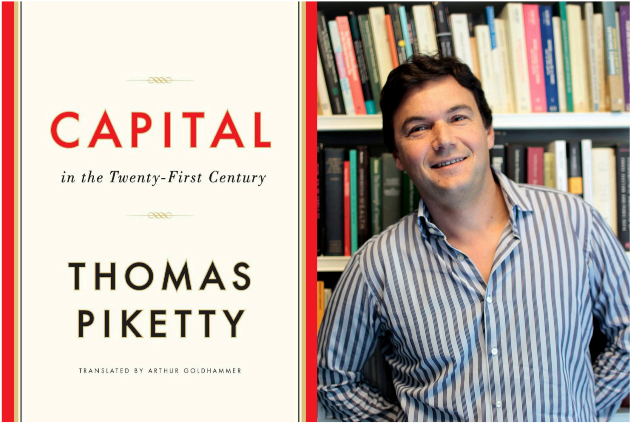
Inequality in a monetary dynamic macroeconomic model

M. R. Grasselli

Mathematics and Statistics, McMaster University
Joint with Gaël Giraud (AFD, CNRS, ENPC)

Seminaire Chaire Énergie et Prospérité, November 21, 2016

The book


CAPITAL
in the Twenty-First Century
THOMAS PIKETTY
TRANSLATED BY ARTHUR GOLDHAMMER

Opening salvo

To put it bluntly, the discipline of economics has yet to get over its childish passion for mathematics and for purely theoretical and often highly ideological speculation, at the expense of historical research and collaboration with the other social sciences.

Piketty's methodology

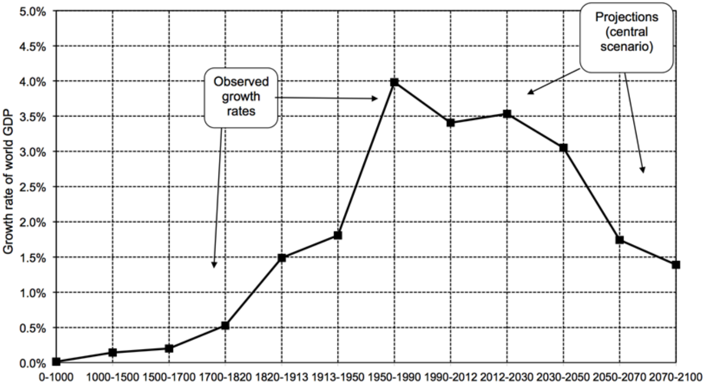
Pikettys model is not a deterministic system from which he attempts to predict all future economic history, but rather a system of interacting mathematical regularities and patterns, **themselves directly measurable from the statistical analysis of historical data**, *intended to give a good match to empirically observed results, and from which we can then make some predictions about the future by extrapolating the most robust trends and incorporating what we know of present economic conditions.* (Dan Kervik, Rugged Egalitarianism)

Key definitions

- $Y_n = (Y_n - W) + W$ (total income equals capital income plus labor income)
- $r_k = \frac{(Y_n - W)}{pK}$ (rate of return on capital)
- $\alpha_k = \frac{Y_n - W}{Y_n}$ (capital share of total income)
- $\beta_k = \frac{pK}{Y_n}$ (capital-to-income ratio)

