Chaotic Interest Rate Model Calibration

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Axiomatic Interest Rate Theory

We follow the axiomatic framework proposed by Hughston and Rafailidis (2005). For this we need:

- a probability space $(\Omega, \mathcal{F}, P)$ (physical measure);
- the augmented filtration $\mathcal{F}_t$ generated by a k–dimensional Brownian motion W_t;
- asset prices S_t given by continuous semimartingales;
- a non–dividend–paying asset with adapted price process $\xi_t > 0$ (natural numeraire).

Axiomatic Interest Rate Theory (continued)

The following axioms define an arbitrage–free interest rate model:

1. There exists a strictly increasing asset with absolutely continuous price process B_t (bank account).
2. If S_t is the price of any asset with an adapted dividend rate D_t then
$$\frac{S_t}{\xi_t} + \int_0^t \frac{D_s}{\xi_s} ds \qquad \text{is a martingale} \tag{1}$$
3. There exists an asset that offers a dividend rate sufficient to ensure that the value of the asset remains constant (floating rate note).
4. There exists a system of discount bond price processes P_{tT} satisfying
$$\lim_{T \to \infty} P_{tT} = 0.$$

The state price density

- Define $V_t = 1/\xi_t$ (state price density).
- Since $B_t V_t$ is a martingale (A2) and B_t is strictly increasing (A1), we have

$$E_t[V_T] = E_t\left[\frac{B_T V_T}{B_T}\right] < E_t\left[\frac{B_T V_T}{B_t}\right] = \frac{B_t V_t}{B_t} = V_t,$$

  which means that V_t is a positive supermartingale.
- Writing $B_t = B_0 \exp\left(\int_0^t r_s ds\right)$ for an adapted process $r_t > 0$ and

$$d(B_t V_t) = -(B_t V_t)\lambda_t dW_t,$$

  for an adapted vector process λ_t, we have that the dynamics for V_t is

$$dV_t = -r_t V_t dt - V_t \lambda_t dW_t. \qquad (2)$$

Conditional variance representation

- Integrating (2), taking conditional expectations and the limit $T \to \infty$ (all well defined thanks to (A3) and (A4)) leads to

$$V_t = E_t \left[\int_t^\infty r_s V_s ds \right].$$

- Now let σ_t be a vector process satisfying $\sigma_t^2 = r_t V_t$ and define the square integrable random variable

$$X_\infty := \int_0^\infty \sigma_s dW_s.$$

- It then follows from the Ito isometry that

$$V_t = E_t \left[(X_\infty - X_t)^2 \right], \qquad (3)$$

where $X_t := E_t[X_\infty] = \int_0^t \sigma_s dW_s$.

Related quantities and bond prices

- Defining $A_t := [X,X]_t = \int_0^t \sigma_s^2 ds = \int_0^t r_s V_s ds$ leads to the Doob-Meyer decomposition

$$V_t = E_t[A_\infty] - A_t \qquad \text{(potential approach)}$$

- Defining the family of martingales $M_{ts} = E_t[\sigma_s^2]$ leads to

$$V_t = \int_t^\infty M_{ts} ds \qquad \text{(Flesaker–Hughston approach)}$$

- In general, bond prices and forward rates are given by

$$P_{tT} = \frac{E_t[V_T]}{V_t} = \frac{\int_T^\infty M_{ts} ds}{\int_t^\infty M_{ts} ds} \tag{4}$$

$$f_{tT} = -\partial_T \log P_{tT} = \frac{M_{tT}}{\int_T^\infty M_{ts} ds}, \tag{5}$$

which are manifestly positive.

Wiener chaos

- Define the Hermite polynomials

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{6}$$

- For $h \in L^2(\mathbb{R}_+^k)$, define the Gaussian random variable

$$W(h) := \int_0^\infty h(s) dW_s.$$

- Then the Wiener chaos of order n,

$$\begin{aligned} \mathcal{H}_n &:= \mathrm{span}\{H_n(W(h)) | h \in L^2(\Delta)\}, \quad n \geq 1, \\ \mathcal{H}_0 &:= \mathbb{C}, \end{aligned}$$

  provide an orthogonal decomposition of square integrable random variables:

$$L^2(\Omega, \mathcal{F}_\infty, P) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Wiener chaos expansion

