Strong conceptual completeness for $\aleph_0$-categorical theories

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Abstract

These are (what started as) notes for my talk at the Harvard logic seminar on 6 February 2018.

Contents

1 Introduction

Let T be a first-order theory. Any formula $\varphi(x)$ of T^{eq} (so a definable set of T quotiented by a definable equivalence relation of T) induces a "functor of points" $\mathrm{ev}_{\varphi(x)}$ on the category $\mathbf{Mod}(T)$ of models of T with maps the elementary embeddings, by sending $M \mapsto \varphi(M)$. In this way the category $\mathbf{Def}(T)$ of 0-definable sets of T embeds into the category of functors $[\mathbf{Mod}(T), \mathbf{Set}]$, via the "evaluation map" $\mathrm{ev} : T \to [\mathbf{Mod}(T), \mathbf{Set}]$.

Here is the motivating problem: how do we recognize, up to isomorphism, the image of ev inside $[\mathbf{Mod}(T), \mathbf{Set}]$? That is, given an arbitrary functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$—some way of attaching a set to every model of T, functorial with respect to elementary embeddings—how can we tell if X was isomorphic to some functor of points $\mathrm{ev}_{\varphi(x)}$ for some formula $\varphi(x) \in T^{\mathrm{eq}}$? We call such functors X *definable*.

A necessary condition for definability is compatibility with ultraproducts. Łos' theorem tells us that evaluation functors $\mathrm{ev}_{\varphi(x)}$ commute with ultraproducts, that is,

$$\varphi\left(\prod_{i \to \mathcal{U}} M_i\right) = \prod_{i \to \mathcal{U}} \varphi(M_i).$$

Strong conceptual completeness for first-order logic, as proved by Makkai in [7], provides a sort of converse to Łos' theorem, and says that the definable functors are precisely the ones which preserve ultraproducts and certain formal comparison maps between ultraproducts, called *ultramorphisms*, which generalize the diagonal embeddings of models into their ultrapowers. This recovers T up to bi-interpretability. To precisely state Makkai's result, we must formalize what it means for an arbitrary functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ to "preserve ultraproducts" and "preserve" these ultramorphisms. We will go into more detail in 3.

Any general framework which recovers theories from their categories of models should be considerably simplified for $\aleph_0$-categorical theories, whose definable sets are exceptionally easy to understand (being precisely the finite disjoint unions of orbits of the automorphism group) and in fact are determined up to bi-interpretability by the automorphism group of the unique countable model topologized by pointwise convergence.

We will show that when T is $\aleph_0$-categorical, we can check definability by checking compatibility with ultraproducts and just diagonal embeddings into ultrapowers, so that for $\aleph_0$-categorical theories, the statement of strong conceptual completeness can indeed be simplified.

2 Preliminaries

Throughout, we will assume our theories eliminate imaginaries, so that $T = T^{\mathrm{eq}}$, and "definable" means "definable without parameters". When we say "sort" or "variable", we will mean an arbitrary finite tuple of sorts and variables.

To a first-order theory T, one can associate two categories: the category of models $\mathbf{Mod}(T)$, and the category of definable sets $\mathbf{Def}(T)$.

Definition 2.1. The **category of models** $\mathbf{Mod}(T)$ of T comprises the following data:

$$\mathbf{Mod}(T) \overset{\text{df}}{=} \begin{cases} \text{Objects: models } M \models T \\ \text{Morphisms: elementary embeddings.} \end{cases}$$

Definition 2.2. The **category of definable sets** $\mathbf{Def}(T)$ of T comprises the following data:

$$\mathbf{Def}(T) \overset{\text{df}}{=} \begin{cases} \text{Objects: equivalence classes of formulas mod } T\text{-provable equivalence} \\ \text{Morphisms: equivalence classes of definable functions mod } T\text{-provable equivalence.} \end{cases}$$

By the completeness theorem, T-provable equivalence $(T \models \varphi(x) \leftrightarrow \psi(x))$ is the same thing as having identical points in every model; by the downward Lowenheim-Skolem theorem, it suffices to check having identical points in models bounded by the size of the theory.

Remark 2.3. Models of T are precisely the functors $M : \mathbf{Def}(T) \to \mathbf{Set}$ which preserve finite limits, finite sups in subobject lattices, and images. In this way, $\mathrm{ev}_{\varphi(x)}$ is literally an evaluation map.

Definition 2.4. Let $(A_i)_{i \in I}$ be an I-indexed collection of nonempty sets. Let $\mathcal{U}$ be an ultrafilter on I. The **ultraproduct** of $(A_i)_{i \in I}$ with respect to the ultrafilter $\mathcal{U}$, which we write as $\prod_{i \to \mathcal{U}} A_i$, is defined as the following quotient,

$$\prod_{i \to \mathcal{U}} A_i \overset{\text{df}}{=} \prod_{i \in I} A_i \Big/ E_{\mathcal{U}},$$

where the equivalence relation $E_{\mathcal{U}}$ is defined by: $(a_i)_{i \in I} \sim_{E_{\mathcal{U}}} (b_i)_{i \in I}$ if and only if the set P of indices $j \in I$ such that $a_j = b_j$ is in the ultrafilter $\mathcal{U}$.

If $(a_i)_{i \in I}$ is a sequence of elements from the A_i, we write $[a_i]_{i \to \mathcal{U}}$ for its equivalence class in $\prod_{i \to \mathcal{U}} A_i$.

Remark 2.5. The above definition fails to produce a non-empty ultraproduct when even a single A_j is empty, although the construction should be unperturbed by anything happening on an ultrafilter-small set of models. We can address this by more generally defining an ultraproduct of sets as the colimit of the following filtered diagram $\mathbf{D}_{A_{i \to \mathcal{U}}}$:

$$\mathbf{D}_{A_{i \to \mathcal{U}}} \overset{\text{df}}{=} \begin{cases} \text{Objects: } \prod_{i \in P} A_i \text{ such that } P \in \mathcal{U} \\ \text{Morphisms: for } P' \subseteq P \text{ both in } \mathcal{U}, \prod_{i \in P} A_i \twoheadrightarrow \prod_{i \in P'} A_i \end{cases}.$$

One verifies that when the A_i are nonempty, $\varinjlim \mathbf{D}_{A_{i \to \mathcal{U}}}$ agrees with the definition 2.4 above, and that when a $\mathcal{U}$-small set of the A_i are empty, $\varinjlim \mathbf{D}_{A_{i \to \mathcal{U}}}$ agrees with $\prod_{i \to \mathcal{U}} A_i$ if, when computing the latter, one replaces the empty sets with arbitrary nonempty sets.

Although this definition via a filtered colimit of a diagram of infinite products works in the category **Set**, it cannot generally work in $\mathbf{Mod}(T)$, because one cannot necessarily form a product of two models of a first-order theory (e.g. fields). However, ultraproducts of models still make sense, and codify compactness arguments (as well as the compactness theorem itself.)

Let me elucidate the previous sentence by spelling out the proofs, using ultraproducts, of some compactness-related statements.

Here's the compactness theorem:

Proposition 2.6. *A theory T has a model if and only if every finite fragment $T' \subseteq T$ has a model.*

Proof. For the non-trivial "if" direction, let $\mathcal{D}$ the directed partial order of finite fragments $T' \subseteq T$. Let $\mathcal{U}$ be an ultrafilter completing the filter base consisting of those sets $P_{T'} \overset{\text{df}}{=} \{T'' \in \mathcal{D} \mid T'' \supseteq T'\}$. Let $M_{T'} \models T'$, and consider the ultraproduct $\prod_{T' \to \mathcal{U}} M_{T'}$. For any sentence $\sigma \in T$, the collection of indices $T' \in \mathcal{D}$ such that $M_{T'} \models \sigma$ contains at least $P_{\{\sigma\}}$, so is in $\mathcal{U}$, so $\prod_{T' \to \mathcal{U}} M_{T'} \models \sigma$; hence, $\prod_{T' \to \mathcal{U}} M_{T'} \models T$. □

Here are some statements proved by a compactness argument:

Proposition 2.7. *Suppose that in every model $M \models T$, the I-indexed definable functions $X_i \xrightarrow{f_i} X$ jointly cover the definable set X. Then finitely many f_i cover X.*

Proof. Let $\mathcal{D}$ be the directed partial order of finite subsets of I. Suppose that for every $F \in \mathcal{D}$, there is a model M_F where the images $\{\mathrm{im}(f_i) \mid i \in F\}$ do not cover $X(M)$, witnessed by an $x_F \in X(M) \setminus \bigvee_{i \in F} \mathrm{im}(f_i)$. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{D}$ as in the proof of 2.6. Consider the element $[x_F]_{F \to \mathcal{U}}$ of the ultraproduct $\prod_{F \to \mathcal{U}} M_F$. Fix an $\overline{F} \in \mathcal{D}$. The set of indices $\{F \mid x_F \notin \bigvee_{i \in \overline{F}} \mathrm{im}(f_i)\}$ contains the $\mathcal{U}$-large set $P_{\overline{F}}$. Since $\overline{F}$ was arbitrary, $[x_F]_{F \to \mathcal{U}}$ is not in any $\mathrm{im}(f_i)$, giving the contrapositive. □

Proposition 2.8. *If a sentence σ is true for all fields of characteristic zero, there is some N such that σ is true for all fields of characteristic $p > N$.*

Proof. Here are two proofs: the first one is just to take an ultraproduct of counterexamples.

The second proves it directly. Let $\mathcal{D}$ enumerate all the complete theories of fields of positive characteristic. Form a filter base $\{P_p\}_{p \text{ prime}}$ where P_p comprises those theories of fields whose characteristics are greater than p. This is closed under finite intersection (and in fact $P_p \cap P_q = P_{\max(p,q)}$). Let $\mathcal{U}$ be any completion of this filter base. For each $T \in \mathcal{D}$, let $M_T \models T$. In the ultraproduct $\prod_{T \to \mathcal{U}} M_T$, for every positive p, it is true that the characteristic is not p on a $\mathcal{U}$-large set, so this has characteristic zero, and by assumption, σ is true on a $\mathcal{U}$-large set.