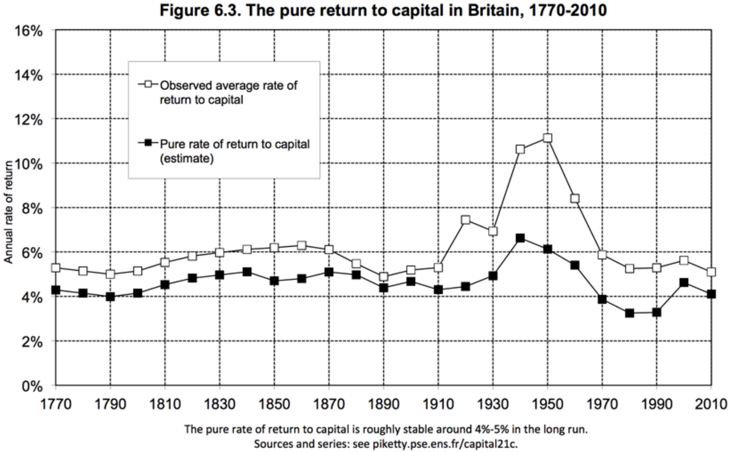
Output growth

Figure 2.5. The growth rate of world output from Antiquity until 2100

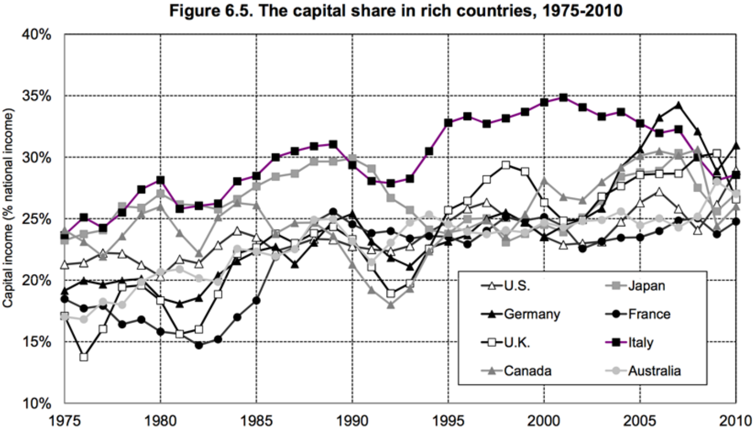


The growth rate of world output surpassed 4% from 1950 to 1990. If the convergence process goes on it will drop below 2% by 2050. Sources and series: see piketty.pse.ens.fr/capital21c.

Rate of return on capital - Britain

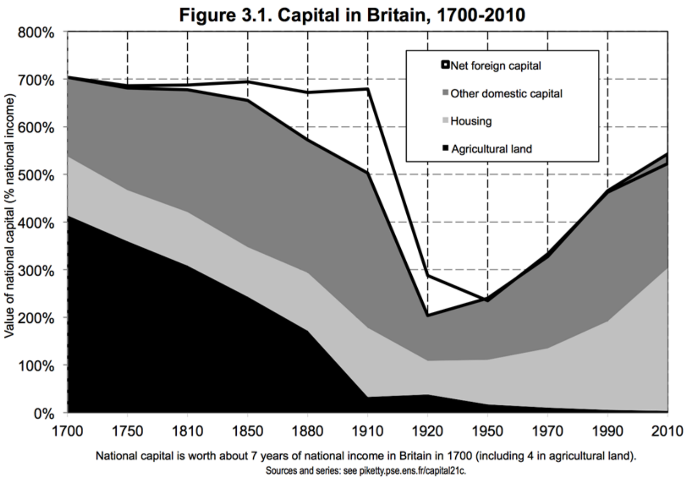


Capital share



Capital income absorbs between 15% and 25% of national income in rich countries in 1970, and between 25% and 30% in 2000-2010. Sources and series: see piketty.pse.ens.fr/capital21c

Capital-to-Income ratio - Britain



The argument in a nutshell

- First Law of Capitalism:

