Stock loans in incomplete markets

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Definitions

- A stock loan is a contract between a bank and a client.
- The client borrows an amount L at t_0 and leaves one share with current market value V_0 as collateral.
- At any time t before maturity T the client can redeem the stock by repaying the amount $e^{\alpha(t-t_0)}L$.
- The bank collects any dividends paid by the stock for the duration of the loan.
- The client pays a one off fee c for the loan at t_0.

Risk–neutral valuation

- In Xia and Zhou (2007), the loan repayment is modeled as a perpetual American option with a time varying strike $e^{\alpha(t-t_0)}L$.
- Denoting the price of this option by C_t, the fair values for the loan parameters at time t_0 are related by

$$c = L + C_{t_0} - V_{t_0}. \tag{1}$$

- They were then able to obtain explicit expressions for C_{t_0} using probabilistic methods in standard Black–Scholes framework.

Market Incompleteness

- The risk–neutral paradigm implicit assumes that the option can be replicated by trading in the underlying stock and the money market.
- This is plausible from the bank's point of view, but arguable for the client.
- If the client had unrestricted access to the money market, he would not have to post collateral in the form of a stock.
- If the client could freely trade the stock, he should simply sell it instead of taking the loan.
- Presumably the client faces selling restrictions, while at the same time being in need of available funds to attend to another financial operation.
- Moreover, the risk neutral price yields the fair price at which the option itself can be traded in the market without introducing arbitrage opportunities.
- But a stock loan typically cannot be sold or bought in a secondary market once it is initiated.

Model set up

- We consider two correlated assets S and V with *discounted* prices given by

$$\begin{aligned} dS_t &= (\mu_1 - r)S_t dt + \sigma_1 S_t dW_t^1 \\ dV_t &= (\mu_2 - r)V_t dt + \sigma_2 V_t(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2), \end{aligned} \tag{2}$$

- The client can hold H_t units of the asset S_t and investing the remaining of his wealth in a bank account $B_t = e^{r(t-t_0)}$.
- His discounted wealth then satisfies

$$dX_t^\pi = \pi_t(\mu_1 - r)dt + \pi_t \sigma_1 dW_t^1, \quad t_0 \le t \le T, \tag{3}$$

where $\pi_t = H_t S_t$.

- The client is a risk–averse economic agent with exponential utility function $U(x) = -e^{-\gamma x}$.

Problem formulation

- At t_0, the client borrows an amount L from the bank leaving V_{t_0} as a collateral and pays a fee c.
- The bank collects the dividends at a rate δ for the duration of the loan.
- The client can redeem the asset with value $e^{r(t-t_0)}V_t$ at time $t \leq T$ by paying an amount $e^{\alpha(t-t_0)}L$.
- At the maturity time T, the client needs to decide between repaying the loan or forfeiting the underlying asset indefinitely.
- We want to compute the indifference value p_{t_0} for the repayment option as well as the optimal repayment strategy.
- Based on that, we can calculate the cost C_{t_0} of this option for the bank.
- As before, the loan parameters are then related by

$$c = L + C_{t_0} - V_{t_0} \tag{4}$$

Part I – Infinite maturity

- Let $T = \infty$ and that $\alpha = r$.
- Having taken the loan at time t_0, we assume that the borrower needs to solve the following optimization problem:

$$G(x,v) = \sup_{(\tau,\pi)\in\mathcal{A}} \mathbb{E}_{x,v}\Big[- e^{\frac{(\mu_1 - r)^2}{2\sigma^2}\tau} e^{-\gamma(X^\pi_\tau + (V_\tau - L)^+)} \Big].$$