Since $\mathcal{U}$ was arbitrary, and the intersection of the non-principal ultrafilters completing the filter base $\{P_p\}_{p \text{ prime}}$ is the filter base $\{P_p\}_{p \text{ prime}}$, σ is true on some P_N. □

Since ultraproducts codify compactness and the compactness theorem is a key feature of first-order logic, it is not unreasonable to expect that invariance with respect to ultraproduct-induced structure characterizes first-order definability in the models of a theory. This is strong conceptual completeness, which we will discuss in the next section.

A special case of strong conceptual completeness is the following definability criterion, which follows from the Beth theorem, though it is not necessary to invoke it.

Theorem 2.9. *Let X be a functor $\mathbf{Mod}(T) \to \mathbf{Set}$ which assigns to each model M a subset of some sort of M, such that for every I, $\mathcal{U}$, and $(M_i)_{i\in I}$ the equality $X\left(\prod_{i\to\mathcal{U}}\right) = \prod_{i\to\mathcal{U}} X(M_i)$ holds. Then X is definable.*

Proof. Expand the language $\mathcal{L}$ of T to the language $\mathcal{L}'$ by adding a new constant symbol c, meant to be interpreted arbitrarily inside $X(M)$ for every $M \models T$. Consider the class of $\mathcal{L}'$-structures (M, c) where $M \models T$ and $c \in X(M)$. By the Chang-Keisler [3] ultraproduct criterion for a class to be elementary and the assumptions on X, this is an elementary class of $\mathcal{L}$'-structures. Let T' axiomatize this class. Since we have only added a constant symbol to the language, the difference between T and T' consists of $\mathcal{L}'$-sentences $\{\varphi_i(c)\}_{i\in I}$, so that $X(M) = \bigcap_{i\in I} \varphi_i(M)$. If I was infinite, then there is a model M, a sequence $(c_i)_{i\in I}$ in M such that for a non-principal ultrafilter $\mathcal{U}$ on I, $[c_i]_{i\to\mathcal{U}}$ is in $\bigcap_{i\in I} \varphi_i\left(M^{\mathcal{U}}\right)$ but each c_i is not in $\varphi_i(M)$, hence not in $X(M)$. Then the inclusion

$$\left(\bigcap_{i\in I} \varphi_i(M)\right)^{\mathcal{U}} \subsetneq \bigcap_{i\in I} \varphi_i\left(M^{\mathcal{U}}\right)$$

is proper and not an equality, a contradiction. □

Strong conceptual completeness, which we will review in the next section 3.19, generalizes the theorem 2.9 by removing the assumptions of "subset" (and therefore also that of being able to talk about "equality").

3 Strong conceptual completeness

3.1 Pre-ultrafunctors

When $X : \mathbf{Mod}(T) \to \mathbf{Set}$ *is* $\mathrm{ev}_{\varphi(x)}$ and one proves the Łos theorem

$$X\left(\prod_{i\to\mathcal{U}} M_i\right) = \prod_{i\to\mathcal{U}} X(M_i),$$

one has the luxury of being able to test the displayed equation above between two subsets of (the interpretation in $\prod_{i\to\mathcal{U}} M_i$ of) the ambient sort of the formula $\varphi(x)$. If X is merely isomorphic to $\mathrm{ev}_{\varphi(x)}$, then $X\left(\prod_{i\to\mathcal{U}} M_i\right)$ and $\prod_{i\to\mathcal{U}} X(M_i)$ might be entirely different sets, with only the isomorphism to $\mathrm{ev}_{\varphi(x)}$ to compare them, so that testing equality as above is not a well-formulated question; rather, one asks for an isomorphism.

Remark 3.1. Given a natural isomorphism $\eta : X \simeq \mathrm{ev}_{\varphi(x)}$ with components $\{\eta_M : X(M) \simeq \varphi(M)\}_{M \in \mathbf{Mod}(T)}$, we have for every ultraproduct $\prod_{i \to \mathcal{U}} M_i$ a commutative square

$$\begin{array}{ccc} X\left(\prod_{i\to\mathcal{U}} M_i\right) & \overset{\Phi_{(M_i)}}{\dashrightarrow} & \prod_{i\to\mathcal{U}} X(M_i) \\ {\scriptstyle \eta_{\prod_{i\to\mathcal{U}} M_i}}\downarrow & & \downarrow{\scriptstyle \prod_{i\to\mathcal{U}} \eta_{M_i}} \\ \varphi\left(\prod_{i\to\mathcal{U}} M_i\right) & = & \prod_{i\to\mathcal{U}} \varphi(M_i). \end{array}$$

where the dashed map $\Phi_{(M_i)}$ is the composition of isomorphisms $\left(\prod_{i\to\mathcal{U}} \eta_{M_i}\right)^{-1} \circ \eta_{\prod_{i\to\mathcal{U}} M_i}$.

It is easy to see that the statement of Łos' theorem is functorial on elementary embeddings. That is, for every I, every ultrafilter $\mathcal{U}$ on I, and every sequence of elementary embeddings $f_i : M_i \to N_i$, the diagram

$$\begin{array}{ccc} \varphi\left(\prod_{i\to\mathcal{U}} M_i\right) & = & \prod_{i\to\mathcal{U}} \varphi(M_i) \\ {\scriptstyle [f_i]_{i\to\mathcal{U}}|_{\varphi(\prod_{i\to\mathcal{U}} M_i)}}\downarrow & & \downarrow{\scriptstyle [f_i|_{\varphi(M_i)}]_{i\to\mathcal{U}}} \\ \varphi\left(\prod_{i\to\mathcal{U}} N_i\right) & = & \prod_{i\to\mathcal{U}} \varphi(N_i) \end{array}$$

commutes.

Definition 3.2. For an arbitrary functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$, if we additionally specify for every $I, \mathcal{U}, (M_i)_{i\in I}$ the data of a *transition isomorphism* $\Phi_{(M_i)} : X\left(\prod_{i\to\mathcal{U}} M_i\right) \to \prod_{i\to\mathcal{U}} X(M_i)$, then we say that (X, Φ) "commutes with ultraproducts" if all diagrams

$$\begin{array}{ccc} X\left(\prod_{i\to\mathcal{U}} M_i\right) & \xrightarrow{\Phi_{(M_i)}} & \prod_{i\to\mathcal{U}} X(M_i) \\ {\scriptstyle X([f_i]_{i\to\mathcal{U}})}\downarrow & & \downarrow{\scriptstyle [X(f_i)]_{i\to\mathcal{U}}} \\ X\left(\prod_{i\to\mathcal{U}} N_i\right) & \xrightarrow[\Phi_{(N_i)}]{} & \prod_{i\to\mathcal{U}} X(N_i) \end{array}$$

commute. We let Φ abbreviate all the transition isomorphisms, and we call a pair (X, Φ) a **pre-ultrafunctor**. We will abuse terminology by referring to Φ as "the" transition isomorphism of the pre-ultrafunctor (X, Φ).

Given two pre-ultrafunctors (X, Φ) and (X', Φ'), we define a map between them, called an **ultratransformation**, to be a natural transformation $\eta : X \to X'$ which satisfies the following additional property: all diagrams

$$\begin{array}{ccc} X\left(\prod_{i\to\mathcal{U}} M_i\right) & \xrightarrow{\Phi_{(M_i)}} & \prod_{i\to\mathcal{U}} X(M_i) \\ {\scriptstyle \eta_{\prod_{i\to\mathcal{U}} M_i}}\downarrow & & \downarrow{\scriptstyle \prod_{i\to\mathcal{U}} \eta_{M_i}} \\ X'\left(\prod_{i\to\mathcal{U}} M_i\right) & \xrightarrow[\Phi'_{(M_i)}]{} & \prod_{i\to\mathcal{U}} X'(M_i) \end{array}$$

must commute.

With this terminology, the theorem 2.9 says that if X is a sub-pre-ultrafunctor of an evaluation functor $\mathrm{ev}_{\varphi(x)}$, then X is definable.

In light of the above definition, we can reformulate our observation about a definable functor $X \overset{\eta}{\simeq} \mathrm{ev}_{\varphi(x)}$ above as saying that the natural isomorphism η canonically equips X with a transition isomorphism such that η is an ultratransformation.

Remark 3.3. Every functor of points $\mathrm{ev}_{\varphi(x)}$ can be canonically viewed as a pre-ultrafunctor with the transition isomorphisms Φ just the identity maps (corresponding to the equality signs in the above diagrams).

One checks that if X and Y are definable sets, and $f : X \to Y$ is a definable function, then the induced natural transformation between evaluation functors $\mathrm{ev}_f : ev_X \to \mathrm{ev}_Y$ is in fact an ultratransformation. (This contains Los' theorem: in the proof, one is really showing that if S is the sort containing a formula $\varphi(x)$, then the canonical definable injection $i : \varphi(x) \hookrightarrow S$ induces an ultratransformation; the fact that the transition isomorphisms are all identities means that one ends up with the usual equality.)

Definition 3.4. The **category of pre-ultrafunctors** $\mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set})$ comprises the following data:

$$\mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set}) \overset{\mathrm{df}}{=} \begin{cases} \text{Objects: pre-ultrafunctors } (X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set} \\ \text{Morphisms: ultratransformations } \eta : (X, \Phi) \to (X, \Phi'). \end{cases}$$

Remark 3.5. By the remark 3.3, the evaluation functor $\mathrm{ev} : \mathbf{Def}(T) \to [\mathbf{Mod}(T), \mathbf{Set}]$ further factors through $\mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set})$:

$$\begin{array}{ccc} \mathbf{Def}(T) & \xrightarrow{\widehat{\mathrm{ev}}} & \mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set}) \\ & \searrow^{\mathrm{ev}} & \downarrow \\ & & [\mathbf{Mod}(T), \mathbf{Set}] \end{array},$$

where the arrow $\mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set})$ is just the forgetful functor $(X, \Phi) \mapsto X$.