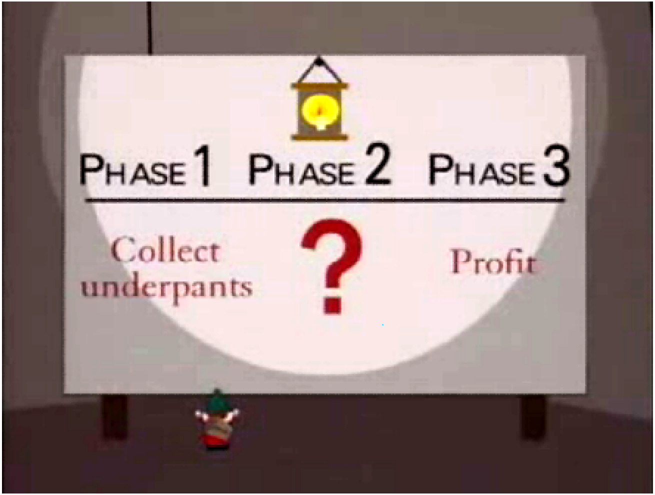
$$\alpha_k = \frac{(Y_n - W)}{Y_n} = \frac{(Y_n - W)}{pK}\frac{pK}{Y_n} = r_k \beta_k$$

- Second Law of Capitalism:

$$\beta_k \to \frac{s}{g}$$

- Therefore, if $r_k > g$, wealth and income inequality tend to increase in time.

Underpants Gnome's Business Plans



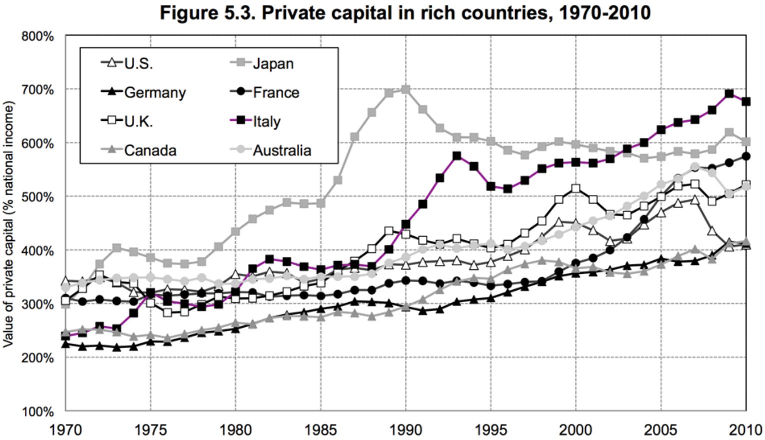
Closing Fanfare

The inequality $r > g$ implies that wealth accumulated in the past grows more rapidly than output and wages. This inequality expresses a fundamental logical contradiction. The entrepreneur inevitably tends to become a rentier, more and more dominant over those who own nothing but their labor. Once constituted , capital reproduces itself faster than output increases. The past devours the future.

Criticisms of Piketty

- Validity of the Second Law of Capitalism
- Stability of the relationship $r_k > g$
- Cambridge Capital Controversies
- Representative Agent
- Nevertheless . . .

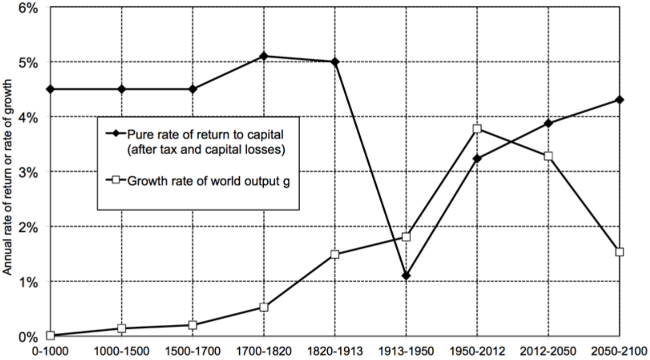
Capital-to-Income - World



Private capital is worth between 2 and 3.5 years of national income in rich countries in 1970, and between 4 and 7 years of national income in 2010. Sources and series: see piketty.pse.ens.fr/capital21c.

