Orlicz spaces in Mathematical Finance

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The Merton Problem

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- The initial constant endowment is $x \in \mathbb{R}$ and the fixed time horizon is $T \in (0, +\infty]$.
- The underlying process S is an $\mathbb{R}^d-$valued càdlàg semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$, which is not assumed to be locally bounded.

Admissible integrands, suitability and compatibility

- Given $W \in L^0_+$, define the W–admissible strategies as

$$\mathcal{H}^W := \left\{ H \in L(S) : \int_0^t H dS \geq -cW, \text{ for some } c > 0 \right\}.$$

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- We say that $W \geq 1$ is suitable for S if for each $i = 1, \dots, d$, there exists a process $H^i \in L(S^i)$ such that

$$P(\{\omega \mid \exists t \geq 0 \text{ such that } H^i_t(\omega) = 0\}) = 0 \tag{3}$$

and

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$$\left| \int_0^t H^i dS^i \right| \leq W, \quad \forall t \in [0, T]. \tag{4}$$
- We say that $W \in L^0_+$ is compatible with the utility function u if
$$E[u(-\alpha W)] > -\infty \text{ for all } \alpha > 0 \tag{5}$$
and that it is weakly compatible with u if
$$E[u(-\alpha W)] > -\infty \text{ for some } \alpha > 0. \tag{6}$$

Terminal values and duality

- Given a suitable and compatible random variable W, define

$$K^W = \left\{ \int_0^T HdS : H \in \mathcal{H}^W \right\} \tag{7}$$

so that the primal problem (1) becomes:

$$\sup_{k \in K^W} E[u(x+k)]. \tag{8}$$

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- For this, we need to choose a Banach spaces and its topological dual in order to define the polar set $(C^W)^0$.
- Classically, the spaces (L^∞, ba) were successfully used when dealing with locally bounded traded assets. In order to accommodate more general markets and inspired by the compatibility conditions above, we argue instead for the use of an appropriate Orlicz space and its dual.

Orlicz spaces

- Consider the Young function $\widehat{u} : \mathbb{R} \to [0, +\infty)$ associated with the utility function u, defined as

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- Its corresponding Orlicz space is

$$L^{\widehat{u}}(P) = \{f \in L^0(P) : E[\widehat{u}(\alpha f)] < +\infty \text{ for some } \alpha > 0\},$$

equipped with the Luxemburg norm

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- In general $M^{\widehat{u}} \subset L^{\widehat{u}}$ (strict inclusion).

Compatibility revisited

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- We can then see that a positive random variable W is compatible (resp. weakly compatible) with the utility function u if and only if $W \in M^{\widehat{u}}$ (resp. $W \in L^{\widehat{u}}$).

Complementary spaces

- The convex conjugate of $\widehat{u}$, called the complementary Young function in the theory Orlicz spaces, is

$$\widehat{\Phi}(y) := \sup_x \{x|y| - \widehat{u}(x)\} = \Phi(|y| + \beta) - \Phi(\beta),$$

where $\beta = u'(0) > 0$ and Φ is the concave conjugate of u.

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- We consider the Orlicz space $L^{\widehat{\Phi}}$ endowed with the Orlicz norm

$$|||g|||_{\widehat{\Phi}} = \sup\{|E[fg]|, f \in L^{\widehat{u}}, E[\widehat{u}(f)] \leq 1\}.$$

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- It then follows that $(M^{\widehat{u}})^* = L^{\widehat{\Phi}}$ in the sense that if $z \in (M^{\widehat{u}})^*$ is a continuous linear functional on $M^{\widehat{u}}$, then there exists a unique $g \in L^{\widehat{\Phi}}$ such that

$$z(f) = \int_\Omega fg dP, \qquad f \in M^{\widehat{u}},$$

with $\|z\|_{(M^{\widehat{u}})^*} := \sup_{\|f\|_{\widehat{u}} \leq 1} |z(f)| = |||g|||_{\widehat{\Phi}}$.

The dual of $L^{\widehat{u}}$

- It follows from the properties of the pair $(\widehat{u}, \widehat{\Phi})$ that each element $z \in (L^{\widehat{u}})^*$ can be uniquely expressed as

$$z = z^r + z^s,$$

where the *regular* part z^r is given by

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- That is, $(L^{\widehat{u}})^* = (M^{\widehat{u}})^* \oplus (M^{\widehat{u}})^\perp$.