Note that whenever there is an isomorphism $\eta : X \simeq Y$ as functors $\mathbf{Mod}(T) \to \mathbf{Set}$, and (X, Φ) is a pre-ultrafunctor, then by conjugating Φ by the isomorphism $X \simeq Y$ (as in the diagram 3.1), one canonically equips Y with a transition isomorphism Φ' such that $\eta : (X, \Phi) \to (Y, \Phi')$ is an ultratransformation.

Remark 3.6. That is, X is definable if and only if there is a transition isomorphism Φ such that (X, Φ) is isomorphic to $(\mathrm{ev}_{\varphi(x)}, \mathrm{id})$ for some formula $\varphi(x) \in T$. We will suppress the canonical transition isomorphism id and just say that (X, Φ) is isomorphic to $\mathrm{ev}_{\varphi(x)}$, understanding that this isomorphism is happening in $\mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set})$.

The pre-ultrafunctor condition 3.2 only stipulates compatibility with respect to ultraproducts of elementary embeddings. However, there are other elementary embeddings which arise purely formally between different ultraproducts with respect to different indexing sets and ultrafilters, and should be viewed as part of the formal structure on $\mathbf{Mod}(T)$ which is induced by being able to take ultraproducts. The canonical example is the diagonal embedding of a model into its ultrapower (which compares an ultrapower M with respect to the trivial indexing set and trivial ultrafilter to an ultrapower $M^{\mathcal{U}}$ with respect to a nontrivial indexing set and a nontrivial ultrafilter).

Definition 3.7. Fix $I, \mathcal{U}$, and a model $M \models T$.

The **diagonal embedding** $\Delta_M : M \to M^{\mathcal{U}}$ is given by sending each $a \in M$ to the equivalence class of the constant sequence $[a]_{i \to \mathcal{U}}$.

We can stipulate that a pre-ultrafunctor furthermore preserves the diagonal embeddings.

Definition 3.8. We say that a pre-ultrafunctor 3.2 (X, Φ) is a **Δ-functor** if for every I, for every $\mathcal{U}$, and for every M and the diagonal embedding $M \xrightarrow{\Delta_M} M^{\mathcal{U}}$, the diagram

$$\begin{array}{ccc} & & X\left(M^{\mathcal{U}}\right) \\ & \overset{X(\Delta_M)}{\nearrow} & \\ X(M) & & \Big\downarrow \Phi_{(M)} \\ & \underset{\Delta_{X(M)}}{\searrow} & \\ & & X(M)^{\mathcal{U}} \end{array}$$

commutes.

Remark 3.9. It is not true in general that the embedding $\widehat{\mathrm{ev}} : \mathbf{Def}(T) \to \mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set})$ is an equivalence of categories. If (X, Φ) is isomorphic to $\mathrm{ev}_{\varphi(x)}$, then (X, Φ) preserves the diagonal embeddings of models into their ultrapowers (in the sense of the definition 3.8). However, later, we will exhibit an example 6.2 of a pre-ultrafunctor which does not preserve diagonal embeddings.

It is not true either that in general being a Δ-functor characterizes the image of $\widehat{\mathrm{ev}}$; we later construct a counterexample 6.1.

Strong conceptual completeness 3.19 says that if we sufficiently generalize the diagonal embeddings to a large-enough class of formal comparison maps between ultraproducts (with respect to possibly different indexing sets and ultrafilters), then we can characterize the image of $\widehat{\mathrm{ev}}$ as precisely those pre-ultrafunctors which additionally preserve all these formal comparison maps. The notion we want is that of an *ultramorphism*.

3.2 Ultramorphisms

Definition 3.10. ([7], Section 3) An **ultragraph** Γ comprises:

(i) Two disjoint sets Γ^f and Γ^b, called the sets of **free nodes** and **bound nodes**, respectively.

(ii) For any pair $\gamma, \gamma' \in \Gamma$, there exists a set $E(\gamma, \gamma')$ of **edges**. This gives the data of a directed graph.

(iii) For any bound node $\beta \in \Gamma^b$, we assign a triple $\langle I, \mathcal{U}, g\rangle \overset{\text{df}}{=} \langle I_\beta, \mathcal{U}_\beta, g_\beta\rangle$ where $\mathcal{U}$ is an ultrafilter on I and g is a function $g : I \to \Gamma^f$.

Definition 3.11. ([7], Section 3) An **ultradiagram of type** Γ in a pre-ultracategory $\underline{\mathbf{S}}$ is a diagram $A : \Gamma \to \mathbf{S}$ assigning an object A to each node $\gamma \in X$, and assigning a morphism in $\mathbf{S}$ to each edge $e \in E(\gamma, \gamma')$, such that

$$A(\beta) = \prod_{i \in I_\beta} A(g_\beta(i))/\mathcal{U}_\beta$$

for all bound nodes $\beta \in \Gamma^b$.

Given this notion of a diagram with extra structure, there is an obvious notion of natural transformations between such diagrams which preserve the extra given structure.

Definition 3.12. ([7], Section 3) Let $A, B : \Gamma \to \mathbf{S}$. A **morphism** of ultradiagrams $\Phi : A \to B$ is a natural transformation Φ satisfying

$$\Phi_\beta = \prod_{i \to \mathcal{U}_\beta} \Phi_{g_\beta(i)}$$

for all bound nodes $\beta \in \Gamma^b$.

Now we define ultramorphisms.

Definition 3.13. ([7], Section 3) Let $\mathrm{Hom}(\Gamma, \underline{\mathbf{S}})$ be the category of all ultradiagrams of type Γ inside $\underline{\mathbf{S}}$ with morphisms the ultradiagram morphisms 3.12 defined above. Any two nodes $k, \ell \in \Gamma$ define *evaluation functors* $(k), (\ell) : \mathrm{Hom}(\Gamma, \underline{\mathbf{S}}) \rightrightarrows \mathbf{S}$, by

$$(k)\left(A \overset{\Phi}{\to} B\right) = A(k) \overset{\Phi_k}{\to} B(k)$$

(resp. ℓ).

An **ultramorphism** of type $\langle \Gamma, k, \ell\rangle$ in $\underline{\mathbf{S}}$ is a natural transformation $\delta : (k) \to (\ell)$.[1]

Let us unravel the definition 3.13 for the prototypical example $\Delta : M \hookrightarrow M^{\mathcal{U}}$ of an ultramorphism.

Example 3.14. Given an ultrafilter $\mathcal{U}$ on I, put:

- $\Gamma^f = \{k\}$,
- $\Gamma^b = \{\ell\}$,
- $E(\gamma, \gamma') = \varnothing$ for all $\gamma, \gamma' \in \Gamma$,
- $\langle I_\ell, \mathcal{U}_\ell, g_\ell\rangle = \langle I, \mathcal{U}, g\rangle$ where g is the constant map to k from I.

[1]Note that in our terminology, an ultramorphism, singular, refers to a collection of possibly many maps (the components of the natural transformation $(k) \to (\ell)$).

By the ultradiagram condition 3.11, an ultradiagram A of type Γ in $\underline{\mathbf{S}}$ is determined by $A(k)$, with $A(\ell) = A(k)^{\mathcal{U}}$.

By the ultradiagram morphism condition 3.12, an ultramorphism of type $\langle \Gamma, k, \ell \rangle$ must be a collection of maps $\left(\delta_M : M \to M^{\mathcal{U}}\right)_{M \in \mathbf{Mod}(T)}$ which make all squares of the form

$$\begin{array}{ccc} M^{\mathcal{U}} & \xrightarrow{f^{\mathcal{U}}} & N^{\mathcal{U}} \\ {\scriptstyle \Delta_M}\uparrow & & \uparrow{\scriptstyle \Delta_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

commute. It is easy to check that setting $\delta_M = \Delta_M$ the diagonal embedding gives an ultramorphism.

Definition 3.15. The next least complicated example of an ultramorphism are the **generalized diagonal embeddings**. Here is how they arise: let $g : I \to J$ be a function between two indexing sets I and J. g induces a pushforward map $g_* : \beta I \to \beta J$ between the spaces of ultrafilters on I and J, by $g_*\mathcal{U} \overset{\text{df}}{=} \{P \subseteq J \,\big|\, g^{-1}(P) \in \mathcal{U}$. Fix $\mathcal{U} \in \beta I$ and put $\mathcal{V} \overset{\text{df}}{=} g_*\mathcal{U}$. Let $(M_j)_{j \in J}$ be a J-indexed family of models.

Then there is a canonical "fiberwise diagonal embedding"

$$\Delta_g : \prod_{j \to \mathcal{V}} M_j \to \prod_{i \to \mathcal{U}} M_{g(i)}$$

given on $[a_j]_{j \to \mathcal{V}}$ by replacing each entry a_j with $g^{-1}(\{a_j\})$-many copies of itself.

In terms of the definition 3.13 of an ultramorphism, the free nodes are J, and there are two bound nodes k and ℓ. To k we assign the triple $\langle J, \mathcal{V}, \mathrm{id}_J \rangle$ and to ℓ we assign the triple $\langle I, \mathcal{U}, g \rangle$. Then Δ_g induces an ultramorphism $(k) \to (\ell)$.

Now we state what it means for ultramorphisms to be preserved. One should keep in mind the special case of the diagonal ultramorphism.

