Introduction

Reconstruction problems in model theory

Reconstructing T from $\mathbf{Mod}(T)$

Reconstructing M from $\mathrm{Aut}(M)$ and $\mathrm{End}(M)$

Grothendieck's formalism of Galois categories

Reconstruction problems for first-order theories

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GSCL 2017

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- For example:
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  - The theory of a structure M is a complete invariant for the isomorphism class of some ultrapower $M^{\mathcal{U}}$: this is the Keisler-Shelah isomorphism theorem.

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- If this happens for a fixed c as above, we say that we can *reconstruct* c from F.

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- *Absolute Galois groups*. We can assign a model M and a parameter set $A \subseteq M$ the Galois group $G(A) \overset{\text{df}}{=} \mathrm{Aut}(\mathrm{acl}(A)/\mathrm{dcl}(A))$.

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An *interpretation* $I : T \to T'$ for T an $\mathcal{L}$-theory and T' an $\mathcal{L}'$-theory assigns to each formula (over $\emptyset$) X of T a definable set $I(X)$ of T' such that the truth of sentences is preserved if you replace all instances X of formulas from T with $I(X)$.

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An *interpretation* $(f, f^*) : M \to M'$ for $M \models T$ an $\mathcal{L}$-structure and $M' \models T'$ an $\mathcal{L}'$-structure is a surjective function $f : U \twoheadrightarrow M$ from some (0-)definable subset $U \subseteq M'$ such that pulling back (0-)definable sets $X \mapsto f^* X$ is an interpretation $T \to T'$.

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- We call it **Th**.

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- We'll see an example of this later.

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Answer (Makkai, 1987)

Ultraproducts (and some other ultra-stuff).

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Definition

An *ultracategory* $\underline{\mathbf{K}}$ is a category together with *ultraproduct functors*

$$[\mathcal{U}] : \underline{\mathbf{K}}^I \to \underline{\mathbf{K}}$$

for every ultrafilter $\mathcal{U}$ on every indexing set I such that the obvious diagrams commute. Together with appropriate notions of ultramorphism-preserving *ultrafunctors* and *ultratransformations*, we can define the 2-category **Ult** of ultracategories.

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whose counit ϵ at any theory T

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Theorem. (Makkai, 1987)

Let $\underline{\mathbf{K}}$ be an ultracategory. Then $\mathbf{Ult}(\underline{\mathbf{K}}, \mathbf{Set})$ is a pretopos. There is a contravariant 2-adjunction

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This is *strong conceptual completeness*. This means we can reconstruct T from $\underline{\mathbf{Mod}}(T)$: if $\underline{\mathbf{Mod}}(T) \simeq \underline{\mathbf{Mod}}(T')$, then strong conceptual completeness gives a bi-interpretation $T \simeq T'$.

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- For example, let $\mathbf{G}$ be a definable group in T and expand each model M of T by an $\mathrm{ev}_{\mathbf{G}}(M)$-torsor. This is easily seen to commute with ultra-stuff. More generally, any internal cover.
- Here's a negative example: let T be the theory of abelian groups, and let $F : \mathbf{Mod}(T) \to \mathbf{Set}$ be the functor $\mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Q}, -)$. This does not commute with ultraproducts, e.g.

$$\prod_p \mathrm{Hom}_{\mathbf{Ab}}(\mathbb{Q}, \mathbb{Z}/p\mathbb{Z}) \,/\mathcal{U} \not\simeq \mathrm{Hom}_{\mathbf{Ab}}\left(\mathbb{Q}, \prod_p \mathbb{Z}/p\mathbb{Z} \,/\mathcal{U}\right)$$

(think about torsion). In general, even the corepresentables $\mathrm{Hom}_{\mathbf{Mod}(T)}(M, -)$ are not ultrafunctors.

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Let **TopMon** *be the 2-category of topological monoids. There is a contravariant 2-functor (which only reverses 1-morphisms)*

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Proposition

Furthermore, if we restrict to the underlying 2-groupoid core$(\mathbf{Struct})$ *of* **Struct**, End$(-)$ *becomes a contravariant 2-functor*

$$\text{core}\left(\mathbf{Struct}\right)^{\text{op}} \overset{\text{Aut}(-)}{\to} \mathbf{TopGrp}$$

to the 2-category of topological groups. In particular, on 2-morphisms $\gamma : (f, f^*) \to (g, g^*)$ *we have* $\text{Aut}(g)(\sigma) = \text{Aut}(\gamma) \circ \text{Aut}(f) \circ \text{Aut}(\gamma)^{-1}$ *for all* $\sigma \in \text{Aut}(B)$.

