84}, \mathbf{1.84})$

Two–stage investment: two firms (continued)

- Using the previous values to calculate the option value at time $t = 0$ leads to:



- Finally, the game played at time $t = 0$ is:

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(9.81, 9.81)$

Sensitivity to model parameters

- Using $R = 0.1$ leads to the following matrices of outcomes at time $t = 1$:

A \ B	invest	wait
invest	$(\mathbf{15}, \mathbf{15})$	$(30, 0)$
wait	$(0, 30)$	$(19.09, 19.09)$

A \ B	invest	wait
invest	$(-5, -5)$	$(-10, 0)$
wait	$(0, -10)$	$(\mathbf{2.05}, \mathbf{2.05})$

- This results in an option value of 10.69 at time $t = 0$, leading to:

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(\mathbf{10.69}, \mathbf{10.69})$

Incomplete Markets

- Consider the two–factor market where the *discounted* project value V and the *discounted* a correlated traded asset S follow:

$$(S_T, V_T) = \begin{cases} (uS_0, hV_0) & \text{with probability } p_1, \\ (uS_0, \ell V_0) & \text{with probability } p_2, \\ (dS_0, hV_0) & \text{with probability } p_3, \\ (dS_0, \ell V_0) & \text{with probability } p_4, \end{cases} \tag{1}$$

  where $0 < d < 1 < u$ and $0 < \ell < 1 < h$, for positive initial values S_0, V_0 and historical probabilities p_1, p_2, p_3, p_4.
- Let the risk preferences be specified through an exponential utility $U(x) = -e^{-\gamma x}$.
- An investment opportunity is model as an option with *discounted* payoff $C_t = (V - e^{-rt}I)^+$, for $t = 0, T$.

European Indifference Price

- Without the opportunity to invest in the project V, a rational agent with initial wealth x will try to solve the optimization problem
$$u^0(x) = \max_H E[U(X_T^{x,H})], \tag{2}$$
where
$$X_T^{x,H} = \xi + HS_T = x + H(S_T - S_0). \tag{3}$$
is the wealth obtained by keeping ξ dollars in a risk–free cash account and holding H units of the traded asset S.
- An agent with initial wealth x who pays a price π for the opportunity to invest in the project will try to solve the modified optimization problem
$$u^C(x - \pi) = \max_H E[U(X_T^{x-\pi,H} + C_T)] \tag{4}$$
- The *indifference price* for the option to invest in the final period as the amount π^C that solves the equation
$$u^0(x) = u^C(x - \pi). \tag{5}$$

Explicit solution

Denoting the two possible pay-offs at the terminal time by C_h and C_ℓ, the European indifference price defined in (5) is given by

$$\pi^C = g(C_h, C_\ell) \tag{6}$$

where, for fixed parameters $(u, d, p_1, p_2, p_3, p_4)$ the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} g(x_1, x_2) &= \frac{q}{\gamma} \log\left(\frac{p_1 + p_2}{p_1 e^{-\gamma x_1} + p_2 e^{-\gamma x_2}}\right) \\ &\quad + \frac{1-q}{\gamma} \log\left(\frac{p_3 + p_4}{p_3 e^{-\gamma x_1} + p_4 e^{-\gamma x_2}}\right), \end{aligned} \tag{7}$$

with

$$q = \frac{1-d}{u-d}.$$

Early exercise

- When investment at time $t = 0$ is allowed, it is clear that immediate exercise of this option will occur whenever its *exercise value* $(V_0 - I)^+$ is larger than its *continuation value* given by π^C.
- That is, from the point of view of this agent, the value at time zero for the opportunity to invest in the project either at $t = 0$ or $t = T$ is given by

$$C_0 = \max\{(V_0 - I)^+, g((hV_0 - e^{-rT}I)^+, (\ell V_0 - e^{-rT}I)^+)\}.$$

One–period investment revisited

- As a first example, consider again the one–period setting with $I = 90,\ V_0 = 100,\ R = 0.06$.
- For the dynamics of S we choose $u = 1.2/1.06$, $d = 0.8/1.06$ (so that $q = 0.65$ as before) and $p_1 = p_4 = 0.4$, $p_2 = p_3 = 0.1$.
- Finally, let us set $\gamma = 0.01$.
- Therefore, using the function g to calculate the *option value* for the "wait–and–see" strategy, we have the matrix of outcomes for this game shown in Table 15.