Definition 3.16. Let $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ be a pre-ultrafunctor, and let δ be an ultramorphism in $\mathbf{Mod}(T)$ and δ' an ultramorphism in $\mathbf{Set}$, both of ultramorphism type $\langle \Gamma, k, \ell \rangle$

Recall that in the terminology of the definition 3.13, δ is a natural transformation $(k) \xrightarrow{\delta} (\ell)$ of the evaluation functors

$$(k), (\ell) : \mathrm{Hom}(\Gamma, \mathbf{Mod}(T)) \to \mathbf{Mod}(T).$$

(Resp. δ', $\mathbf{Set}$.)

Note that for any ultradiagram $\mathscr{M} \in \mathrm{Hom}(\Gamma, \mathbf{Mod}(T))$, $X \circ \mathscr{M}$ is an ultradiagram in $\mathrm{Hom}(\Gamma, \mathbf{Set})$. We say that X **carries** δ **into** δ' (prototypically, δ and δ' will both be canonically defined in the same way in both $\mathbf{Mod}(T)$ and $\mathbf{Set}$ and in this case we say that

δ has been preserved) if for every ultradiagram $\mathscr{M} \in \operatorname{Hom}(\Gamma, \mathbf{Mod}(T))$, the diagram

$$\begin{array}{ccc} X(\mathscr{M}(k)) & \xrightarrow{X(\delta_{\mathscr{M}})} & X(\mathscr{M}(\ell)) \\ {\scriptstyle \Phi_{\mathscr{M}(k)}}\downarrow & & \downarrow{\scriptstyle \Phi_{\mathscr{M}(\ell)}} \\ (X\mathscr{M})(k) & \xrightarrow[\delta'_{X\mathscr{M}}]{} & (X\mathscr{M})(\ell) \end{array}$$

commutes. (We are abusing notation and understand that in the above if k is not a bound node, then the ultraproduct on the bottom left becomes trivial and $\Phi_{\mathscr{M}(k)}$ is actually the identity map $\mathrm{id}_{X(\mathscr{M}(k))}$ (resp. ℓ, ultraproduct on the bottom right).)

Note that what is really happening is that we are applying the covariant Hom-functor $\operatorname{Hom}(X, -)$ to push forward each ultradiagram $\mathscr{M}$ to an ultradiagram $X \circ \mathscr{M}$, and then asking that the pushed-forward ultramorphism $X(\delta)$ is isomorphic to $\delta'_{X\mathscr{M}}$ via X's transition isomorphism Φ.

3.3 Stating strong conceptual completeness

Just as Δ-functors 3.8 are pre-ultrafunctors which additionally preserve the diagonal embedding ultramorphisms, we define *ultrafunctors* to be pre-ultrafunctors which preserve *all* ultramorphisms.

Definition 3.17. ([7], Section 3) An **ultrafunctor** $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is a pre-ultrafunctor which respects the fibering over $\mathbf{Set}$: for every $\delta \in \Delta(\mathbf{Set})$, X carries $\delta_{\mathbf{Mod}(T)}$ into $\delta_{\mathbf{Set}}$ (in the sense of the definition 3.16 above) for all $\delta \in \Delta(\mathbf{Set})$.

Definition 3.18. A map between ultrafunctors is just an ultratransformation 3.2 of the underlying pre-ultrafunctors. Write $\mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set})$ for the category of ultrafunctors $\mathbf{Mod}(T) \to \mathbf{Set}$.

There is a canonical evaluation functor

$$\widetilde{\mathrm{ev}} : \mathbf{Def}(T) \to \mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set})$$

sending each definable set $A \in T$ to its corresponding ultrafunctor $\widetilde{\mathrm{ev}}_A$, and we now have the following picture of factorizations of the original evaluation map $\mathrm{ev} : \mathbf{Def}(T) \to [\mathbf{Mod}(T) \to \mathbf{Set}]$:

$$\begin{array}{ccc} \mathbf{Def}(T) & \xrightarrow{\widetilde{\mathrm{ev}}} & \mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set}) \\ & {\scriptstyle \widehat{\mathrm{ev}}}\searrow & \big\downarrow \\ & & \mathbf{PUlt}(\mathbf{Mod}(T), \mathbf{Set}) \\ & {\scriptstyle \mathrm{ev}}\searrow & \big\downarrow \\ & & [\mathbf{Mod}(T), \mathbf{Set}] \end{array}$$

Now, we can state strong conceptual completeness.

Theorem 3.19. *([7], Section 4)* $\widetilde{\mathrm{ev}} : \mathbf{Def}(T) \to \mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set})$ *is an equivalence of categories.*

4 Strong conceptual completeness for $\aleph_0$-categorical theories

Of course, the point of all this is that when the theory is nice enough, we can ignore the more general ultramorphisms and still obtain a statement of strong conceptual completeness. In this section, we show that when the theory T is additionally assumed to be $\aleph_0$-categorical, we can replace "ultrafunctor" with Δ-functor 3.8 in the statement of strong conceptual completeness, so that only the simplest ultramorphisms 3.13 suffice to state strong conceptual completeness for $\aleph_0$-categorical theories.

4.1 Preliminaries on $\aleph_0$-categorical theories

Definition 4.1. A theory T is $\aleph_0$-categorical if T is countable and has, up to isomorphism, a single countable model.

The definable sets of $\aleph_0$-categorical theories are exceptionally easy to understand: they are precisely the finite disjoint unions of orbits of the automorphism group; furthermore, $\aleph_0$-categorical theories are determined up to bi-interpretability by the automorphism group of the unique countable model topologized by pointwise convergence.

Theorem 4.2. *(Ryll-Nardzewski) T is $\aleph_0$-categorical if and only if it has only finitely many types in each sort (this implies that all the types are isolated.)*

Corollary 4.3. *If $M \models T$ is the unique countable model of the $\aleph_0$-categorical theory T, then M has only finitely many $\mathrm{Aut}(M)$-orbits in each sort (each corresponding to the points of an isolated type.)*

Theorem 4.4. *(Coquand-Ahlbrandt-Ziegler) Let T and T' be $\aleph_0$-categorical with countable models M and M'. Then T and T' are bi-interpretable if and only if there is an isomorphism of topological groups $\mathrm{Aut}(M) \simeq \mathrm{Aut}(M')$, where $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M')$ are topologized by pointwise convergence.*

It follows that in the unique countable model $M \models T$ of an $\aleph_0$-categorical theory, a subset of a sort in M is definable if and only if it is invariant under the action of $\mathrm{Aut}(M)$. In fact, any $\mathrm{Aut}(M)$-invariant *quotient* of a definable subset of M is definable in T^{eq}, since the kernel relation of the quotient will be a definable set.

4.2 Diagonal embeddings and the finite support property

As a warm-up to the theorem 4.8, we will show in general that if $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is a Δ-functor, X must map $\mathrm{Aut}(M)$ *continuously* to $\mathrm{Sym}(X(M))$.

Proposition 4.5. *Let T be any theory, and let $(X, \Phi) : \mathbf{Mod}(T) \to \mathbf{Set}$ be a Δ-functor. Then for any model $M \models T$, the restriction of X to a map $\mathrm{Aut}(M) \to \mathrm{Sym}(X(M))$ is a continuous group homomorphism (where both groups are topologized by pointwise convergence).*

Proof. Since X is a functor, its restriction to $\mathrm{Aut}(M)$ is a group homomorphism. To check continuity, let $\mathcal{D}$ be a directed partial order indexing a net of automorphisms $[\sigma_\alpha]_{\alpha \in \mathcal{D}}$. It suffices to check that if $[\sigma_\alpha]_{\alpha \in \mathcal{D}} \to \sigma$ in $\mathrm{Aut}(M)$, then $[X\sigma_\alpha]_{\alpha \in \mathcal{D}} \to X\sigma$ in $\mathrm{Sym}(X(M))$.

We will suppose not and take an ultraproduct of counterexamples. So suppose that $[X\sigma_\alpha]_{\alpha \in \mathcal{D}}$ does not converge to $X\sigma$. The basic open neighborhoods $B_{c \mapsto d}$ of $X\sigma$ are parametrized by tuples c, d of the same sort, and they look like this:

$$B_{c \mapsto d} \stackrel{\mathrm{df}}{=} \{\rho : X(M) \to X(M) \,|\, \rho(c) = d\}.$$

Since $[X\sigma_\alpha]_{\alpha \in \mathcal{D}}$ does not converge to $X\sigma$, then there exists some neighborhood $B_{c \mapsto d}$ such that for every $\alpha \in \mathcal{D}$, there exists an $\alpha' \geqslant \alpha \in \mathcal{D}$ such that $X\sigma_{\alpha'} \notin B_{c \mapsto d}$.

Now, let I be the underlying set of $\mathcal{D}$, and consider the collection of subsets $\{P_\alpha \subseteq I\}_{\alpha \in \mathcal{D}}$, where each P_α is the set of all $\beta \in \mathcal{D}$ such that $\beta \geqslant \alpha$. Since $\mathcal{D}$ was a directed partial order, $\{P_\alpha\}_{\alpha \in \mathcal{D}}$ has the finite intersection property, and can therefore be completed to an ultrafilter $\mathcal{U}$.

