# Indifference Price of Insurance Contracts: stochastic volatility, stochastic interest rates

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# Part 1: Stochastics Volatility

- We consider two factor stochastic volatility models where the financial asset satisfies:

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma(t, Y_t) S_t dW_t^1 \\
dY_t &= a(t, Y_t) dt + b(t, Y_t)[\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2]
\end{aligned}
$$

- Here $\mu$ and $-1 < \rho < 1$ are constants, $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ are deterministic functions, and $W_t^1$ and $W_t^2$ are independent one dimensional $P$–Brownian motions.
- In addition, we assume the existence of a risk-free bank account paying a constant interest rate $r = 0$.

# Optimal Hedging and Investment

- We assume that, after selling an insurance contract $B_T$ maturing at a future time $T$, the insurance company tries to solve the stochastic control problem

$$u^B(x, S, y, t) = \sup_{\pi \in \mathcal{A}} E^{x,S,y,t} \left[ U \left( X_T^{\pi,x} - B_T \right) \right],$$

  where $X_t^{H,x}$ is value of a self–financing portfolio with initial wealth x and $\pi_t$ dollars invested in the stock, with the remaining value invested in the bank account.
- When $B = 0$, this reduces to the Merton problem:

$$u^0(x, y, t) = \sup_{H \in \mathcal{A}} E^{x,y,t} \left[ U \left( X_T^{\pi,x} \right) \right]$$

# Utility based pricing

- The sellers indifference price for the claim B is the solution $\pi^s$ to the equation

$$u^0(x,y,t) = u^B(x + P(x,S,y,t), S, y, t)$$

- From now on, we consider an exponential utility function of the form:

$$U(x) = -e^{-\gamma x}, \quad \gamma > 0.$$

- We can then write

$$\begin{aligned} u^B(x,S,y,t) &= -e^{-\gamma x} G(S,y,t) = -e^{-x} e^{\phi(S,y,t)} \\ u^0(x,y,t) &= -e^{\gamma x} F(y,t) = -e^{-\gamma x} e^{\psi(y,t)} \end{aligned}$$

- The indifference price is then given by

$$P(S,y,t) = \frac{1}{\gamma} \log\left(\frac{G(S,y,t)}{F(y,t)}\right) = \frac{1}{\gamma}(\phi(S,y,t) - \psi(y,t)).$$

# Life insurance

- Consider now a claim of the form $B_T = \mathbf{1}_{\{\tau \leq T\}}$.
- Here $\tau$ is the arrival time of the first jump of an inhomogeneous Poisson process with intensity $\lambda(t)$, and is assumed to be independent of $(W^1, W^2)$.
- In this case, we have

$$
\begin{aligned}
u^B(x+P, S, y, t) &= \sup_{H \in \mathcal{A}} E\left[-e^{\gamma(x+P+\int_0^T H_s dS_s - B_T)}\right] \\
&= e^{-\gamma P} E\left[e^{\gamma B_T}\right] \sup_{H \in \mathcal{A}} E\left[-e^{\gamma(x+\int_0^T H_s dS_s)}\right] \\
&= e^{-\gamma P} E\left[e^{\gamma B_T}\right] u^0(x, S, y, t).
\end{aligned}
$$

- Therefore, the *excess hedge* is zero and the indifference price in this case is given by

$$
P = \frac{1}{\gamma} \log E\left[e^{\gamma B_T}\right].
$$

# Equity–linked contracts

- Consider now an insurance contract that pays $B(S_\tau)$ at time $\tau$ for some deterministic function $B(\cdot)$.
- In this case, the wealth process satisfies

$$\begin{cases} dX_s = \pi_s dS_s = \pi_s[(\mu - r)ds + \sigma(s, Y_s)dW_s] \\ X_\tau = X_{\tau-} - B(S_\tau), \quad \tau < T \\ X_t = x \end{cases}$$

- To obtain the equation satisfied by $u^s$ in this case, consider the interval $[t, t+h)$ and observe that,

$$\begin{aligned} u^B(x, s, y, t) \quad \geq \quad & E[u^B(X_{t+h}, S_{t+h}, Y_{t+h}, t+h)]p(h) \\ & + E[u^0(X_{t+h} - B(S_{t+h}), Y_{t+h}, t+h)]q(h) \end{aligned}$$

where $p(h) = P(\tau > t + h | \tau > t)$ and $q(h) = 1 - p(h)$.