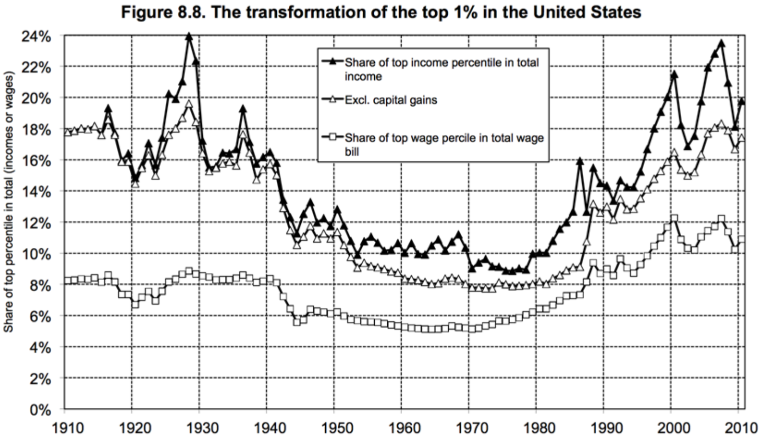
Return on capital versus growth

Figure 10.10. After tax rate of return vs. growth rate at the world level, from Antiquity until 2100



The rate of return to capital (after tax and capital losses) fell below the growth rate during the 20th century, and may again surpass it in the 21st century. Sources and series : see piketty.pse.ens.fr/capital21c

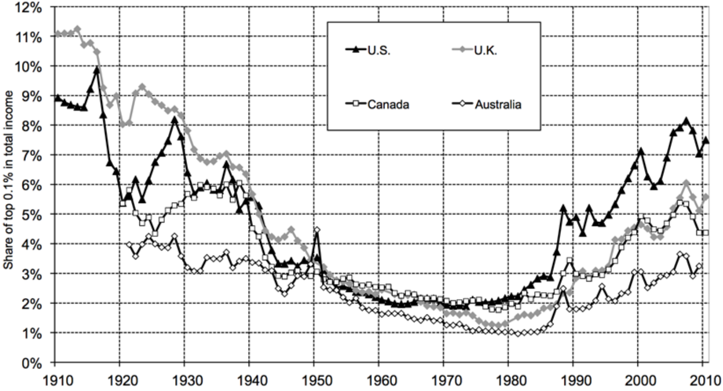
Income inequality - top 1%



The rise in the top 1% highest incomes since the 1970s is largely due to the rise in the top 1% highest wages. Sources and series: see piketty.pse.ens.fr/capital21c.

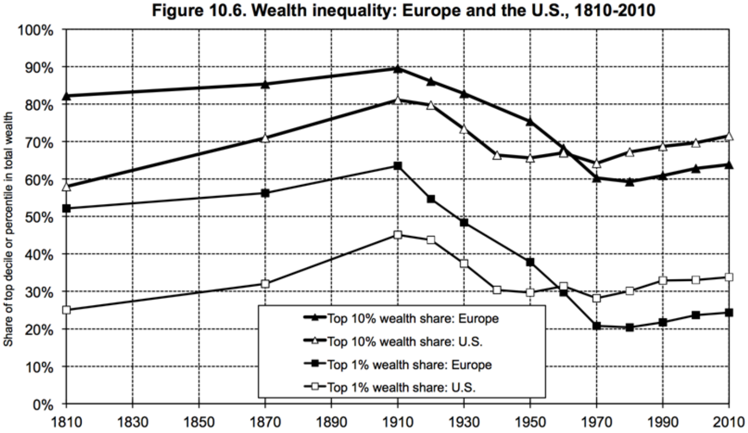
Income inequality - top 0.1%

Figure 9.5. The top 0.1% income share in Anglo-saxon countries, 1910-2010



The share of the top 0.1% highest incomes in total income rose sharply since the 1970s in all Anglo-saxon countries, but with varying magnitudes. Sources and series: see piketty.pse.ens.fr/capital21c.