Then consider the ultraproduct of automorphisms

$$[X\sigma_{\alpha'}]_{\alpha \to \mathcal{U}} : X(M)^{\mathcal{U}} \to X(M)^{\mathcal{U}}.$$

Let $\Delta_{X(M)}$ be the diagonal embedding of $X(M)$ into $X(M)^{\mathcal{U}}$. Since every $X\sigma_{\alpha'}$ sends c to $d' \neq d$, $[X\sigma_{\alpha'}]_{\alpha \to \mathcal{U}}$ sends $\Delta_{X(M)}(c)$ to $\Delta_{X(M)}(d') \neq \Delta_{X(M)}(d)$. Therefore,

$$[X\sigma_{\alpha'}]_{\alpha \to \mathcal{U}} \circ \Delta_{X(M)} \neq [X\sigma]_{\alpha \to \mathcal{U}} \circ \Delta_{X(M)}.$$

By the definition 3.8 of a Δ-functor, we can replace $\Delta_{X(M)}$ with $\Phi_{(M)} \circ X(\Delta_M)$. By the definition 3.2 of a pre-ultrafunctor, we can replace $[X\sigma_{\alpha'}]_{\alpha \to \mathcal{U}}$ and $[X\sigma]_{\alpha \to \mathcal{U}}$ with

$$\Phi_{(M)} \circ X([\sigma_{\alpha'}]_{\alpha \to \mathcal{U}}) \circ \Phi^{-1}_{(M)} \text{ and } \Phi_{(M)} \circ X([\sigma]_{\alpha \to \mathcal{U}}) \circ \Phi^{-1}_{(M)}.$$

Substituting into the displayed inequality above and letting inverse transition isomorphisms cancel out, we obtain

$$\Phi_{(M)} \circ X([\sigma_{\alpha'}]_{\alpha \to \mathcal{U}}) \circ X(\Delta_M) \neq \Phi_{(M)} \circ X([\sigma]_{\alpha \to \mathcal{U}}) \circ X(\Delta_M)$$

and since $\Phi_{(M)}$ is a bijection, we may omit it:

$$X([\sigma_{\alpha'}]_{\alpha \to \mathcal{U}}) \circ X(\Delta_M) \neq X([\sigma]_{\alpha \to \mathcal{U}}) \circ X(\Delta_M).$$

Since X is a functor, we conclude that

$$X\left([\sigma_{\alpha'}]_{\alpha\to\mathcal{U}}\circ\Delta_M\right)\neq X\left([\sigma]_{\alpha\to\mathcal{U}}\circ\Delta_M\right)$$

and since X is certainly a function from $\mathbf{Mod}(T)(M,M^{\mathcal{U}})\to\mathbf{Set}\left(X(M),X\left(M^{\mathcal{U}}\right)\right)$, this means that

$$[\sigma_{\alpha'}]_{\alpha\to\mathcal{U}}\circ\Delta_M\neq[\sigma]_{\alpha\to\mathcal{U}}\circ\Delta_M.$$

But this inequality says that there is some $a\in M$ such that for every α, there is an α' such that $\{\sigma_{\alpha'}(a)\}_\alpha$ disagrees with $\{\sigma(a)\}_\alpha$ on some $\mathcal{U}$-large set of indices P. Letting $c=a$ and $d=\sigma(c)$, we have that a $\mathcal{U}$-large subset of $\{\sigma_{\alpha'}(a)\}_\alpha$ lies outside of the basic open $B_{c\mapsto d}\ni\sigma$. Since $\mathcal{U}$ contains all the principal filters in $\mathcal{D}$, we have that for every $\alpha\in\mathcal{D}$, the intersection $P\cap P_\alpha$ is nonempty. So, for the basic open $B_{c\mapsto d}\ni\sigma$, we have that for every α we can find some $\alpha''\in P\cap P_\alpha$ such that $\sigma_{\alpha''}\notin B_{c\mapsto d}$. Therefore, $[\sigma_\alpha]_{\alpha\in\mathcal{D}}$ does not converge to σ, which is the contrapositive. □

Definition 4.6. Fix X a functor $\mathbf{Mod}(T)\to\mathbf{Set}$ which restricts to continuous maps on automorphism groups. Fix $M\models T$. From the continuity we can associate to every tuple $x\in X(M)$ a tuple $a_x\in M$ as follows: the preimage $X^*\operatorname{Stab}(x)$ of the basic open subgroup $\operatorname{Stab}(x)\subseteq\operatorname{Sym}(X(M))$ must be open, and must therefore be covered by the cosets of a basic open subgroup of $\operatorname{Aut}(M)$, which is of the form $\operatorname{Stab}(a_x)$ for some tuple a_x.

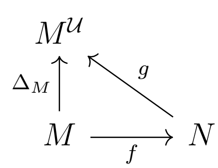
We call the tuple a_x the **support** of x. It satisfies the following property: whenever $\sigma_1,\sigma_2\in\operatorname{Aut}(M)$ agree on a_x, then $X\sigma_1, X\sigma_2$ agree on x. By sending $a_x\mapsto x$ and letting $\operatorname{Aut}(M)$ act, this induces an $\operatorname{Aut}(M)$-equivariant surjection from the orbit of a_x to the orbit of x.

Lemma 4.7. *Let T be any theory, and let $X:\mathbf{Mod}(T)\to\mathbf{Set}$ be a Δ-functor. Then X preserves filtered colimits of models: for any model N, if N can be written as the filtered colimit $N\simeq\varinjlim M_i$, then $X(N)\simeq\varinjlim X(M_i)$.*

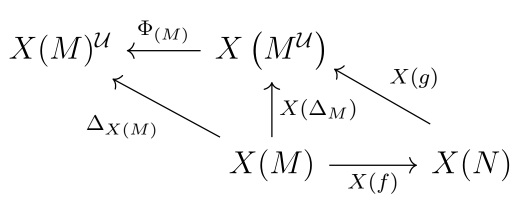
Proof. First, we'll show that being a Δ-functor implies that elementary embeddings are sent to injective functions:

Claim: Let $f:M\to N$ be an elementary embedding. Then $X(f):X(M)\to X(N)$ is injective.

Proof of claim. By Scott's lemma (see e.g. [1] for a proof), there is an ultrapower $M^{\mathcal{U}}$ of M and an elementary map $g:N\to M^{\mathcal{U}}$ such that the diagram



commutes. Since X was assumed to be a Δ-functor, the diagram



commutes. Since $\Delta_{X(M)} : X(M) \hookrightarrow X(M)^{\mathcal{U}}$ is injective and $\Phi_{(M)}$ is a transition isomorphism, $X(\Delta_M)$ is injective, and therefore the composite $X(g) \circ X(f)$ is injective. Therefore, $X(f)$ was injective. $\square$

Claim: For any $N \models T$, the collection of maps $\{X(f) \mid f : M \to N,\ M \text{ countable}\}$ jointly surject onto $X(N)$.

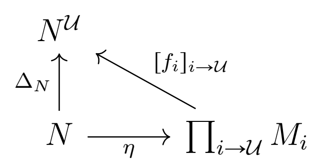
Proof of claim. Since N is covered by copies of countable models, we do know that $\{f \mid f : M \to N,\ M \text{ countable}\}$ jointly covers N.

Let I index the elementary embeddings from (representatives of isomorphism classes of) all countable models to N. Let $\mathcal{U}$ be a non-principal ultrafilter on I which contains the sets $P_{\vec{n}} \overset{\mathrm{df}}{=} \{i \in I \mid \mathrm{im}(f_i) \ni \vec{n}\}$, which has the finite intersection property by the downward Lowenheim-Skolem theorem.

Consider the map

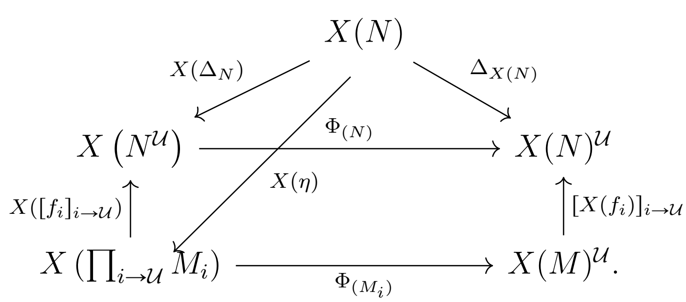
$$\prod_{i \to \mathcal{U}} M_i \overset{[f_i]_{i\to\mathcal{U}}}{\longrightarrow} N^{\mathcal{U}}.$$

The diagonal copy of N in $N^{\mathcal{U}}$ is in the image of this map: if $[n]_{i\to\mathcal{U}} \in N^{\mathcal{U}}$, then $\{i \in I \mid \exists m_i \text{ s.t. } f_i(m_i) = n\}$ is in $\mathcal{U}$, so $[f_i]_{i\to\mathcal{U}}[m_i]_{i\to\mathcal{U}} = [n]_{i\to\mathcal{U}}$. Pulling back $\Delta_N(N)$ along $[f_i]_{i\to\mathcal{U}}$, we obtain a map η from N into $\prod_{i\to\mathcal{U}} M_i$ such that the diagram



commutes.

Now apply X, obtaining the commutative diagram (it is easy to check that the extra subdiagrams involving $X(\eta)$ commute by $\Phi_{(N)}$ and $\Phi_{(M_i)}$ being isomorphisms):



In particular,

$$\Delta_{X(N)} = [X(f_i)]_{i\to\mathcal{U}} \circ \Phi_{(M_i)} \circ X(\eta).$$

This implies that $\Delta_{X(N)}$ is contained inside the image of $[X(f_i)]_{i\to\mathcal{U}}$.

Now, suppose that the $X(f_i)$ did not cover $X(N)$. That is, suppose that there exists an $x \in X(N)$ such that x lies outside of the image of $X(f_i)$ for every $i \in I$. Then for any $[m_i]_{i \to \mathcal{U}} \in \prod_{i \to \mathcal{U}} M_i$, $f_i(m_i) \neq x$ for all $i \in I$. Therefore, $\Delta_{X(N)}(x)$ is not contained in the image of $[X(f_i)]_{i \to \mathcal{U}}$, a contradiction.

We conclude that $\{X(f) \,|\, f : M \to N\}$ jointly surjects onto $X(N)$. $\square$

Claim: Present N as a filtered colimit of its countable submodels M_i. Then $X(N) \simeq \varinjlim X(M_i)$.