- Let $\Delta_n := \{(s_1, \ldots, s_n) \in \mathbb{R}_+^n | 0 \leq s_n \leq \cdots \leq s_2 < s_1 \leq \infty\}$.
- Each $\mathcal{H}_n$ can be identified with $L^2(\Delta_n)$ via the isometries

$$J_n : L^2(\Delta_n) \to \mathcal{H}_n$$

given by

$$\phi_n \mapsto J_n(\phi_n) = \int_{\Delta_n} \phi_n(s_1, \ldots, s_n) dW_{s_1} \ldots dW_{s_n}, \tag{7}$$

- With these ingredients, one is then led to the result that any $X \in L^2(\Omega, \mathcal{F}_\infty, P)$ can be represented as a Wiener chaos expansion

$$X = \sum_{n=0}^{\infty} J_n(\phi_n), \tag{8}$$

where the deterministic functions $\phi_n \in L^2(\Delta_n)$ are uniquely determined by the random variable X.

First order chaos

- In a first order chaos model we have

$$X_\infty = \int_0^\infty \phi(s) dW_s.$$

- In this case $\sigma_s = \phi(s)$, so that $M_{ts} = E_t[\sigma_s^2] = \phi^2(s)$ and

$$V_t = \int_t^\infty M_{ts} ds = \int_t^\infty \phi^2(s) ds$$

- This corresponds to a deterministic interest rate theory, since

$$P_{tT} = \frac{\int_T^\infty \phi^2(s) ds}{\int_t^\infty \phi^2(s) ds}, \quad f_{tT} = \frac{\phi^2(T)}{\int_T^\infty \phi^2(s) ds} = r_T.$$

- The remaining asset prices can be stochastic, however. Indeed, for a derivative with payoff H_T we have

$$H_t = \frac{E_t[V_T H_T]}{V_t} = \frac{V_T}{V_t} E_t[H_T] = P_{tT} E_t[H_T]$$

Second order chaos: definition

- In a second order chaos model we have

$$X_\infty = \int_0^\infty \phi_1(s) dW_s + \int_0^\infty \int_0^s \phi_2(s,u) dW_u dW_s$$

- In this case $M_{ts} = E_t[\sigma_s^2]$ where

$$\sigma_s = \phi_1(s) + \int_0^s \phi_2(s,u) dW_u.$$

- Using the Ito isometry we find that

$$M_{ts} = \left(\phi_1(s) + \int_0^t \phi_2(s,u) dW_u \right)^2 + \int_t^s \phi_2^2(s,u) du,$$

which, for each t, is a squared Gaussian RV plus a constant.

Second order chaos: bond and option prices

- Defining $Z_{tT} = \int_T^\infty M_{ts} ds$ (integral of a parametric family of squared Gaussian RVs plus a constant), we see that bond prices are given by
$$P_{tT} = \frac{Z_{tT}}{Z_{tt}}.$$
- In particular, since $M_{0s} = \phi_1^2(s) + \int_0^s \phi_2^2(s,u)du$, it follows that
$$P_{0T} = \frac{\int_T^\infty \left(\phi_1^2(s) + \int_0^s \phi_2^2(s,u)du\right) ds}{\int_0^\infty \left(\phi_1^2(s) + \int_0^s \phi_2^2(s,u)du\right) ds}.$$
- Moreover, the price at time zero of an option with payoff $(P_{tT} - K)^+$ is
$$ZBC(0,t,T,K) = \frac{1}{V_0} E\left[V_t \left(P_{tT} - K\right)^+\right] = \frac{1}{V_0} E\left[\left(Z_{tT} - KZ_{tt}\right)^+\right],$$
which can be calculated in terms of the joint distribution of Z_{tT_1} and Z_{tT_2}.

Factorizable second order chaos: definition

- Consider $\phi_1(s) = \gamma(s)$ and $\phi_2(s, u) = \beta(s)\gamma(u)$.
- Then $\sigma_s = \phi(s) + \beta(s)R_s$ where

$$R_t = \int_0^t \gamma(s)dW_s$$

  is a martingale with quadratic variation $Q(t) = \int_0^t \gamma^2(s)ds$.
- Therefore

$$\begin{aligned} M_{ts} &= (\alpha^2(s) + \beta(s)R_t)^2 + \beta^2(s)[Q(s) - Q(t)] \\ &= \alpha^2(s) + \beta^2(s)Q(s) + 2\alpha(s)\beta(s)R_t + \beta^2(s)(R_t^2 - Q(t)) \end{aligned}$$

- Notice that the scalar random variable R_t is the sole state variable for the interest rate model at time t, even in the case of a multidimensional Brownian motion W_t.