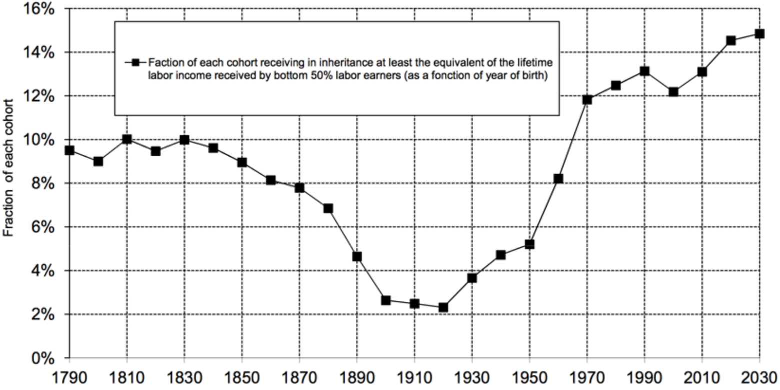
Wealth inequality



Until the mid 20th century, wealth inequality was higher in Europe than in the United States.
Sources and series: see piketty.pse.ens.fr/capital21c.

Inheritance

Figure 11.11. Which fraction of a cohort receives in inheritance the equivalent of a lifetime labor income?



Within the cohorts born around 1970-1980, 12-14% of individuals receive in inheritance the equivalent of the lifetime labor income received by the bottom 50% less well paid workers. Sources and series : see piketty.pse.ens.fr/capital21c

SFC table for the dual Keen model

	Households	Firms (Current)	Firms (Capital)	Banks	Sum
Balance sheet					
Capital stock		$+pK$			pK
Deposits	$+M_h$	$+M_f$		$-(M_h+M_f)$	0
Loans	$-L_h$	$-L_f$		$+(L_h+L_f)$	0
Sum (Net worth)	X_h	X_f		X_b	X
Transactions		Current	Capital		
Consumption	$-pC_h$	$+pC$		$-pC_b$	0
Investment		$+pI$	$-pI$		0
Accounting memo [GDP]		$[pY]$			
Depreciation		$-p\delta K$	$+p\delta K$		0
Wages	$+\mathrm{w}\ell$	$-\mathrm{w}\ell$			0
Interest on loans	$-rL_h$	$-rL_f$		$+r(L_h+L_f)$	0
Interest on deposits	$+rM_h$	$+rM_f$		$-r(M_h+M_f)$	0
Dividends	$+\Delta_b$			$-\Delta_b$	0
Financial balances	S_h	S_f	$-pI+p\delta K$	S_b	0
Flows of funds					
Change in capital stock		$+p(I-\delta K)$			$+p(I-\delta K)$
Change in deposits	$+\dot{M}_h$	$+\dot{M}_f$		$-(\dot{M}_h+\dot{M}_f)$	0
Change in loans	$-\dot{L}_h$	$-\dot{L}_f$		$+(\dot{L}_h+\dot{L}_f)$	0
Column sum	S_h	S_f		S_b	$+p(I-\delta K)$
Change in net worth	$\dot{X}_h=S_h$	$\dot{X}_f=S_f+\dot{p}K$		$\dot{X}_b=S_b$	$\dot{X}=\dot{p}K+p\dot{K}$

Table: SFC table for the dual Keen model.

Dual Keen model - definitions

- Let $D_h = L_h - M_h$ and $D_f = L_f - M_f$ and assume that $\Delta_b = r(D_h + D_f)$ and $C_b = 0$.
- This leads to $S_b = 0$, so we take $X_b = x_0 = 0$, so that $D_h = -D_f$.
- Therefore

$$\begin{aligned}\dot{D}_h &= pC_h - \mathrm{w}\ell + rD_h - r(D_h + D_f) \\ &= pY - pI - \mathrm{w}\ell - rD_f = -\dot{D}_f.\end{aligned}$$

- Denoting $\omega = W/(pY)$, $d_h = D_h/(pY)$, assume that consumption is given be $C := c(\omega - rd)Y$ for a function c of disposable income $(\omega - rd)$.
- Letting $I = Y - C$, we have that

$$\dot{K} = Y - C - \delta K = \left(\frac{1 - c(\omega - rd)}{\nu} - \delta\right) K$$

where $\nu = K/Y$ is a constant capital-to-output ratio.