Proof of claim. Our two previous claims show that we may view $X(N)$ as the union of the $X(M_i)$'s. $\varinjlim X(M_i)$ can be canonically written as

$$\left(\bigsqcup_{i \in I} X(M_i)\right)/E$$

where $(x \in X(M_i)) \sim_E (y \in X(M_j))$ if and only if x and y become the same element in some $X(M_k)$ for M_k amalgamating M_i and M_j. It is easy to check that sending an $x \in X(N)$ to the E-class of an arbitrary lift $x' \in X(M_i)$ (for a choice of some $X(M_i)$ containing x') gives a bijection

$$X(N) \simeq \varinjlim X(M_i) \text{ by } x \mapsto [x']_E,$$

compatible over the $X(M_i)$'s. $\square$

So far, we have shown that X preserves filtered colimits of countable models. But every model is a filtered colimit of countable models. Explicitly, if we have $N = \varinjlim_i N_i$ where the N_i are possible uncountable, we have that each $N_i = \varinjlim_j N_j^i$, so that we have written N as a filtered colimit of countable models N_j^i:

$$N = \varinjlim_i \varinjlim_j N_j^u = \varinjlim_{(i,j)} N_j^i$$

Then

$$X(N) \simeq \varinjlim_{(i,j)} X(N_j^i) \simeq \varinjlim_i \varinjlim_j X(N_j^i) \simeq \varinjlim_i X(N_i).$$

$\square$

Theorem 4.8. *Let T be $\aleph_0$-categorical. A functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is definable if and only if there is a transition isomorphism Φ such that (X, Φ) is a Δ-functor.*

Proof. If X is definable, then its isomorphism to an evaluation functor φ pulls back φ's transition isomorphism Φ' to a transition isomorphism Φ for X, and since (φ, Φ') was an ultrafunctor (X, Φ) is also (these are diagrammatic conditions on Φ' and so are invariant under conjugation by isomorphisms).

On the other hand, suppose that (X, Φ) is a Δ-functor. $\mathrm{Aut}(M)$ acts via X on $X(M)$, and so $X(M)$ splits up into $\mathrm{Aut}(M)$-orbits. For each representative x of these orbits, we know from the remarks following 4.5 that there is a tuple $a_x \in M$ which supports x, and the map $a_x \mapsto x$ induces an $\mathrm{Aut}(M)$-equivariant map from the orbit (type) of a_x to the orbit of x.

Therefore, each $\mathrm{Aut}(M)$-orbit of $X(M)$ is a quotient of an $\mathrm{Aut}(M)$-orbit of M by some $\mathrm{Aut}(M)$-invariant equivalence relation. Since M is ω-categorical, these equivalence relations are definable and all types are isolated by formulas, so we can write:

$$X(M) \simeq \bigvee_{i \in I} M(\varphi_i(x_i)) \simeq \bigsqcup_{i \in I} M(\varphi_i(x_i)).$$

By the previous lemma 4.7 and the fact that colimits always commute with colimits and definable functors always commute with filtered colimits of models, we conclude (writing $N = \varinjlim_j M_j$):

$$\begin{aligned} X(N) &\simeq \varinjlim_j \left(\bigsqcup_{i \in I} \varphi_i(M_j) \right) && (1) \\ &\simeq \bigsqcup_{i \in I} \left(\varinjlim_j \varphi_i(M_j) \right) && (2) \\ &\simeq \bigsqcup_{i \in I} \left(\varphi_i(\varinjlim_j M_j) \right) && (3) \\ &\simeq \bigsqcup_{i \in I} \varphi_i(N). && (4) \end{aligned}$$

Now we will show that the I indexing the φ_i must be finite.

In the pre-ultrafunctor condition

$$\begin{array}{ccc} X\left(\prod_{\mathcal{U}} M_i\right) & \xrightarrow{\Phi_{\mathcal{U},(M_i)}} & \prod_{\mathcal{U}}\left(X(M_i)\right) \\ {\scriptstyle X(\prod_{\mathcal{U}} f_i)}\downarrow & & \downarrow{\scriptstyle \prod_{\mathcal{U}} X(f_i)} \\ X\left(\prod_{\mathcal{U}} N_i\right) & \xrightarrow[\Phi_{\mathcal{U},(N_i)}]{} & \prod_{\mathcal{U}}\left(X(N_i)\right), \end{array}$$

restricting our attention to just ultraproducts of automorphisms tells us that $\Phi_{(M_i)} : X\left(\prod_{i \to \mathcal{U}}\right) M_i \to \prod_{i \to \mathcal{U}} X(M_i)$ is a $\prod_{i \to \mathcal{U}} \mathrm{Aut}(M_i)$-equivariant bijection, and therefore induces a bijection on the orbits of the action on either side.

Let $\mathcal{U}$ be some ultrafilter such that $|I^{\mathcal{U}}| > |I|$. Then, at the countable model M, we have the bijection:

$$X\left(M^{\mathcal{U}}\right) \overset{\Phi_{(M)}}{\simeq} \left(X(M)\right)^{\mathcal{U}}.$$

Now, the left hand side is $\bigsqcup_{i \in I} \varphi_i\left(M^{\mathcal{U}}\right)$. Each $\varphi_i\left(M^{\mathcal{U}}\right)$ is actually an $\mathrm{Aut}(M)^{\mathcal{U}}$-orbit, since $\varphi_i(M)$ was an $\mathrm{Aut}(M)$-orbit. Therefore, the number of $\mathrm{Aut}(M)^{\mathcal{U}}$-orbits on the left hand side is $|I|$.

On the right hand side, we have $(\bigsqcup_{i \in I} \varphi_i(M))^{\mathcal{U}}$. Two points $[x_i]_{i \to \mathcal{U}}$ and $[y_i]_{i \to \mathcal{U}}$ are $\mathrm{Aut}(M)^{\mathcal{U}}$-conjugate if and only if there exists a $P \in \mathcal{U}$ such that for all $j \in P$, $\varphi_{x_j} = \varphi_{y_j}$ (where φ_{x_i} means which φ_k x_i came from.) But, this is the same as saying $[\varphi_{x_j}]_{j \to \mathcal{U}} = [\varphi_{y_j}]_{j \to \mathcal{U}}$. So the number of orbits on the right hand side is $|I|^{\mathcal{U}}$.

Therefore, $|I^{\mathcal{U}}| = |I|$, so I must be finite. Hence there is a formula $\varphi(x)$ such that $X(N) \simeq \varphi(N)$ for all $N \models T$. Since for each N, this isomorphism $X(N) \simeq \varphi(N)$ is induced via filtered colimits by $X(M) \simeq \varphi(M)$, this is a natural isomorphism, so X is definable. □

5 An ultraproduct coherence criterion for objects in the classifying topos

In this section, we will prove, for objects B in the *classifying topos* $\mathscr{E}(T)$ of T—which is a natural enlargement of $\mathbf{Def}(T)$ whose "models" are the same as T's, and whose objects pick out a subcategory of evaluation functors $\mathbf{Mod}(T) \to \mathbf{Set}$ containing the image of $\mathrm{ev} : \mathbf{Def}(T) \to \mathbf{Set}$—that ev_B being a pre-ultrafunctor characterizes whether or not $B \in \mathbf{Def}(T)$.

We will see that this generalizes the theorem 4.8.

5.1 Preliminaries on the classifying topos

For the construction and standard facts about the classifying topos of a first-order (or generally, a coherent) theory, see e.g. Part D of [5] or Volume III of [2]. For our convenience we will repeat the essentials we will need in the rest of this section.

Definition 5.1. The **classifying topos** of a first-order theory T is a topos $\mathscr{E}(T)$ equipped with a fully faithful functor $\mathbf{y} : \mathbf{Def}(T) \to \mathscr{E}(T)$ which is also a model in the sense of 2.3 (the definition given there only involves the preservation of certain categorical properties, so makes sense for functors into any topos instead of $\mathbf{Set}$). $\mathscr{E}(T)$ additionally satisfies the following universal property: for any other topos $\mathscr{S}$ and any model $M : \mathbf{Def}(T) \to \mathscr{S}$ of $\mathbf{Def}(T)$ in $\mathscr{S}$, there exists a unique $\widetilde{M} : \mathscr{E}(T) \to \mathscr{S}$ such that the diagram

$$\begin{array}{ccc} \mathscr{E}(T) & & \\ {\scriptstyle \mathbf{y}}\uparrow & \searrow^{\widetilde{M}} & \\ \mathbf{Def}(T) & \xrightarrow[M]{} & \mathscr{S} \end{array}$$

commutes.

This characterizes $\mathscr{E}(T)$ up to equivalence. We call $\widetilde{M}$ the **inverse image functor** associated to the model M. We also call objects of $\mathscr{E}(T)$ which are, up to isomorphism, in the image of **y representable** (echoing the standard construction of $\mathscr{E}(T)$ as a certain category of sheaves on $\mathbf{Def}(T)$.)

As the definition indicates, the extension $\widetilde{M}$ of M from $\mathbf{Def}(T)$ to $\mathscr{E}(T)$ should be determined by what M does on the objects of $\mathbf{Def}(T)$. The following discussion is meant to make this intuition explicit, and to give a formula for computing what $\widetilde{M}$ is outside of the image of $\mathbf{y}$ inside $\mathscr{E}(T)$.

5.1.1 Computing the associated inverse image functor $\widetilde{M}$

Definition 5.2. (3.7.1 of [2]) Let $F : A \to B$ and $G : A \to C$ be functors. The **left Kan extension** of G along F, if it exists, is a pair (K, α) where $K : B \to C$ is a functor and $\alpha : G \to K \circ F$ is a natural transformation satisfying the following universal property if (H, β) is another pair with $H : B \to C$ a functor and $\beta : G \to H \circ F$ a natural transformation, then there exists a unique natural transformation $\gamma : K \to H$ satisfying the equality $(\gamma F) \circ \alpha = \beta$, as in the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ {\scriptstyle G}\downarrow & \underset{H}{\overset{K}{\swarrow\!\swarrow}} & \\ C & & \end{array} \quad , \qquad \gamma : K \xrightarrow{!} H.$$

We write $\operatorname{Lan}_F G$ for the left Kan extension of G along F. Right Kan extensions are defined dually, and are written $\operatorname{Ran}_F G$.