# Dynamic Programming

- Using Ito's lemma, we obtain

$$
\begin{aligned}
u^B(x,S,y,t) \geq & \left( u^B(x,S,y,t) + E\left[\int_t^{t+h} \mathcal{L}^{X,S,Y} u^B ds\right]\right) p(h) \\
& + \left( u^0(x-B(S),y,t) + E\left[\int_t^{t+h} \mathcal{L}^{X,Y} u^0 ds\right]\right) q(h),
\end{aligned}
$$

  where $\mathcal{L}$ denotes the generator for the corresponding Markov process.
- Subtracting $u^B(x,S,y,t)p(h)$ from both sides, dividing by $h$ and taking the limit $h \longrightarrow 0$ gives

$$
\lambda(t)\left[u^0(x-B(S),y,t) - u^B(x,S,y,t)\right] + \mathcal{L}^{X,S,Y} u^B(x,S,y,t) \leq 0, \tag{1}
$$

  with equality holding at the optimal wealth process.

# The HJB equation

- Taking the maximum in (1) and using a function of the form $u^B(x,S,y,t) = -e^{-\gamma x} e^{\phi(S,y,t)}$ leads to

$$\begin{cases} \phi_t + \frac{1}{2}\sigma^2 S^2 \phi_{SS} + \rho\sigma b S \phi_{yS} + \frac{1}{2} b^2 \phi_{yy} + \left(a - \frac{\mu b \rho}{\sigma}\right)\phi_y \\ + \frac{1}{2} b^2 (1-\rho^2)\phi_y^2 + \lambda(t)\left[e^{\gamma B + \psi - \phi} - 1\right] = \frac{\mu^2}{2\sigma^2} \\ \phi(y,S,T) = 0 \end{cases}, \tag{2}$$

where, as it is well-known,

$$\psi(y,t) = \frac{1}{1-\rho^2} \log \widetilde{E}^{y,t}\left[e^{-\int_0^T \frac{(1-\rho^2)\mu^2}{2\sigma^2(Y_s)} ds}\right],$$

with $\widetilde{E}[\cdot]$ denoting an expectation with respect to the *minimal martingale measure* for this market.

# Optimal hedge

- In terms of $\phi$, the optimizer for (1) is a portfolio of the form

$$\pi_t^B = \frac{1}{\gamma}\left[\frac{\mu}{\sigma^2(y)} + \phi_S S + \frac{b(y,t)\rho}{\sigma(y)}\phi_y\right]$$

- By comparison, the optimal Merton portfolio is

$$\pi_t^0 = \frac{1}{\gamma}\left[\frac{\mu}{\sigma^2(y)} + \frac{b(y,t)\rho}{\sigma(y)}\psi_y\right]$$

- Subtracting one from the other we obtain the *excess hedge*

$$\pi_t^B - \pi_t^0 = P_S(S,y,t)S_t + \frac{b(y,t)\rho}{\gamma\sigma(y)}P_y(S,y,t),$$

  which has the form of a *delta* hedge plus a volatility correction.

# Fast-mean reversion asymptotics

- Let us now take

$$dY_t = \alpha(m - Y_t)dt + \beta(\rho dW_t + \sqrt{1-\rho^2}dZ_t)$$

  and consider the regime $\frac{1}{\alpha} = \varepsilon << 1$, with $\beta = \sqrt{2\nu}/\sqrt{\varepsilon}$ where $\nu^2$ is a fixed variance for the invariant distribution of $Y_t$.
- We then look for expansion of the form

$$\phi^{\varepsilon} = \phi^{(0)}(y, S, t) + \sqrt{\varepsilon}\phi^{(1)}(y, S, t) + \varepsilon\phi^{(2)}(y, S, t) + ...$$

# Operators

- The previous PDE can be rewritten in compact notation as

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right)\phi + NL^{\phi} = \frac{\mu^2}{2\sigma^2} \tag{3}$$

where $NL^{\phi} = \lambda(t)\left[e^{\gamma B+\psi-\phi} - 1\right] + \frac{\mu^2}{\varepsilon}(1-\rho^2)\phi_y^2$.