Dual Keen model - Differential Equations

- Assume further a wage-price dynamics of the form

$$\frac{\dot{w}}{w} = \Phi(\lambda) + \gamma\left(\frac{\dot{p}}{p}\right)$$
$$i(\omega) = \frac{\dot{p}}{p} = \eta_p(m\omega - 1),$$

for a constant mark-up factor $m \geq 1$.

- The model can now be described by the following system

$$\begin{aligned}
\dot{\omega} &= \omega[\Phi(\lambda) - \alpha - (1-\gamma)i(\omega)] \\
\dot{\lambda} &= \lambda\left[\frac{1-c(\omega - rd_h)}{\nu} - (\alpha+\beta+\delta)\right] \\
\dot{d}_h &= d_h\left[r - \frac{1-c(\omega-rd_h)}{\nu} + \delta - i(\omega)\right] + c(\omega - rd_h) - \omega.
\end{aligned}$$

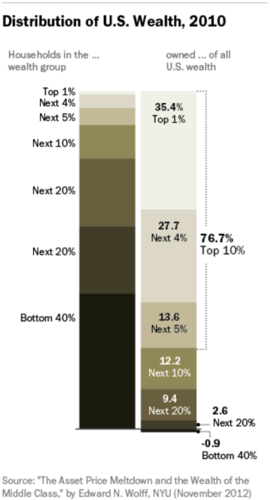
Dual Keen model - Equilibria

- Analogously to the original Keen model, this model exhibits a good equilibrium characterized by

$$
\begin{aligned}
\overline{\omega}_1 &= \eta + r\left[\frac{1-\eta-\nu(\alpha+\beta+\delta)}{\alpha+\beta+i(\overline{\omega}^1)}\right]. \\
\overline{\lambda}_1 &= \Phi^{-1}\big(\alpha+(1-\gamma)i(\overline{\omega}^1)\big). \\
\overline{d}_1 &= \frac{1-\eta-\nu(\alpha+\beta+\delta)}{\alpha+\beta+i(\overline{\omega}^1)},
\end{aligned}
$$

  where $\eta_1 := c^{-1}\big(1-\nu(\alpha+\beta+\delta)\big)$.
- It also exhibits a bad equilibrium of the form $(0, 0, +\infty)$.
- Both equilibria can be locally stable for some parameter values, but *not* at the same time.
- There's also an equilibrium of the form $(\overline{\omega}_3, 0, \overline{d_{h3}})$.

Workers versus investors - motivation



Workers versus investors - modelling

- Consider now two different classes of households, namely workers and investors, with wealth given by

$$X_w = -D_w$$
$$X_i = p^e E - D_i.$$

- It follows from the budget constraint that

$$\begin{aligned}\dot{D}_w &= pC_w - \mathrm{w}\ell + rD_w \\ \dot{D}_i - p^e \dot{E} &= pC_i - r_k pK + rD_i - \Delta_b \\ &= pC_i - r_k pK + rD_i - r(D_f + D_w + D_i) \\ &= pC_i - r_k pK + rD_i.\end{aligned}$$

- Finally, assume that consumption is of the form $C_w = c_w(y_w, x_w)Y$ and $C_i = c_i(y_i, x_i)Y$ with

$$\frac{\partial c_w}{\partial y_w}(\omega - rd_w, x_w) > \frac{\partial c_i}{\partial y_i}(r_k\nu - rd_i, x_i).$$