Factorizable second order chaos: bond prices

- Integrating the previous expression gives

$$Z_{tT} = \int_T^\infty M_{ts} ds = A(T) + B(T)R_t + C(T)(R_t^2 - Q(t)),$$

where

$$\begin{aligned} A(T) &= \int_t^\infty (\alpha^2(s) + \beta^2(s)Q(s))ds \\ B(T) &= 2\int_T^\infty \alpha(s)\beta(s)ds, \quad C(T) = \int_T^\infty \beta^2(s)ds \end{aligned}$$

- Therefore

$$V_t = A(t) + B(t)R_t + C(t)(R_t^2 - Q(t))$$

and

$$P_{tT} = \frac{A(T) + B(T)R_t + C(T)(R_t^2 - Q(t))}{A(t) + B(t)R_t + C(t)(R_t^2 - Q(t))}$$

Factorizable second order chaos: option prices

- Fixing t, T and K, it follows that

$$Z_{tT} - KZ_{tt} = A + BY + CY^2,$$

where $Y = R(t)/\sqrt{Q(t)} \sim N(0,1)$ and

$$\begin{aligned} A &= [A(T) - KA(t)] - [C(T) - KC(t)]Q(t) \\ B &= [B(T) - KB(t)]\sqrt{Q(t)}, \quad C = [C(T) - KC(t)]Q(t) \end{aligned}$$

- Therefore, defining $p(y) = A + By + Cy^2$, we have

$$ZBC(0,t,T,K) = \frac{1}{A(0)\sqrt{2\pi}} \int_{p(y)\geq 0} p(y) e^{-\frac{1}{2}y^2} dy,$$

which can be calculated explicitly in terms of the roots of the polynomial $p(y)$.

- Analogous expressions can be derived for puts, swaptions, caps, floors, etc...

One–variable second order chaos

- Consider now

$$\begin{aligned} X_\infty &= \int_0^\infty \alpha(s)dW_s + \int_0^\infty \int_0^s \beta(s)dW_u dW_s \\ &= \int_0^\infty [\alpha(s) + \beta(s)W_s]dW_s \end{aligned}$$

- As far as fitting the initial term structure, this behaves like a first order chaos model with $\phi^2(s) = \alpha^2(s) + \beta^2(s)s$
- However, the stochastic evolution of bond prices is now

$$P_{tT} = \frac{A(T) + B(T)W_t + C(T)(W_t^2 - t)}{A(t) + B(t)W_t + C(t)(W_t^2 - t)}$$

- Option prices are determined by the same expression as before by setting $Q(t) = t$.

One–variable third order chaos

- Motivated by the previous example, we consider

$$X_\infty = \int_0^\infty \alpha(s)dW_s + \int_0^\infty\int_0^s \beta(s)dW_u dW_s + \int_0^\infty\int_0^s\int_0^u \delta(s)dW_v dW_u dW_s$$
$$= \int_0^\infty \left[\alpha(s) + \beta(s)W_s + \frac{1}{2}\delta(s)(W_s^2 - s)\right] dW_s$$

- Again, for fitting P_{0T} this behaves like a first order chaos model with $\phi(s) = \alpha^2(s) + \beta^2(s)s + \delta^2(s)s^2/2$.
- Moreover, since

$$Z_{tT} = a(T) + b(T)W_t + c(T)W_t^2 + d(T)W_t^3 + e(T)W_t^4,$$

general bond prices are expressed as the ratio of 4th–order polynomials in W_t.
- Similarly, option prices can be found explicitly by integrating a 4th–order polynomial of a standard normal random variable.