- Here

$$\mathcal{L}_0 = \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}$$
$$\mathcal{L}_1 = \sqrt{2}\rho\nu\left(\sigma(y)S\frac{\partial^2}{\partial y\partial S} - \frac{\mu}{\sigma(y)}\frac{\partial}{\partial y}\right)$$
$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2(y)S^2\frac{\partial^2}{\partial S^2}$$

# Main result

- The insurer's indifference price satisfy:

$$|P(y,S,t) - P^{(0)}(S,t) - \widetilde{P1}(y,S,t)| = \mathcal{O}(\varepsilon) \qquad (4)$$

where

$$\widetilde{P1}(y,S,t) = -(T-t)(V_3 S^3 P^{(0)}_{SSS} + V_2 S^2 P^{(0)}_{SS})$$

- Here $P^{(0)}$ satisfies

$$\begin{cases} P^{(0)}_t + \frac{1}{2}\sigma_\star^2 P^{(0)}_{SS} + \frac{\lambda(t)}{\gamma}\left[e^{\gamma(g-P^{(0)})} - 1\right] = 0 \\ P^{(0)}(S,T) = 0 \end{cases} \qquad (5)$$

where $\sigma_\star^2 = \langle \sigma^2 \rangle$.

# Part 2: Stochastic Interest Rates

- Consider now the *discounted* price of a financial asset given by

$$\begin{cases} dS_s = (\mu - r_s)S_s ds + \sigma S_s dW_s^1 \\ S_t = S \end{cases}$$

- We model the short rate as

$$\begin{cases} dr_s = (a_0(s)r_s + b_0(s))ds + \sqrt{c(s)r_s + d(s)}dZ_s \\ r_t = r \end{cases},$$

  where $Z_t = \rho W_t^1 + \sqrt{1-\rho^2}dW_t^2$.

- It then follows that the price of a zero–coupon bond with maturity $T_1$ is given by

$$F_{tT_1} = e^{A(t,T_1) - C(t,T_1)r_t},$$

  for deterministic functions $A(\cdot,\cdot)$ and $C(\cdot,\cdot)$.

# Portfolio choice

- In this context, the insurance company can invest $\pi_t$ dollars in the stock $S_t$ and $\eta_t$ dollars in the bond $F_{tT_1}$, with the remaining of its wealth in a bank account paying the interest rate $r_t$.
- We assume the market for bonds of different maturities has a market price of risk of the form

$$q(r_s, s) = \frac{(a_0(s) - a(s))r_s + (b_0(s) - b(s))}{\sqrt{c(s)r_s + d(s)}} \tag{6}$$

- Under this assumption, one can show that the dynamics of the discounted bond price is

$$\begin{aligned} \frac{d(e^{-\int_0^s r_u du} F_{sT_1})}{e^{-\int_0^s r_u du} F_{sT_1}} &= -C(s, T_1)\left[(\Delta a(s) r_s + \Delta b(s))dt \right. \\ &\left. + \sqrt{c(s)r_s + d(s)}dZ_s\right] \end{aligned}$$

# Path–dependent claims

- We consider path–dependent claims of the form $B_t = B(S_t, r_t, v_t)$, where

$$V_t = \int_0^t f(S_s, r_s, s)ds.$$

- In this case, the wealth process satisfies

$$\begin{cases} dX_s = \pi_s \frac{dS_s}{S_s} + \eta_s \frac{d(e^{-\int_0^s r_u du} F_{sT_1})}{e^{-\int_0^s r_u du} F_{sT_1}} \\ dX_s = [\pi_s(\mu - r) - \eta_s C(s, T_1)(\Delta a(s) r_s + \Delta b(s))]ds \\ +\pi_s \sigma dW^1 - \eta_s C(s, T_1)\sqrt{c(s) r_s + d(s)} dZ_s \\ X_\tau = X_{\tau-} - B(S_\tau, r_\tau, V_\tau), \quad \tau < T \\ X_t = x \end{cases}$$