SFC table for the dual Keen model

	Workers	Investors	Firms		Banks	Sum
Balance sheet						
Capital stock			$+pK$			pK
Deposits	$+M_w$	$+M_i$	$+M_f$		$-(M_w + M_i + M_f)$	0
Loans	$-L_w$	$-L_i$	$-L_f$		$+(L_w + L_i + L_f)$	0
Equities		$+p^e E$	$-p^e E$			0
Sum (Net worth)	X_w	X_i	X_f		X_b	X
Transactions			Current	Capital		
Consumption	$-pC_w$	$-pC_i$	$+pC$		$-pC_b$	0
Investment			$+pI$	$-pI$		0
Accounting memo [GDP]			[pY]			
Wages	$+\mathrm{w}\ell$		$-\mathrm{w}\ell$			0
Depreciation			$-p\delta K$	$+p\delta K$		0
Interest on loans	$-rL_w$	$-rL_i$	$-rL_f$		$+r(L_w + L_i + L_f)$	0
Interest on deposits	$+rM_w$	$+rM_i$	$+rM_f$		$-r(M_w + M_i + M_f)$	0
Dividends		$+r_k pK + \Delta_b$	$-r_k pK$		$-\Delta_b$	0
Financial balances	S_w	S_i	S_f	$-pI + p\delta K$	S_b	0
Flows of funds						
Change in capital stock			$+p(I - \delta K)$			$p(I - \delta K)$
Change in deposits	$+\dot{M}_w$	$+\dot{M}_i$	$+\dot{M}_f$		$-(\dot{M}_w + \dot{M}_i + \dot{M}_f)$	0
Change in loans	$-\dot{L}_w$	$-\dot{L}_i$	$-\dot{L}_i$		$+(\dot{L}_w + \dot{L}_i + \dot{L}_f)$	0
Change in equities		$+p^e \dot{E}$	$-p^e \dot{E}$			0
Column sum	S_w	S_i	S_f		S_b	$p(I - \delta K)$
Change in net worth	$\dot{X}_w = S_w$	$\dot{X}_i = S_i + \dot{p}^e E$	$\dot{X}_f = S_f - \dot{p}^e E + \dot{p}K$		$\dot{X}_b = S_b$	$\dot{X} = \dot{p}K + p\dot{K}$

Table: SFC table for the workers and investors model.

Return on capital and external financing

- We assume the firms retain profits according to a constant retention rate Θ, leading to an endogenous return on capital given by

$$r_k := r_k(\omega, d_w, d_i) = \frac{\Theta(pY - \mathrm{w}\ell - rD_f - p\delta K)}{pK}$$
$$= \frac{\Theta}{\nu}\left(1 - \omega + r(d_w + d_i) - \delta\nu\right),$$

- Savings by the firms are then given by

$$S_f = (1-\Theta)(pY - \mathrm{w}\ell - rD_f - p\delta K) = pY - \mathrm{w}\ell - rD_f - p\delta K - r_k pK$$

- Therefore, the amount to be raised externally by firms is

$$p(I - \delta K) - S_f = pI - pY + \mathrm{w}\ell + rD_f + r_k pK$$
$$= \left(\omega - r(d_i + d_w) - c + r_k\nu\right)pY,$$

- We assume that a fraction ϖ of this amount is raised from new debt and $(1 - \varpi)$ from new equities.

Equity market equilibrium

- Assume further that investors allocate a fraction φ of their wealth to equities and a fraction $(1-\varphi)$ to net deposits so that $p^e E = \frac{-\varphi}{1-\varphi} D_i$
- It then follows that

$$\frac{\dot{p}^e}{p^e} = \frac{\dot{D}_i}{D_i} - \frac{\dot{E}}{E}.$$

- Inserting the supply of equities from firms into the savings equation for investors gives

$$\frac{\dot{D}_i}{D_i} = \frac{c_i - r_k\nu + rd_i + (1-\varpi)\big(\omega - r(d_i + d_w) - c + r_k\nu\big)}{d_i}.$$

- On the other hand

$$\frac{\dot{E}}{E} = \frac{p^e\dot{E}}{p^e E} = -\frac{(1-\varpi)(1-\varphi)\big(\omega - r(d_i + d_w) - c + r_k\nu\big)}{\varphi d_i},$$

from which we can find $\frac{\dot{p}^e}{p^e}$.