- Here $\mathcal{A}$ is a set of admissible pairs (τ, π), where $\tau \in [0, \infty]$ is a stopping time and π is a portfolio process.
- Because of time–homogeneity, the borrower should decide to pay back the loan at the first time that V reaches a stationary threshold V^*, that is

$$\tau^* = \inf\{s \geq t_0 : V_s = V^*\}.$$

- We follow Hodges and Neuberger (1989) and define the indifference value for the option to pay back the loan as the amount $p(v)$ satisfying

$$G(x, 0) = G(x - p(v), v). \quad (5)$$

The Henderson (2007) solution

- Let $\beta = 1 - \frac{2}{\sigma_2}\left(\frac{\mu_2 - r}{\sigma_2} - \rho\frac{\mu_1 - r}{\sigma_1}\right)$. If $\beta > 0$, the threshold $V^* > L$ is the unique solution to

$$V^* - L = \frac{1}{\gamma(1-\rho^2)} \log\left[1 + \frac{\gamma(1-\rho^2)V^*}{\beta}\right] \qquad (6)$$

and

$$G(x,v) = \begin{cases} -e^{-\gamma x}\left[1 - (1 - e^{-\gamma(V^*-L)(1-\rho^2)})\left(\frac{v}{V^*}\right)^{\beta}\right]^{\frac{1}{1-\rho^2}}, v < V^* \\ -e^{\gamma x}e^{-\gamma(v-L)}, v \geq V^*. \end{cases} \qquad (7)$$

- In this case, the indifference value $p(v)$ is given by

$$p(v) = \begin{cases} -\frac{1}{\gamma(1-\rho^2)}\log\left[(e^{-\gamma(V^*-L)(1-\rho^2)} - 1)\left(\frac{v}{V^*}\right)^{\beta} + 1\right], v < V^* \\ (v - L), v \geq V^*. \end{cases} \qquad (8)$$

- Alternatively, if $\beta \leq 0$, then $V^* = \infty$ and the option to repay the loan is never exercised.

Cost for the bank

- Assume that S is the discounted price of the market portfolio.
- It follows from CAPM that

$$\frac{\overline{\mu}_2 - r}{\sigma_2} = \rho \frac{\mu_1 - r}{\sigma_1}, \tag{9}$$

  where $\overline{\mu}_2$ is the equilibrium rate of return on the asset V.
- The dividend rate paid by V is then $\delta = \overline{\mu}_2 - \mu_2$ and

$$\beta = 1 - \frac{2}{\sigma_2}\left(\frac{\mu_2 - r}{\sigma_2} - \rho \frac{\mu_1 - r}{\sigma_1}\right) = 1 + \frac{2\delta}{\sigma_2^2} > 0. \tag{10}$$

- Proposition

  Assuming that the borrower exercises the repayment option optimally. Then the cost of this option for the bank is given by

$$C(v) = \begin{cases} (V^* - L)\mathbb{E}^Q\left[\mathbf{1}_{\{\tau^* < \infty\}}\right] = (V^* - L)\left(\frac{v}{V^*}\right)^{\beta}, v < V^* \\ v - L, \qquad\qquad v \geq V^* \end{cases}$$

Loan fee

- We can now use (1) and the previous proposition to determine the loan fee c.

- Proposition

The loan fee:

1. *decreases as the risk aversion γ increases;*
2. *decreases as the dividend rate δ increases;*
3. *increases as ρ^2 increases.*

Moreover, its limiting values either as $\rho^2 \to 1$ or $\gamma \to 0$ coincide and are given by

$$c = \begin{cases} L + (\widetilde{V} - L)\left(\frac{V_{t_0}}{\widetilde{V}}\right)^{\beta} - V_{t_0}, & \text{if} \quad V_{t_0} < V^* \\ 0, & \text{if} \quad V_{t_0} \geq V^*. \end{cases} \tag{11}$$

where $\widetilde{V} = \frac{\beta}{\beta - 1} L = \left(1 + \frac{\sigma_2^2}{2\delta}\right) L.$

Numerical Examples

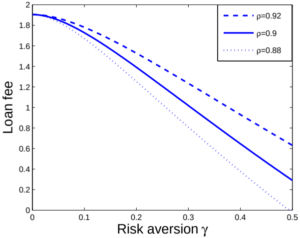
Table: Loan fee c as for different loan amounts L (infinite maturity)