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(8.02, 8.02)$

- As expected, the utility–based option value is smaller than the one obtained under risk–neutrality.

Two–period investment revisited

- For the two–period investment game we find

A \ B	invest	wait
invest	$(\mathbf{15}, \mathbf{15})$	$(30, 0)$
wait	$(0, 30)$	$(15.39, 15.39)$

A \ B	invest	wait
invest	$(-5, -5)$	$(-10, 0)$
wait	$(0, -10)$	$(\mathbf{1.66}, \mathbf{1.66})$

- This gives an *indifference* option value of 8.86 at time $t = 0$, leading to

A \ B	invest	wait
invest	$(\mathbf{5}, \mathbf{5})$	$(10, 0)$
wait	$(0, 10)$	$(8.86, 8.86)$

One–period expansion option under monopoly

- Suppose now that a firm faces the decision to expand capacity for a product with uncertain demand:

$$Y_1 = \begin{cases} hY_0 & \text{with probability } p \\ \ell Y_0 & \text{with probability } 1-p \end{cases}, \qquad (8)$$

  correlated with a traded asset
- The expansion requires a discounted sunk cost I.
- The state of the firm after the investment decision at time t_i is

$$x(i) = \begin{cases} 1 & \text{if the firm invests at time } t_i \\ 0 & \text{if the does not invest at time } t_i \end{cases} \qquad (9)$$

- The discounted cash flow per unit demand for the firm is denoted by $D_{x(i)}$.

Definition of project values

- We denote by $V^{(x(i))}(i+1, Y_{i+1})$ the project value at time t_{i+1} given that the state of the firm at time t_i was $x(i)$ and that the firm will act optimally from time t_{i+1} onwards.
- Next, denote by $v^{(x(i))}(i, Y_i)$ the sum of the discounted cash flow from time t_i to t_{i+1} plus the indifference value of the project at time t_{i+1}, that is

$$v^{(x(i))}(i, Y_i) = D_{x(i)} Y_i + g(V^{(x(i))}(i+1, hY_i), V^{(x(i))}(i+1, \ell Y_i))$$

- For simplicity, we assume in this section that the project terminates one period after time t_1 so that

$$v^{(x(1))}(1, Y_1) = D_{x(1)} Y_1.$$

The NPV solution

- Assume first that the decision has to be taken at time t_0.
- If no expansion occurs, then $V^{(0)}(1, Y_1) = D_0 Y_1$ and

$$v^{(0)}(0, Y_0) = D_0 Y_0 + g(D_0 h Y_0, D_0 \ell Y_0).$$

- If expansion occurs, then $V^{(1)}(1, Y_1) = D_1 Y_1$ and

$$v^{(1)}(0, Y_0) = D_1 Y_0 + g(D_1 h Y_0, D_1 \ell Y_0).$$

- Accordingly, the firm should expand provided $v^{(1)} - I \geq v^{(0)}$, that is, provided $Y_0 \geq Y^{NPV}$ where Y^{NPV} solves

$$(D_1 - D_0) y = g(D_0 h y, D_0 \ell y) - g(D_1 h y, D_1 \ell y) + I.$$

The Real Options solution

- Assume now that the decision can be taken either at t_0 or t_1.
- If expansion occurs at t_0, then we still have

$$v^{(1)}(0, Y_0) = D_1 Y_0 + g(D_1 h Y_0, D_1 \ell Y_0).$$

- Conversely, if no expansion occur at t_0, then $V^{(0)}(1, Y_1) = \max\{D_1 Y_1 - I, D_0 Y_1\}$ and

$$v^{(0)}(0, Y_0) = D_0 Y_0 + g(V^{(0)}(1, hY_0), V^{(0)}(1, \ell Y_0)).$$

- Accordingly, the firm should expand provided $Y_0 \geq Y^{RO}$ where Y^{RO} solves

$$\begin{aligned}(D_1 - D_0)y = {} & g(\max\{D_1 hy - I, D_0 hy\}, \max\{D_1 \ell y - I, D_0 \ell y\}) \\ & - g(D_1 hy, D_1 \ell y) + I.\end{aligned}$$

- It is easy to show that $Y^{RO} \geq Y^{NPV}$, so that the firm is less likely to expand at time t_0.