# The solution to Merton's Problem

- The Merton problem for the insurance company is now
$$u^0(x,r,t) = \sup_{\pi,\eta\in\mathcal{A}} E^{x,r,t}\left[U\left(X_T\right)\right].$$
- Using the same reasoning as before for the function $u^0(x,r,t) = -e^{-\gamma x}e^{\psi(r,t)}$ we arrive at the following PDE:
$$\psi_t+(ar+b)\psi_r+\frac{1}{2}\psi_{rr}(cr+d)-\left[\frac{1}{2}\left(\frac{\mu-r-\sigma\rho q}{\sqrt{1-\rho^2}\sigma}\right)^2+\frac{q^2}{2}\right]=0,$$
subject to $\psi(r,T)=0$.
- Using Feynmann-Kac we obtain that
$$\psi(r,t) = -\int_t^T \widehat{E}^{t,r}\left[\left(\frac{\mu-r-\sigma\rho q}{2\sqrt{1-\rho^2}\sigma}\right)^2+\frac{q^2}{2}\right],$$
where $\widehat{E}[\cdot]$ denotes expectation with respect to the (unique) martingale measure for bond prices defined by the market price of risk $q$.

# The value function with the claim

- Similarly, the hedging problem for the insurance company is now

$$u^B(x, S, r, v, t) = \sup_{\pi,\eta \in \mathcal{A}} E^{x,S,r,v,t}\left[U\left(X_T\right)\right]. \tag{7}$$

- For a function of the form $u^B(x, S, y, t) = -e^{-\gamma x} e^{\phi(S,r,v,t)}$, we obtain that $\phi$ satisfies the PDE

$$\begin{cases} \phi_t + (ar+b)\phi_r + \frac{1}{2}(cr+d)\phi_{rr} + \rho\sigma\sqrt{cr+d}S\phi_{Sr} + \frac{1}{2}\sigma^2 S^2 \phi_{SS} \\ +f(S,r,t)\phi_v - \left[\frac{1}{2}\left(\frac{\mu - r - \sigma\rho q}{\sqrt{1-\rho^2}\sigma}\right)^2 + \frac{q^2}{2}\right] - \lambda(t)\left(1 - e^{\gamma B + \psi - \phi}\right) = \\ \phi(S, r, T) = 0 \end{cases} \tag{8}$$

subject to $\phi(S, r, v, T) = 0$.

# Optimal hedge

- In terms of $\phi$, the optimizers for (7) are

$$\begin{aligned}
\pi_t^B &= \frac{1}{\gamma}\left[\frac{\mu - r - q\rho\sigma}{(1 - rho^2)\sigma^2} + \phi_S S\right] \\
\eta_t^B &= \frac{1}{\gamma C\sqrt{cr + d}}\left[\frac{\rho\sigma(\mu - r) - q\sigma^2}{(1 - \rho^2)\sigma^2} - \sqrt{cr + d}\,\phi_r\right]
\end{aligned}$$

- By comparison, the optimal Merton portfolio is

$$\begin{aligned}
\pi_t^0 &= \frac{1}{\gamma}\left[\frac{\mu - r - q\rho\sigma}{(1 - rho^2)\sigma^2}\right] \\
\eta_t^0 &= \frac{1}{\gamma C\sqrt{cr + d}}\left[\frac{\rho\sigma(\mu - r) - q\sigma^2}{(1 - \rho^2)\sigma^2} - \sqrt{cr + d}\,\Psi_r\right]
\end{aligned}$$

- Subtracting one from the other we obtain the *excess hedge*

$$\begin{aligned}
\pi_t^B - \pi_t^0 &= P_S(S, r, v, t)S_t \\
\eta_t^B - \eta_t^0 &= -\frac{1}{C}P_r(S, r, v, t)
\end{aligned}$$

# The pricing equation

- Therefore, $P$ satisfies the following nonlinear PDE:

$$\begin{cases} P_t + (ar+b)P_r + \frac{1}{2}(cr+d)P_{rr} + \rho\sigma\sqrt{cr+d}SP_{Sr} + \frac{1}{2}\sigma^2 S^2 P_{SS} \\ +f(S,r,t)P_v - \frac{\lambda(t)}{\gamma}\left(1 - e^{\gamma B - \gamma P}\right) = 0 \\ P(S,r,T) = 0 \end{cases}$$