Initial term structure for a generic chaos model

- In a general chaos model we that

$$
\begin{aligned}
Z_{0T} &= \int_T^\infty \mathbb{E}\left[\left(\phi_1(s_1) + \int_0^{s_1} \phi_2(s_1, s_2) dW_{s_2} + \cdots\right)^2\right] ds_1 \\
&= \int_T^\infty \left(\phi_1^2(s_1) + \int_0^{s_1} \phi_2^2(s_1, s_2) ds_2 + \cdots\right) ds_1.
\end{aligned}
$$

- Therefore

$$
P_{0T} = \frac{Z_{0T}}{Z_{0t}} = \frac{\int_T^\infty \psi(s) ds}{\int_t^\infty \psi(s) ds},
$$

where

$$
\psi(s_1) = \begin{cases}
\phi_1^2(s_1) & \text{first chaos} \\
\phi_1^2(s_1) + \int_0^{s_1} \phi_2^2(s_1, s_2) ds_2 & \text{second chaos} \\
\phi_1^2(s_1) + \int_0^{s_1} \phi_2^2(s_1, s_2) ds_2 + \int_0^{s_1} \int_0^{s_2} \phi_3^2(s_1, s_2, s_3) ds_3 ds_2 & \\
\vdots &
\end{cases}
$$

Data

- For P_{0T} we use clean prices of treasury coupon strips in the Gilt Market using data from the UK Debt Management Office (DMO).
- We consider bond prices at 146 dates (every other business day) from Jan 1998 to Jan 1999 with 50 maturities for each date.
- We also consider bond prices at 10 dates (every 3 months) from Sep 2006 to March 2008 with 150 maturities for each date.
- For interest rate options we consider (initially) ATM caps/floors quotes from ICAP (via Bloomberg) on March 4th, 2009 with 10 maturities, as well as 148 bond prices for the same date.

Parametric specification

- Motivated by the vast literature on forward rate curve fitting (so-called descriptive–form interest rate models), we consider the exponential–polynomial family (Bjork and Christensen 99):

$$\phi(s) = \sum_{i=1}^{n} \left(\sum_{j=1}^{\mu_i} b_{ij} s^j \right) e^{-c_i s}$$

- Special cases in this family are the Nelson–Sigel (87), Svensson (94) and Cairns (98) models:

$$\begin{aligned}
\phi_{NS}(s) &= b_0 + (b_1 + b_2 s)e^{-c_1 s} \\
\phi_{Sv}(s) &= b_0 + (b_1 + b_2 s)e^{-c_1 s} + b_3 s e^{c_2 s} \\
\phi_C(s) &= \sum_{i=1}^{4} b_1 e^{c_i s}
\end{aligned}$$

Term structure calibration

For calibration to P_{0T} we consider the following cases (N is number of parameters):

1. 1st chaos ($N = 3$):

$$\phi(s) = (b_1 + b_2 s)e^{-c_1 s}$$

2. 1st chaos ($N = 6$):

$$\phi^2(s) = [(b_1 + b_2 s)e^{-c_1 s}]^2 + [(b_3 + b_4 s)e^{-c_2 s}]^2$$

3. One–variable 2nd chaos ($N = 6$):

$$\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = (b_3 + b_4 s)e^{-c_2}$$

4. One–variable 2nd chaos ($N = 6$):

$$\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s} + b_3 e^{-c_3 s}$$

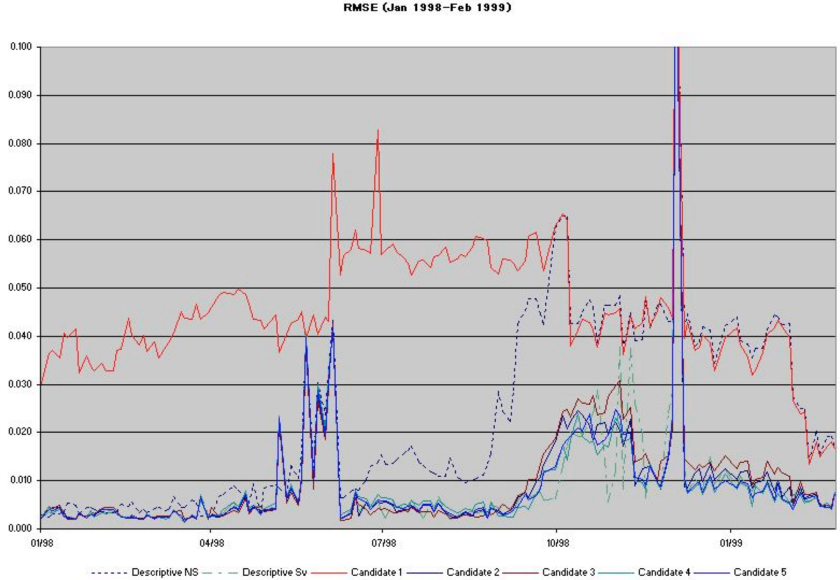
5. One–variable 3rd chaos ($N = 6$):

$$\alpha(s) = b_1 e^{-c_1 s}, \quad \beta(s) = b_2 e^{-c_2 s}, \quad \delta(s) = b_3 e^{-c_3 s}$$