One–period expansion game under duopoly

- Consider now two firms A and B facing the same decision as before.
- The state of the firm m after the investment decision at time t_i is

$$x_m(i) = \begin{cases} 1 & \text{if firm } m \text{ invests at time } t_i \\ 0 & \text{if firm } m \text{ does not invest at time } t_i \end{cases} \tag{10}$$

- Let $D_{x_A(t_i)x_B(t_i)}$ denote the cash–flow per unit of demand of firm A and $D_{x_B(t_i)x_A(t_i)}$ the cash–flow per unit of demand of firm B.
- Assume that $D_{10} > D_{11} > D_{00} > D_{01}$.
- We say that there is FMA is $(D_{10} - D_{00}) > (D_{11} - D_{01})$ and that there is SMA otherwise.

Definition of project values

- $V_m^{(x_A(i),x_B(i))}(i+1, Y_{i+1})$ the value of the project for firm m at time t_{i+1} given that the state of the firms at time t_i was $(x_A(i), x_B(i))$ and assuming that both firms will follow an equilibrium strategy from t_{i+1} onwards.
- Next denote by $v_m^{(x_A(i),x_B(i))}(i, Y_i)$ the sum of the cash–flows for firm m from time t_i to time t_{i+1} with the indifference value of the project at time t_{i+1}, that is

$$v_m^{(x_A(i),x_B(i))}(i,Y_i)=D_{x_{m(i)}x_{m'(i)}}Y_i\Delta t+g\left(V_m^{(x_A(i),x_B(i))}(i+1,hY_i),V_m^{(x_A(i),x_B(i))}(i+1,\ell Y_i)\right),$$

  where $m' = B$ whenever $m = A$ and vice-versa.
- For simplicity, we still assume that the project terminates one period after time t_1 so that

$$v_m^{(x_A(1),x_B(1))}(1, Y_1) = D_{x_{m(1)}x_{m'(1)}}Y_1.$$

NPV analysis

- Assume for now that firm A decides first and firm B observes the decision of A before reaching it own (this will be dropped later !).
- If firm A invests at t_0 we have that

$$v_B^{(1,1)}(0, Y_0) = D_{11} Y_0 + g(D_{11} h Y_0, D_{11} \ell Y_0),$$

  and

$$v_B^{(1,0)}(0, Y_0) = D_{01} Y_0 + g(D_{01} h Y_0, D_{01} \ell Y_0).$$

- Therefore, firm B should also invest provided $Y_0 \geq Y_B^{NPV}$, where Y_B^{VPN} solves

$$(D_{11} - D_{01})y = g(D_{01} hy, D_{01} \ell y) - g(D_{11} hy, D_{11} \ell y) + I$$

- Similarly, if firm A does not invest t_0, then firm B should invest provided $Y_0 \geq Y_A^{NPV}$, where Y_A^{NPV} solves

$$(D_{10} - D_{00})y = g(D_{00} hy, D_{00} \ell y) - g(D_{10} hy, D_{10} \ell y) + I$$

NPV equilibirum

Proposition

Under first mover advantage (FMA) and assuming that the investment decision can only be made at time t_0, we have that $Y_A^{NPV} \leq Y_B^{NPV}$ and:

1. *If $Y_0 \geq Y_B^{NPV}$, then the optimal strategy at time zero is $(x_A(0), x_B(0)) = (1, 1)$.*
2. *If $Y_A^{NPV} \leq Y_0 < Y_B^{NPV}$, then the optimal strategy at time zero is $(x_A(0), x_B(0)) = (1, 0)$.*
3. *If $Y_0 < Y_A^{NPV}$, then the optimal strategy at time zero is $(x_A(0), x_B(0)) = (0, 0)$.*

In other words, under FMA, the demand thresholds for firms A and B are Y_A^{NPV} and Y_B^{NPV}, respectively.