Before proceeding, we give two definitions around the category of points of a (contravariant) functor.

Definition 5.3. Consider the diagram of functors $\begin{array}{ccc} C & & D. \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle G} & \\ & E & \end{array}$ The **comma category** $(F \downarrow G)$ is given by:

Objects: (c, d, α) where $c \in C, d \in D, \alpha : F(c) \to G(d) \in E$.

Morphisms: $\operatorname{Hom}_{(F\downarrow G)}((c_1, d_1, \alpha_1), (c_2, d_2, \alpha_2))$ is defined to be the set

$$\left\{ (\beta_1, \beta_2) \,\middle|\, \beta_1 : c_1 \to c_2, \beta_2 : d_1 \to d_2, \text{ and } \begin{array}{ccc} F(c_1) & \xrightarrow{F(\beta_1)} & F(c_2) \\ {\scriptstyle\alpha_1}\downarrow & & \downarrow{\scriptstyle\alpha_2} \\ G(d_1) & \xrightarrow{G(\beta_2)} & G(d_2) \end{array} \text{ commutes.} \right\}$$

.

Definition 5.4. If $F \colon C \to \mathbf{Set}$ is a **Set**-valued functor on a locally small category C, the **category of (global) points of** F, written $\int^{c\in C} F(c)$, is the comma category $(1 \downarrow F)$.

Explicitly, it is given by:

Objects: $\{(c, x) \,|\, c \in C, x \in F(C)\}$.

Morphisms: $\mathrm{Hom}_{\int^{c \in C} F(c)}((c_1, x_1), (c_2, x_2))$ is defined to be the set

$$\{f \mid f : c_1 \to c_2 \text{ and } F(f)(x_1) = x_2.\}$$

If $F : C^{\mathrm{op}} \to D$ is a contravariant functor, we write $\int_{c \in C} F(c)$ for the opposite of $\int^{c \in C} F(c)$.

The category of points of a functor $F : C \to D$ is equipped with a projection (forgetful) functor π back to C.

Lemma 5.5. *(3.7.2 of [2]) Consider two functors $F : A \to B$ and $G : A \to C$ with A small and C cocomplete. Then the left Kan extension of G along F exists, and is given pointwise by a colimit*

$$(b \to b') \mapsto \varinjlim \left(\int^{a \in A} B(a, b) \xrightarrow{\pi} A \xrightarrow{G} C \right) \to \varinjlim \left(\int^{a \in A} B(a, b') \xrightarrow{\pi} A \xrightarrow{G} C \right)$$

Lemma 5.6. *(3.7.3 of [2]) Let $F : A \to B$ be a full and faithful functor with A a small category. Let C be a cocomplete category. Then for any functor $A \to C$, the canonical natural transformation $G \xrightarrow{\alpha} (\mathrm{Lan}_F\, G) \circ F$ is an isomorphism (so that the inner triangle from 5.2 "commutes").*

Corollary 5.7. *Every model M : $\mathbf{Def}(T) \to \mathbf{Set}$ extends uniquely along* $\mathbf{y}$ $\mathbf{Def}(T) \xhookrightarrow{\mathbf{y}} \mathscr{E}(T)$ *to an inverse image functor $\widetilde{M}$, as in*

$$\begin{array}{ccc} \mathscr{E}(T) & & \\ {\scriptstyle \mathbf{y}}\big\uparrow & \searrow^{\widetilde{M}} & \\ \mathbf{Def}(T) & \xrightarrow[M]{} & \mathbf{Set} \end{array} .$$

The extension to $\mathscr{E}(T)$ is given by a pointwise Kan extension, so that for any $B \in \mathscr{E}(T)$, $\widetilde{M}(B)$ can be computed as the colimit

$$\varinjlim \left(\int_{A \in \mathbf{Def}(T)} \mathscr{E}(T)(A, B) \xrightarrow{\pi} \mathbf{Def}(T) \xrightarrow{M} \mathbf{Set} \right).$$

5.1.2 Coherent and compact objects in the classifying topos

Now, thinking of $\mathbf{Def}(T)$ as a full subcategory of $\mathscr{E}(T)$, we introduce some definitions which categorically characterize the objects of $\mathscr{E}(T)$ which correspond to quotients of definable sets by $\bigvee$-definable equivalence relations and definable sets.

Definition 5.8. An object A of a topos $\mathscr{E}$ is **compact** if every covering (jointly epimorphic) family of maps $\{f_i \mid i \in I\}$ of maps into A contains a finite subcover.

Definition 5.9. An object A of a topos $\mathscr{E}$ is **stable** if for every morphism $f : B \to A$ where B is compact, the domain K of the kernel relation $K \rightrightarrows B \xrightarrow{f} A$ is also compact.

Definition 5.10. An object A of a topos $\mathscr{E}$ is **coherent** if it is both compact and stable.

Remark 5.11. In a coherent topos, the pretopos of coherent objects is not necessarily closed under arbitrary finite colimits. This is because coequalizers are quotients by (at least) transitive closures of certain relations, so if one has a relation $R \rightrightarrows X$ whose transitive closure is properly ind-definable, the coequalizer $\mathbf{y}(R) \rightrightarrows \mathbf{y}(X) \twoheadrightarrow Y$ will not be definable.

Lemma 5.12. *(D3.3.7, [5]) An object B of the classifying topos $\mathscr{E}(T)$ of a first-order theory T is representable (i.e. isomorphic to an object from $\mathbf{Def}(T) \hookrightarrow \mathscr{E}(T)$) if and only if it is coherent.*

As we saw in 5.11, the prototypical example in a coherent topos of a compact non-coherent object is the coequalizer of a definable relation $R \rightrightarrows X$ on a definable set X with a properly ind-definable transitive closure. Our aim in this section is to prove the lemma 5.16, which says that this obstruction to coherence actually characterizes the compact non-coherent objects in a coherent topos.

An important basic category-theoretic fact is the canonical coproduct-coequalizer decomposition of colimits (whose proof can be found, for example, in [6]).

Fact 5.13. *Let $\mathcal{D}$ be a subcategory of $\mathbf{C}$ a category with all colimits.*

Then the colimit $\varinjlim(\mathcal{D})$ of $\mathcal{D}$ is isomorphic to the coequalizer of the following diagram:

$$\left(\coprod_{f \in \mathcal{D}_1} s(f)\right) \underset{G}{\overset{F}{\rightrightarrows}} \left(\coprod_{d \in \mathcal{D}_0} d\right)$$

where on each component $s(f) \in \mathcal{D}_0$ of the left hand side, F sends $s(f)$ to itself $d = s(f)$ by the identity map of $d = s(f)$, and on each $s(f) \in \mathcal{D}_0$ of the left hand side, G sends $s(f)$ to $t(f)$ by the map f.

We apply this fact to show the following:

Lemma 5.14. *An object B of a coherent topos $\mathscr{E}(T)$ is compact if and only if every covering of B whose domains are representables admits a finite subcover.*

Proof. The implication "⇒" is immediate.

Conversely, suppose that $\{B_i \to B\}$ is a covering of B. By the Kan extension colimit formula and the coproduct-coequalizer decomposition of colimits, each B_i is covered by (possibly infinitely many) representables. The collection of all these representables across all B_i form a covering of representables of B. By assumption, this covering admits a finite subcovering. Therefore, only finitely many of these B_i were needed since all these representable coverings factored through some B_i. □

We recount the following fact from [4], closely related to the lemma 5.16:

Fact 5.15. *(Lemma 7.36 of [4]). Let $\mathscr{E}$ be a topos generated by compact objects. Let X be a coherent object of E, and let $R \rightrightarrows X$ be an equivalence relation with coequalizer $R \rightrightarrows X \twoheadrightarrow X$.*

Then Y is coherent if and only if R is compact.

The lemma 5.16 is a sharpening of the fact 5.15: not only will we show that a compact non-coherent object is the quotient of a coherent object by a non-compact congruence, but we will explicitly describe the non-compact congruence as an infinite join of coherent objects.

Lemma 5.16. *Let $B \in \mathscr{E}(T)$ be a compact non-coherent object. Then B is the quotient of a coherent object A by a non-compact equivalence relation E which is a join of infinitely many coherent equivalence relations on A.*

Proof. Write B as a colimit of a diagram $\mathcal{D}$ whose objects are representables A_i. By the coproduct-coequalizer decomposition, B is a quotient of the coproduct $\bigsqcup_{A \in \mathcal{D}_l} A$ and therefore the maps $A_i \hookrightarrow \bigsqcup_{A \in \mathcal{D}_l} A \overset{p_B}{\twoheadrightarrow} B$ are a covering family for B. Since B is compact, finitely many A_i, say $A_1, \ldots, A_n$ suffice to cover B.

What we have said so far amounts to saying that B is a quotient of the coherent object $\bigsqcup_{i \leqslant n} A_i$, since the obvious map

$$\left(\bigsqcup_{i \leqslant n} A_i\right) \overset{i}{\hookrightarrow} \left(\bigsqcup_{A \in \mathcal{D}_0} A\right) \overset{p_B}{\twoheadrightarrow} B$$

covers B.

It now remains to calculate the kernel relation K' of $p_B \circ i$ and show that it is an infinite union of coherent relations on $\bigsqcup_{i \leqslant n} A_i$.

We break the remainder of the proof into the following steps:

1. The kernel relation K' of $p_B \circ i$ is the pullback of the kernel relation K of p_B along the inclusion
$$i \times i : \left(\bigsqcup_{i \leqslant n} A_i\right) \times \left(\bigsqcup_{i \leqslant n} A_i\right) \hookrightarrow \left(\bigsqcup_{A \in \mathcal{D}_0} A\right) \times \left(\bigsqcup_{A \in \mathcal{D}_0} A\right)$$
and therefore in every model consists of those pairs $(a_1, a_2) \in K$ such that both a_1 and a_2 are in $\bigsqcup_{i \leqslant n} A_i$.