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$\text{End}(-)$ *reflects 2-isomorphisms: if* $f \xrightarrow{\gamma} g$ *becomes an isomorphism after applying* $\text{End}(-)$*, then* $\text{End}(\gamma)$ *is invertible, so* γ *must have been invertible.*

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- Thus, $\mathrm{End}(-)$ reflects equivalences.
- However, $\mathrm{End}(-)$ does not reflect 1-isomorphisms: if we have mutual interpretations $f : A \leftrightarrows B : g$ with $\mathrm{End}(f)$ and $\mathrm{End}(g)$ forming an isomorphism of topological monoids $\mathrm{End}(g) : \mathrm{End}(A) \leftrightarrows \mathrm{End}(B) : \mathrm{End}(f)$, it is not generally true that f and g *invert* each other.

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What if we instead restict our attention to ω-categorical structures, which are "highly symmetric" and have a nice structure theory determined by the action of their automorphism group?

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Answer (Coquand-Ahlbrandt-Ziegler, 1986)

Yes. In fact, M is bi-interpretable with the canonical structure $\mathrm{Inv}\left(\mathrm{Aut}(M) \curvearrowright M\right)$.

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Answer (Bodirsky, Evans, Kompatscher, Pinsker, 2015)

Nope.

Implications of the BEKP counterexample

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Theorem (BEKP, 2015)

There exists an ω-categorical structure M such that $\mathrm{End}(M)$ fails to determine M up to bi-interpretability. (Equivalently, there is another ω-categorical structure M' such that $\mathrm{End}(M') \simeq \mathrm{End}(M)$ as monoids, but not as topological monoids.)

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Observation (Lascar, '80s)

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- We can actually see this very concretely.

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- Either way, F fails to preserve an ultraproduct of endomorphisms.

Remark

This gives an example of an equivalence of categories $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ *which was not induced by a bi-interpretation* $T \simeq T'$*.*

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- Recall the *Ryll-Nardzewski theorem*: in an ω-categorical structure, there are only finitely many types in any given tuple of (sorted) variables.
- We can use this to apply much of the formalism to the countable model $M : T \to \mathbf{Set}$ of an ω-categorical theory T.

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Remark

As a functor, M is left-exact and isomorphism reflecting: it preserves all finite left limits (products, pullbacks, etc.) and if f becomes a bijection after taking points in M, then f was a definable bijection.

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- θ *is an epimorphism since every definable set splits into finitely many types.*

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- *For any* $(A, a) \in \mathbf{I}$*, there is a canonical natural transformation* $\mathrm{Hom}_T(A, -) \to M$*, induced by evaluation: we send* $f : A \to X$ *to* $f(a) \in M(X)$*.*
- *This induces (glues together into) a universal map* θ*:*

$$\theta : G \overset{\mathrm{df}}{=} \varinjlim_{\mathbf{I}} (\mathrm{Hom}_T(A_i, -)) \to M.$$

- θ *is an epimorphism since every definable set splits into finitely many types.*
- θ *is a monomorphism: if two germs* x, y *in* $G(X)$ *are equalized by* θ_X*, then we can represent them by* $x', y' : A \to X$ *for some* (A, a) *such that* $x'(a) = y'(a)$*.*

Proof of theorem

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- This defines a functor $\mathbf{I} \to \mathbf{Grp}$, hence a projective system of finite groups, whose projective limit is a profinite group $\mathbf{G}$.

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- We can always obtain a canonical embedding $\mathcal{C}_{\mathbf{C}} \hookrightarrow \mathbf{Def}(T)$.

Introduction

Reconstruction problems in model theory

Reconstructing T from $\mathbf{Mod}(T)$

Reconstructing M from $\mathrm{Aut}(M)$ and $\mathrm{End}(M)$

Grothendieck's formalism of Galois categories

The category $\mathcal{C}_{\mathbf{G}}$

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$$F : \mathcal{C}_{\mathbf{G}} \hookrightarrow \mathbf{Def}(T).$$

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- *Set $F(\mathbf{G}/\mathbf{H}) \stackrel{\mathrm{df}}{=} A//\overline{\mathbf{H}}$.*
- *Define F similarly on $\mathbf{G}$-equivariant maps by doing the above to their graphs.*

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- What's the relationship of $\mathbf{G}$ with $\mathrm{Aut}(M)$ and $\widehat{\mathrm{Aut}(M)}$? (the latter should be the profinite fundamental group of the classifying topos of T...)

Thank you!