Real Option analysis at time t_1

- Suppose now that both firms can either invest at time t_0 or postpone investment to time t_1 and are perfectly symmetric.
- We start with time t_1, where

$$V_A^{(1,1)}(1, Y_1) = V_B^{(1,1)}(1, Y_1) = D_{11} Y_1 \tag{11}$$

$$V_B^{(1,0)}(1, Y_1) = V_A^{(0,1)}(1, Y_1) = \max\{D_{11} Y_1 - I, D_{01} Y_1\} \tag{12}$$

$$V_A^{(1,0)}(1, Y_1) = V_B^{(0,1)}(1, Y_1) = \begin{cases} D_{11} Y_1 & \text{if } D_{11} Y_1 - I \geq D_{01} Y_1 \\ D_{10} Y_1 & \text{otherwise} \end{cases} . \tag{13}$$

- Finally, the values $V_m^{(0,0)}(1, Y_1)$ corresponds to the game:

A \ B	invest	wait
invest	$(D_{11} Y_1 - I, D_{11} Y_1 - I)$	$(D_{10} Y_1 - I, D_{01} Y_1)$
wait	$(D_{01} Y_1, D_{10} Y_1 - I)$	$(D_{00} Y_1, D_{00} Y_1)$

- When multiple equilibria occur, we select one at random with equal probabilities.

Real Option analysis at time t_0

- The conditional values at time t_0 are

$$v_m^{(1,1)}(0, Y_0) = D_{11} Y_0 + g\left(V_m^{(1,1)}(1, hY_0), V_m^{(1,1)}(1, \ell Y_0)\right)$$

$$v_B^{(1,0)}(0, Y_0) = v_A^{(0,1)}(0, Y_0) = D_{01} Y_0 + g\left(V_B^{(1,0)}(1, hY_0), V_B^{(1,0)}(1, \ell Y_0)\right)$$

$$v_A^{(1,0)}(0, Y_0) = v_B^{(0,1)}(0, Y_0) = D_{10} Y_0 + g\left(V_A^{(1,0)}(1, hY_0), V_A^{(1,0)}(1, \ell Y_0)\right)$$

$$v_m^{(0,0)}(0, Y_0) = D_{00} Y_0 + g\left(V_m^{(0,0)}(1, hY_0), V_m^{(0,0)}(1, \ell Y_0)\right)$$

- Since by definition both firms still have the option to invest at time t_0, they play the game

A \ B	invest	wait
invest	$(v_A^{(1,1)} - I, v_B^{(1,1)} - I)$	$(v_A^{(1,0)} - I, v_B^{(1,0)})$
wait	$(v_A^{(0,1)}, v_B^{(0,1)} - I)$	$(v_A^{(0,0)}, v_B^{(0,0)})$

- Again, when multiple equilibria occur, we select one at random with equal probabilities.

The N–period game

- Consider now a continuous-time model of the form

$$\begin{aligned} dS_t &= (\mu_1 - r)S_t dt + \sigma_1 S_t dW \\ dY_t &= (\mu_2 - r)Y_t dt + \sigma_2 Y_t(\rho dW + \sqrt{1-\rho^2} dZ). \end{aligned}$$

- Next take $\Delta t = \frac{T}{N}$ and

$$p_1 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(\frac{\nu_1}{\sigma_1} + \frac{\nu_2}{\sigma_2}\right)\right] \tag{14}$$

$$p_2 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(\frac{\nu_1}{\sigma_1} - \frac{\nu_2}{\sigma_2}\right)\right] \tag{15}$$

$$p_3 = \frac{1}{4}\left[1 - \rho + \sqrt{\Delta t}\left(-\frac{\nu_1}{\sigma_1} + \frac{\nu_2}{\sigma_2}\right)\right] \tag{16}$$

$$p_4 = \frac{1}{4}\left[1 + \rho + \sqrt{\Delta t}\left(-\frac{\nu_1}{\sigma_1} - \frac{\nu_2}{\sigma_2}\right)\right] \tag{17}$$

$$u = e^{\Delta y_1} = e^{\sigma_1\sqrt{\Delta t}}, \qquad d = 1/u = e^{-\sigma_1\sqrt{\Delta t}} \tag{18}$$

$$h = e^{\Delta y_2} = e^{\sigma_2\sqrt{\Delta t}}, \qquad \ell = 1/h = e^{-\sigma_2\sqrt{\Delta t}}, \tag{19}$$

where $\nu_i = \mu_i - r - \sigma_i^2/2$.