The main dynamical system

- Define

$$c(\omega, d_w, d_i) = c_w(\omega - rd_w, -d_w) + c_i(r_k\nu + rd_w, -(1-\phi)d_i),$$

- We then get

$$\begin{aligned}
\dot{\omega} &= \omega[\Phi(\lambda) - \alpha - (1-\gamma)i] \\
\dot{\lambda} &= \lambda\left[\tfrac{1-c}{\nu} - (\alpha+\beta+\delta)\right] \\
\dot{d}_w &= d_w\left[r + \delta - \tfrac{1-c}{\nu} - i\right] + c_w - \omega \\
\dot{d}_i &= d_i\left[r\varpi(1-\Theta) + \delta - \tfrac{1-c}{\nu} - i\right] + c_i - \varpi\Theta(1-\omega) \\
&\quad -\big(1 - \varpi(1-\Theta)\big) rd_w + (1-\varpi)(\omega - c) + \varpi\Theta\delta\nu
\end{aligned}$$

Equilibria

- With considerable more work, it is possible to show that the system exhibits a class of good equilibria of the form $(\overline{\omega}_1, \overline{\lambda}_1, \overline{d}_{w1}, \overline{d}_{i1})$ typically (but not always) satisfying $\overline{d}_{w1} > 0$ and $\overline{d}_{i1} < 0$.
- In addition, the system admits a class of bad equilibria to the form $(\overline{\omega}_2, \overline{\lambda}_2, \overline{d}_{w2}, \overline{d}_{i2}) = (0, 0, +\infty, \pm\infty)$
- Finally, it also exhibits equilibria of the form $(\overline{\omega}_3, 0, \overline{d}_{w3}, \overline{d}_{i3})$, where $\overline{d}_{w3}$ and $\overline{d}_{i3}$ can be either finite of infinite.

Long-run inequality

- The growth rate of real net income $(\omega - rd_w)Y$ for workers is given by

$$g_w = \frac{(\dot{\omega - rd_w})}{\omega - rd_w} + \frac{\dot{Y}}{Y}.$$

- The growth rate of real net income $(r_k\nu - rd_i)Y$ for investors is

$$g_i = \frac{(\dot{r_k\nu - rd_i})}{r_k\nu - rd_i} + \frac{\dot{Y}}{Y}.$$

- At the good equilibrium, both rates equal $\alpha + \beta$ and the income ratio for the two classes converge to a constant.
- At the bad equilibria, on the other hand, it is clear that both classes of households have zero income asymptotically (since $Y \to 0$), BUT the ratio of capital income to labour income goes to infinity.

Endogenous portfolio change

- One way to generalize the model is to assume that

$$\dot{\varphi} = \mu\left(\overline{\varphi}(r^e) - \varphi\right) \qquad \overline{\varphi}' > 0, \mu > 0$$

where $\overline{\varphi}(\cdot)$ is the desired share of equity and r^e is the expected rate of return on equity.

- Furthermore, assume that expectations are adaptive, namely,

$$\dot{r}^e = \rho(r_e - r_e^e) \qquad \rho > 0,$$

where r_e is the current rate of return on equity, namely

$$r_e = \frac{r_k pK}{p^e E} + \frac{\dot{p}^e}{p^e}.$$

- We expect that, similarly to the introduction of Ponzi speculation in the Keen model, this reduces the basin of attraction for the good equilibrium.

Concluding remarks

- We provided a stock-flow consistent model for debt dynamics of workers and investors.
- When the economy converges to an equilibrium with finite debt ratios, the income ratio between the two classes is constant.
- Increasing income (and wealth) inequality is a signature of convergence to the bad equilibrium with infinite debt ratios.
- In future work we explore the effects of default and of migration between classes a la Acemoglu (2014).
- MERCI!