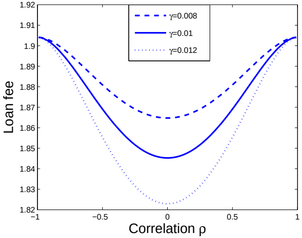
L		50	60	70	80	90	100	110	120
Case 1	c	50	60	70	80	90	100	110	120
Case 2	c	31	40	48	57	66	75	84	93
	V^*	264	293	320	346	370	394	417	440
Case 3	c	0	0	0	0	2	7	15	23
	a_0	61	74	86	98	110	122	135	147
Case 4	c	0	0	0	0	2	7	15	23
	V^*	61	73	85	98	110	122	134	146

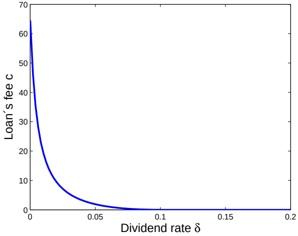
1. (complete) $\sigma_2 = 0.15, \delta = 0, r = \alpha = 0.05, V_0 = 100.$
2. (incomplete) $\sigma_2 = 0.15, \delta = 0, r = \alpha = 0.05,\ V_0 = 100, \rho = 0.9, \gamma = 0.01.$
3. (complete) $\sigma_2 = 0.15, \delta = 0.05, r = \alpha = 0.05, V_0 = 100.$
4. (incomplete) $\sigma_2 = 0.15, \delta = 0.05, r = \alpha = 0.05,\ V_0 = 100, \rho = 0.9, \gamma = 0.01.$

Fee behavior

For the next figure, σ_2=0.15, δ=0.05, $r = \alpha = 0.05$, L=90, $V_0 = 100$, $\rho = 0.9$ and $\gamma = 0.01$.







Part II - Finite maturity

- Let $T < \infty$ and define

$$M(t,x) = \sup_{\pi \in \mathcal{A}_{[t,T]}} \mathbb{E}[-e^{-\gamma X_T^\pi} | X_t^\pi = x] = -e^{-\gamma x} e^{-\frac{(\mu_1 - r)^2}{2\sigma^2}(T-t)},$$

- The borrower now needs to solve:

$$u(t_0, x, v) = \sup_\tau \sup_\pi \mathbb{E}_{x,v}[M(\tau, X_\tau^\pi + (V_\tau - e^{(\alpha - r)(\tau - t_0)} L)^+)].$$

- The indifference value for the repayment option is p satisfying

$$M(t_0, x) = u(t_0, x - p, v).$$

The free boundary problem

- It follows from DP that u solves

$$\begin{cases} \dfrac{\partial u}{\partial t} + \sup_{\pi} \mathcal{L}^{\pi} u \leq 0, \\ u(t,x,v) \geq \Lambda(t,x,v), \\ \left(\dfrac{\partial u}{\partial t} + \sup_{\pi} \mathcal{L}^{\pi} u \right) \cdot (u - \Lambda) = 0, \end{cases} \tag{12}$$

- Here $\mathcal{L}^{\pi}$ is the infinitesimal generator of (X^{π}, V) and

$$\Lambda(t,x,v) = M(t, x + (v - e^{(\alpha - r)(t - t_0)} L)^{+})$$

  is the utility obtained from exercising the repayment option at time t.

- The boundary conditions are

$$\begin{aligned} u(T,x,v) &= -e^{-\gamma[x + (v - e^{(\alpha - r)(T - t_0)} L)^{+}]} \\ u(t,x,0) &= -e^{-\gamma x} e^{-\frac{(\mu_1 - r)^2}{2\sigma^2}(T - t)}. \end{aligned} \tag{13}$$