Calibration results: bonds from Jan/98 to Feb/99

Model	N	Speed	-L	RMSE
1st chaos	3	68	13.5930	0.0454
1st chaos	6	211	0.3801	0.0092
One–var 2nd chaos (a)	6	289	0.4008	0.0100
One–var 2nd chaos (b)	6	114	0.3806	0.0087
One–var 3rd chaos	6	129	0.3721	0.0088
Descriptive NS	4	150	3.5579	0.0228
Descriptive Sv	6	251	0.3499	0.0091

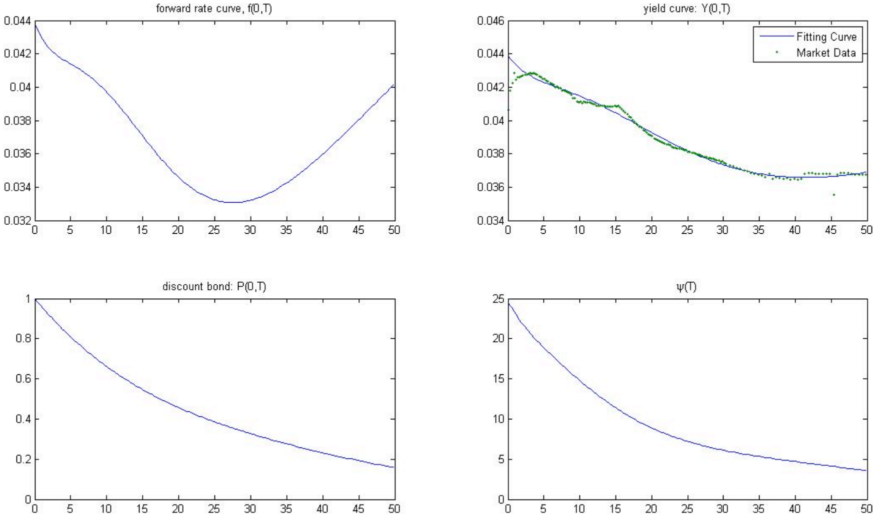
Stability of parameters



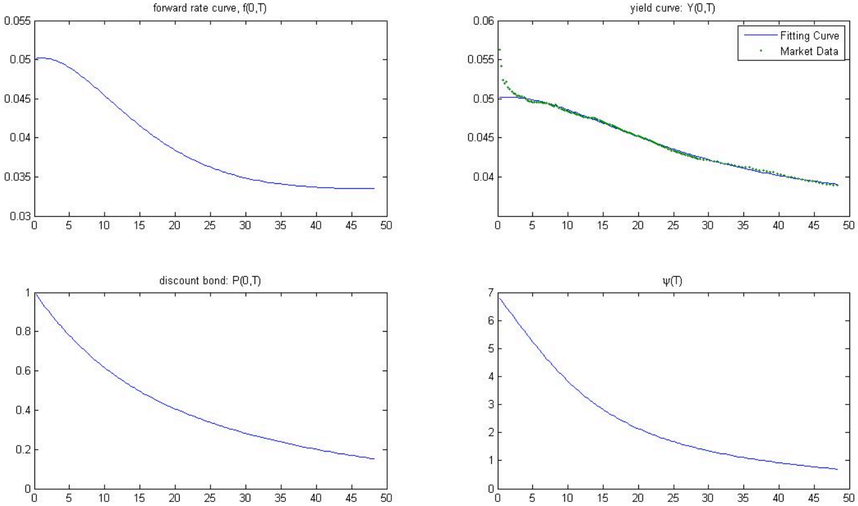
Calibration results: bonds from Sept/06 to March/08