Positive singular functionals

- Consider the concave integral functional

$$\begin{array}{rcl} I_u : L^{\widehat{u}} & \rightarrow & [-\infty, \infty) \\ f & \mapsto & E[u(f)] \end{array}$$

with effective domain

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- One consequence of choosing the correct Orlicz spaces is that the norm of a *non negative* singular element $0 \leq z \in (M^{\widehat{u}})^{\perp}$ satisfies

$$\|z\|_{(L^{\widehat{u}}(P))^*} := \sup_{\|f\|_{\widehat{u}} \leq 1} |z(f)| = \sup_{f \in \mathcal{D}(P)} z(-f),$$

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$$(C^W)^0 := \left\{ z \in (L^{\widehat{u}})^* \mid z(f) \leq 0, \quad \forall f \in C^W \right\}, \tag{12}$$

which satisfies $(C^W)^0 \subseteq (L^{\widehat{u}})^*_+$, since $(-L^{\widehat{u}}_+) \subseteq C^W$.

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$$\mathcal{M}^W := \{ Q \in (C^W)^0 \mid Q(\mathbf{1}_\Omega) = 1 \}. \tag{13}$$
- It was shown in Biagini-Frittelli (2006) that
$$\mathcal{M}^W \cap L^1(P) = \mathbb{M}_\sigma \cap L^{\widehat{\Phi}}. \tag{14}$$

Indifference Pricing

- Following Hodges and Neuberger (1989), we define the *indifference price* $\pi(B)$ for the seller of a claim B as the the implicit solution of the equation

$$\begin{aligned} \sup_{H \in \mathcal{H}^W} E\left[u\left(x + \int_0^T HdS\right)\right] &= \\ \sup_{H \in \mathcal{H}^W} E[u(x + \pi(B) &+ \int_0^T HdS - B)]. \end{aligned} \tag{15}$$

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- We consider the set $\mathcal{B}$ of claims $B \in \mathcal{F}_T$ satisfying

$$E[u(-(1+\varepsilon)B^+)] > -\infty, \quad E[u(\varepsilon B^-)] > -\infty, \qquad (16)$$

for some $\varepsilon > 0$

Main result

Theorem (Biagini-Fritelli-G)

Suppose that $B \in \mathcal{B}$ and that there exists $W \in \mathbb{S} \cap L^{\widehat{u}}$ satisfying

$$\sup_{H \in \mathcal{H}^W} E\left[u\left(x + \int_0^T HdS - B\right)\right] < u(\infty). \tag{17}$$

Then

$$\begin{aligned} &\sup_{H \in \mathcal{H}^W} E\left[u\left(x + \int_0^T HdS - B\right)\right] \\ &= \min_{\lambda > 0,\ Q \in \mathcal{M}^W} \left\{\lambda x - \lambda Q(B) + E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] + \lambda \|Q^s\|\right\}, \end{aligned}$$

where $Q(B) = E_{Q^r}[B] + Q^s(B)$. If $B \in M^{\widehat{u}}$, then $Q^s(B) = 0$. Moreover, if $W \in M^{\widehat{u}}$ and $B \in M^{\widehat{u}}$ then $\mathcal{M}^W$ can be replaced by $\mathbb{M}_\sigma \cap L^{\widehat{\Phi}}$ and no singular term appears.

Properties of the indifference price

Corollary

The indifference price $\pi : \mathcal{B} \to \mathbb{R}$ is well defined, convex, monotone, translation invariant, norm continuous, subdifferentiable and admits the representation:

$$\pi(B) = \max_{Q \in \mathcal{M}^W} \left[Q(B) - \alpha(Q) \right], \tag{18}$$

where the penalty term is given by:

$$\alpha(Q) := x + \|Q^s\| + \inf_{\lambda > 0} \left\{ \frac{E\left[\Phi\left(\lambda \frac{dQ^r}{dP}\right)\right] - U_0^W(x)}{\lambda} \right\}. \tag{19}$$

Exponential Utility

- For an exponential utility function $u(x) = -e^{-\gamma x}$, $\gamma > 0$, we have

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\begin{aligned}
\Phi(y) &= \frac{y}{\gamma}\log\frac{y}{\gamma} - \frac{y}{\gamma} \\
\widehat{u}(x) &= e^{\gamma|x|} - \gamma|x| - 1 \\
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- Moreover, the boundedness condition on B is equivalent to

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- In this case, the properties of the indifference price can be proved using a change of measure technique based on Pistone and Rogantin (1999)

Further work

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