The Zariphopoulou transformation

- Use the factorization
$$u(t,x,v) = M(t,x)F(t,v)^{\frac{1}{1-\rho^2}}. \tag{14}$$
- The problem for F becomes
$$\begin{cases} \dfrac{\partial F}{\partial t} + \mathcal{L}^0 F \geq 0, \\ F(t,v) \leq \kappa(t,v), \\ \left(\dfrac{\partial F}{\partial t} + \mathcal{L}^0 F\right) \cdot (F-\kappa) = 0, \end{cases} \tag{15}$$
- Here
$$\mathcal{L}^0 = \left[\mu_2 - r - \rho\frac{\mu_1 - r}{\sigma_1}\sigma_2\right] v\frac{\partial}{\partial v} + \frac{\sigma_2^2 v^2}{2}\frac{\partial^2}{\partial v^2}$$
and
$$\kappa(t,v) = e^{-\gamma(1-\rho^2)(v-e^{(\alpha-r)(t-t_0)}L)^+}. \tag{16}$$
- The boundary conditions for Problem (15) are
$$F(T,v) = e^{-\gamma(1-\rho^2)(v-e^{(\alpha-r)(T-t_0)}L)^+} \qquad F(t,0) = 1.$$

Optimal exercise

- Since problem (15) is independent of X and S, we define the borrower's optimal exercise boundary as the function

$$V^*(t) = \inf \{v \geq 0 : F(t,v) = \kappa(t,v)\} \tag{17}$$

and the optimal repayment time as

$$\tau^* = \inf \{t_0 \leq t \leq T : V_t = V^*(t)\}. \tag{18}$$

- It follows from the definition (13) and the factorization (14) that the indifference value for the repayment option is given by $p = p(t_0, V_{t_0})$ where

$$p(t,v) = -\frac{1}{\gamma(1-\rho^2)} \log F(t,v). \tag{19}$$

Cost for the bank

- Once we find $V^*(t)$, we can calculate the cost for the bank as

$$C_{t_0} = E_v^Q \left[e^{-r(\tau - t_0)} \left(e^{r(\tau - t_0)} V^*(t) - e^{\alpha(\tau - t_0)} L \right)^+ \mathbf{1}_{\{\tau^* < \infty\}} \right]$$
$$= E_v^Q \left[e^{-\widehat{r}(\tau - t_0)} \left(\widehat{V}^*(t) - L \right)^+ \mathbf{1}_{\{\tau^* < \infty\}} \right]$$

where $\widehat{r} = r - \alpha$ and $\widehat{V}^*(t) = e^{\widehat{r}(\tau - t_0)} V^*(t)$.

- Denoting $\widehat{V}_t = e^{(r-\alpha)(\tau - t_0)} V_t$, we have

$$\tau^* = \inf \{ t : V_t = V^*(t) \} = \inf \left\{ t : \widehat{V}_t = \widehat{V}^*(t) \right\} \qquad (20)$$

- Therefore $C(t, v)$ satisfies the Black–Scholes PDE

$$\frac{\partial C}{\partial t} + (r - \alpha - \delta) v \frac{\partial C}{\partial v} + \frac{\sigma_2^2 v^2}{2} \frac{\partial^2 C}{\partial v^2} = (r - \alpha) C \qquad (21)$$

with boundary conditions

$$C(t, 0) = 0, \qquad C(t, \widehat{V}^*(t)) = (\widehat{V}^*(t) - L)^+,$$
$$C(T, v) = (v - L)^+, \qquad 0 \le v \le \widehat{V}^*(T)$$

Properties of the fee

- We now fix r, μ_1, σ_1,α, and L and vary γ, δ, ρ, and σ_2.
- Observe that μ_2 is given by the CAPM condition as

$$\mu_2 = \rho \frac{\mu_1 - r}{\sigma_1} \sigma_2 + r - \delta. \tag{22}$$

- Using the same technique as Leung and Sircar (2009) we have:

- Proposition

  The loan fee c:

  1. *decreases as the risk aversion γ increases;*
  2. *decreases as the dividend rate δ increases;*
  3. *increases as ρ^2 increases;*

- Proposition

  If $\alpha = r$, the loan fee is an increasing function of the maturity T.