Model	N	Speed	-L	RMSE
1st chaos	3	65	110.8482	0.0393
1st chaos	6	185	2.6163	0.0098
One–var 2nd chaos (a)	6	206	2.7283	0.0080
One–var 2nd chaos (b)	6	396	2.6995	0.0089
One–var 3rd chaos	6	234	2.3534	0.0096
Descriptive NS	4	215	3.8481	0.0124
Descriptive Sv	6	417	2.3668	0.0078

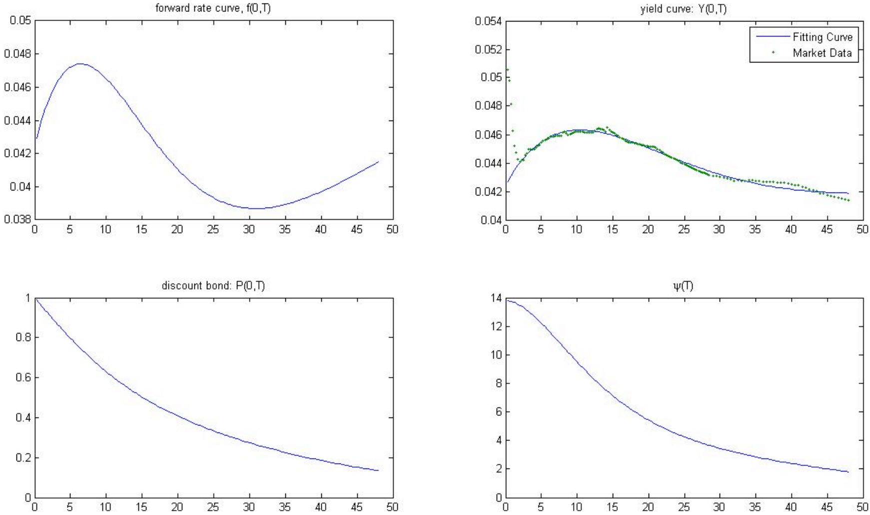
Fitted curves on Feb 3rd, 2006



Fitted curves on June 7th, 2007



Fitted curves on September 7th, 2007



Term structure and option price calibration

For calibration to P_{0T} and caps/floor prices we consider

1. (1st chaos, 3 par.)
$$\phi(s) = (b_1 + b_2 s)e^{-c_1 s}$$
2. (1st chaos, 5 par.)
$$\phi(s) = (b_1 + b_2 s)e^{-c_1 s} + b_3 s e^{-c_2 s}$$
3. (one–variable 2nd chaos, 5 par.)
$$\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \quad \beta(s) = b_3 e^{-c_2 s}$$
4. (factorizable 2nd chaos, 6 par.)
$$\alpha = (b_1 + b_2 s)e^{-c_1 s}, \beta(s) = b_3 e^{-c_2 s}, \quad \gamma(s) = e^{-c_3 s}$$
5. (one–variable 2nd chaos, 7 par)
$$\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \beta(s) = b_3 e^{-c_2 s} + b_4 e^{-c_3 s}$$
6. (one–variable 3rd chaos, 7 par)
$$\alpha(s) = (b_1 + b_2 s)e^{-c_1 s}, \beta(s) = b_3 e^{-c_2 s}, \delta(s) = b_3 e^{-c_3 s}$$

Calibration results: bonds and options on March 6th, 2009

Model	N	Bond error	Option error	RMSE
1st chaos	3	0.0745	0.0044	0.2808
1st chaos	5	0.0354	0.0069	0.2056
One–var 2nd chaos	5	0.0037	0.0053	0.0944
Factorizable 2nd chaos	6	0.0033	0.0002	0.0586
One–var 2nd chaos	7	0.0034	0.0001	0.0590
One–var 3rd chaos	7	0.0030	0.0000	0.0553

Conclusions

1. We propose a systematic way to calibrate interest rate model in the chaotic approach.
2. For term structure calibration, the performance of first order chaos models is comparable to their deterministic descriptive form analogues (Nelson–Sigel and Svensson models).
3. One–variable higher order chaos models slightly improve the performance, but have the advantage of being fully stochastic and automatically consistent with non-arbitrage and positivity conditions.
4. Preliminary results on option calibration reinforce the need for at least a second order chaos model.
5. Further work will compare (factorizable) second and (2–variable) third order chaos models for option calibration.
6. Higher order chaos models are likely to be unnecessary (and possibly made illegal anyway...)
7. THANK YOU !