Numerical experiments

- In what follows, we use $I = 200,\ r = 0.03,\ T = 1,\ N = 500.$
- For the dynamics of S_t we choose $\mu_1 = 0.10$ and $\sigma_1 = 0.30$.
- For the demand Y_t we fix $\sigma_2 = 0.20$ and calculate μ_2 as

$$\mu_2 = \overline{\mu}_2 - \delta, \tag{20}$$

  where $\overline{\mu}_2$ is an equilibrium expected rate of return on the non-traded asset and $\delta = 0.04$ is the *below-equilibirum shortfall rate*

- For the equilibrium rate $\overline{\mu}_2$ we use the CAPM relation

$$\lambda = \frac{\mu_1 - r}{\sigma_1} \tag{21}$$

$$\overline{\mu}_2 = r + \lambda\rho\sigma_2 \tag{22}$$

- Finally we consider FMA with $D_{10} = 8,\ D_{00} = 3,\ D_{01} = 0.$

Dependence on risk aversion

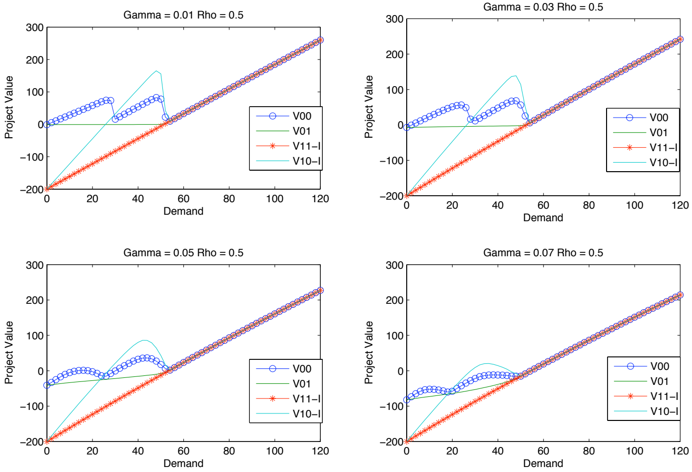


Figure: Project values in FMA case for different risk aversions.

Dependence on correlation.

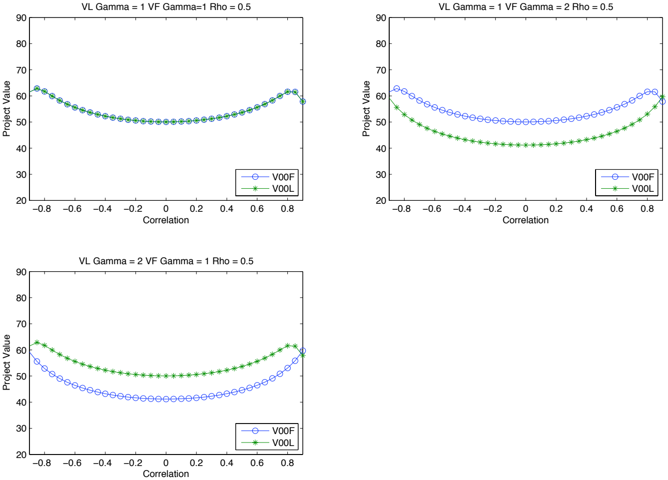


Figure: Project values in FMA case as function of correlation.

Conclusions

- Real options and game theory can be combined in a dynamic framework for decision making under uncertainty and competition.
- The effects of incompleteness and risk aversion can be incorporated using the concept of indifference pricing.
- Analytic expressions for exponential utility lead to numerical schemes with the same computational complexity as a binomial model.
- We have fully implemented a generic example of two firms and uncertain demand and finite maturity in discrete time.
- Continuous–time versions with infinite maturity are also possible (extensions of Grenadier (1996)).
- Much more work is necessary for a large number of firms.
- Merci !