Numerical results

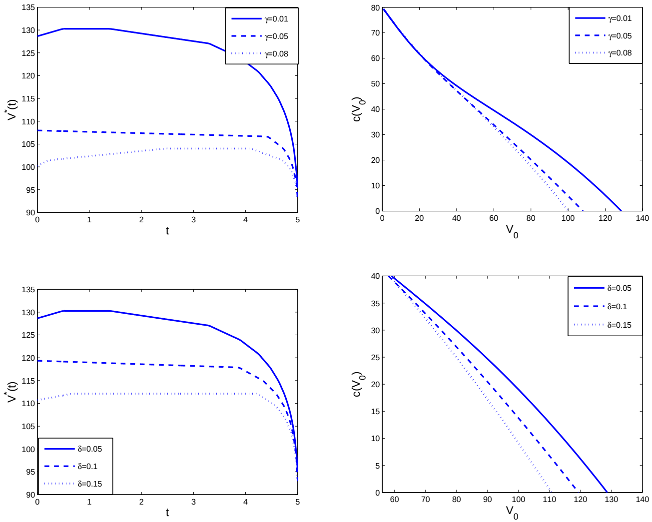
- We first we use finite differences with projected successive–over–relaxation (PSOR) to solve the linear free boundary problem (15).
- This yields a threshold function $V^*(t)$, which we then use to solve equation (21) subject to the boundary conditions (17), again by finite differences.
- For the next table we use $\sigma_2 = 0.4, \rho = 0.4, \gamma = 0.01, \delta = 0.05, r = 0.05, \alpha = 0.07, V_{t_0} = 100$ and $T = 5$ (in years).

Table: Loan fee c for different loan amounts L (finite maturity)

L	50	60	70	80	90	100	110	120
c	0	0	0	1	4	9	16	24

Fee behavior

For the next figure we use $T = 5$, $L = 80$, $\sigma_2 = 0.4$, $r = 0.05$, $\alpha = 0.07$, $\delta = 0.05$, $\rho = 0.4$ and $V_0 = 100$.



Fee behavior (continued)

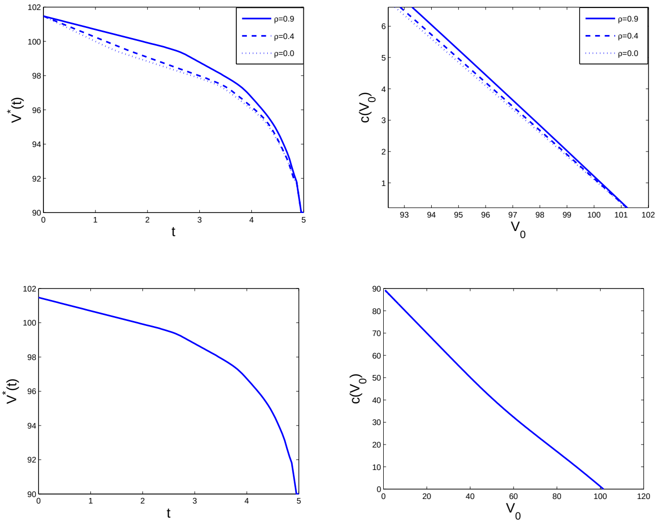


Figure: Dependence on model parameters for finite maturity

Concluding remarks

- We have extended the analysis of Xia and Zhou (2007) for stock loans in incomplete markets.
- An explicit expression for the loan fee can still be found in the infinite–horizon case provided $r = \alpha$.
- In the finite–horizon case, the loan fee can be characterized in terms of a free–boundary problem and calculated numerically.
- In both cases, we analyzed how the loan fee depends on the underlying model parameters.
- We found that the complete–market, risk–neutral value of a stock loan is an upper bound for the fee to be charged by the bank.
- By following our model a bank can quantify the effects of the restrictions faced by the client and charge a smaller fee for the loan, presumably increasing its competitiveness.
- Thank you !