2. Fix an arbitrary model. There is no harm in working with points and sets in a generic model since by Deligne's completeness theorem we can then lift our calculations to the classifying topos.

   Now, K is by definition the smallest equivalence relation containing "$\exists b : F(b) = a_1$ and $G(b) = a_2 \implies a_1 \sim_K a_2$." By how F and G are constructed, this means that $a \sim_K a'$ if and only if there are finitely many other points $a_1, \ldots, a_n$ and maps linking a to a_1, each a_i to a_{i+1}, and a_n to a', where the maps may point in either direction.

   It follows that K' is finer than just the kernel relation of the coequalizer of the pullback of $F, G : \bigsqcup_{A \in \mathcal{D}_0} A \to \bigsqcup_{A \in \mathcal{D}_0} A$ along the inclusion i, and is given by the following union:
$$K' = \bigvee_{n \in \omega} R_n$$

where R_0 is the diagonal copy of $\coprod_{i \leqslant n} A_i$, R_1 consists of those pairs (a_1, a_2) such that there is some a_0' in $\coprod_{A \in \mathcal{D}_0} A$ such that there is a map f in $\mathcal{D}_1$ that moves a_1 to a_0' or vice-versa, and there is a map g in $\mathcal{D}_1$ that moves a_0' to a_2 or vice-versa, etc.

3. R_1 is the infinite union $\bigvee_{A \in \mathcal{D}_0} S_A$, where each S_{A_k} corresponds to the A containing a particular witness $a_k = a_0'$ as above.

4. Each S_{A_k} looks like this:

$$\bigvee_{(f,f',g,g')} \left\{(a_i, a_j) \in A_i \times A_j \,\middle|\, \exists a_k \in A_k \big((a_i, a_k) \in \Gamma(f) \vee \Gamma(f') \text{ and } (a_j, a_k) \in \Gamma(g) \vee \Gamma(g')\big)\right\},$$

where the 4-tuple of maps (f, f', g, g') ranges over definable maps

$$\mathbf{Def}(T)(A_i, A_k) \times \mathbf{Def}(T)(A_k, A_i) \times \mathbf{Def}(T)(A_j, A_k) \times \mathbf{Def}(T)(A_k, A_j)$$

and therefore each S_{A_k} is $\bigvee$-coherent.

Therefore, R_1 is $\bigvee$-coherent.

5. Let us inductively assume that R_k is $\bigvee$-coherent as the union $\bigvee_{i \in I} T_i$. Then R_{k+1} is the following subset of $R_k \times R_1$:

$$R_{k+1} = \left\{(a, b) \,\middle|\, \bigvee_{(T_i, S_A) \in I \times \mathcal{D}_0} \exists c \;\text{ s.t. } (a, c) \in T_i \wedge (a, b) \in S_A\right\}$$

and is therefore also $\bigvee$-coherent.

We conclude that K' is $\bigvee$-coherent. □

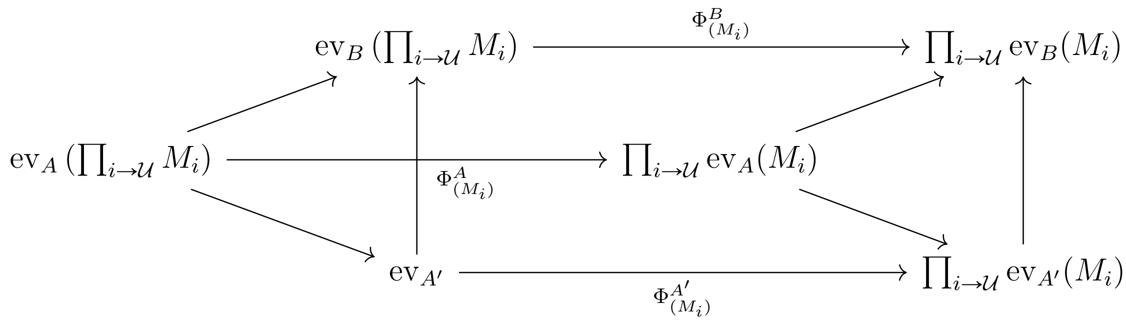
5.2 The coherence criterion

Theorem 5.17. *Let $\mathscr{E}(T)$ be the classifying topos of a first-order theory. Let B be an object of $\mathscr{E}(T)$. The following are equivalent:*

1. *B is coherent.*
2. *$\mathrm{ev}_B : \mathbf{Mod}(T) \to \mathbf{Set}$ is the underlying functor of a pre-ultrafunctor (ev_B, Φ) such that, if B is canonically the colimit of representables A_i, then each canonical map $A_i \to B$ induces an ultratransformation of the pre-ultrafunctors $(\mathrm{ev}_{A_i}, \mathrm{id}) \to (\mathrm{ev}_B, \Phi)$.*

Proof. $(1 \implies 2)$ If B is coherent, then it is representable and $(\mathrm{ev}_B, \mathrm{id})$ is a pre-ultrafunctor,, and since $\mathbf{y} : \mathbf{Def}(T) \to \mathscr{E}(T)$ is full and faithful, every map $A_i \to B$ corresponds to a definable function, which induces an ultratransformation $\mathrm{ev}(A_i) \to \mathrm{ev}(B)$.

$(2 \implies 1)$ First, we note that under the assumptions, ev_B's transition isomorphism is uniquely determined by the transition isomorphisms of the representables appearing in the Kan extension colimit formula for B: all diagrams of the form



commute, and since the Kan extension colimit formula is computed pointwise, the transition isomorphism $\Phi^B_{(M_i)}$ is a unique comparison map from the colimit $\mathrm{ev}_B\left(\prod_{i\to\mathcal{U}} M_i\right)$ of the $\mathrm{ev}_A(\prod_{i\to\mathcal{U}} M_i)$'s into $\prod_{i\to\mathcal{U}} \mathrm{ev}_B(M_i)$.

Now, knowing this, suppose B is not coherent. Then either B cannot be covered by finitely many definables, or it can. If it can be covered by the finitely many definables $A_1, \ldots, A_n$, then the associated map $A_1 \sqcup \cdots \sqcup A_n \twoheadrightarrow B$ does not have a definable kernel relation, and in fact by 5.16, the kernel relation is properly ind-definable.

In either case, we know what the transition isomorphism $\Phi^B_{(M_i)}$ looks like. In the first case, if B cannot be covered by finitely many definables, we still know from the Kan extension colimit formula that it can be covered by infinitely many $(A_i)_{i\in I}$. Fix a model M and take a sequence $(a_i)_{i\in I}$ such that for every A_j, cofinitely many a_i are not in (the image of) A_j (in B). Then for a non-principal ultrafilter $\mathcal{U}$ on I, $[a_i]_{i\in\mathcal{U}}$ is not in any of the (images of the) $(M^{\mathcal{U}})(A_j)$'s. Therefore, it is not in the image of the transition isomorphism $\Phi^B_{(M)}$, a contradiction.

In the second case, if B looks like a definable set A quotiented by a properly ind-definable equivalence relation $R = \bigcup_{i\in I} R_i$, then once again we know that the transition isomorphism

$$\left(\prod_{i\to\mathcal{U}} M_i\right)(A/R) \to \prod_{i\to\mathcal{U}}(M_i(A/R))$$

is the "obvious" one. Here's what the "obvious" map is: since A is definable, we are really comparing two equivalence relations on the same set. On the left hand side, we have that $[a_i]_{i\to\mathcal{U}} \sim [b_i]_{i\to\mathcal{U}}$ if and only if there exists some R_j such that $(\prod_{i\to\mathcal{U}} M_i)(R_j)$ contains $([a_i],[b_i])_{i\to\mathcal{U}}$. On the right hand side, we have that $[a_i]_{i\to\mathcal{U}} \sim [b_i]_{i\to\mathcal{U}}$ if and only if $a_i \sim_R b_i$ $\mathcal{U}$-often. Since R is properly ind-definable, the equivalence relation on the left is properly contained in the equivalence relation on the right. This containment induces an obvious map between the quotients, and since the containment is proper, the obvious map is not injective, and cannot be a bijection.

□

Now we use this result 5.17 to prove a stronger statement than 4.8. The difference is that in the original statement of 4.8, we only concluded that X was definable, without saying anything about the transition isomorphism Φ which allowed us to view (X, Φ) as a Δ-functor. In fact, we can show that (X, Φ) is isomorphic to $\mathrm{ev}_{\varphi(x)}$, and must therefore be an ultrafunctor.

Corollary 5.18. *Let T be $\aleph_0$-categorical. Let (X, Φ) be a pre-ultrafunctor. Then the underlying functor X is definable if and only if for some $\varphi(x) \in T$, (X, Φ) is isomorphic as a pre-ultrafunctor to* $\mathrm{ev}_{\varphi(x)}$.

Proof. By applying the lemma 4.7 that Δ-functors preserve filtered colimits and arguing as in the proof of 4.8, we conclude that X is isomorphic to a possibly infinite disjoint union of representables $\bigsqcup_{i \in I} A_i$. In this way, X is canonically the colimit of the representables A_i. It remains to verify the rest of item **??**, i.e. the canonical inclusions $A_k \hookrightarrow \bigsqcup_{i \in I} A_i \simeq X$ induce ultratransformations.

Before proceeding, we reduce the problem of verifying this for all ultraproducts to just verifying this for all ultrapowers. This is because, in general, every ultraproduct is a filtered colimit of ultraproducts of countable models: for every $[x_i]_{i \to \mathcal{U}}$ in some ultraproduct $\prod_{i \to \mathcal{U}} N_i$, take a countable elementary model $M_i \stackrel{f_i}{\hookrightarrow} N_i$ which contains x_i; then there is an embedding $\prod_{i \to \mathcal{U}} f_i : \prod_{i \to \mathcal{U}} M_i \hookrightarrow \prod_{i \to \mathcal{U}} N_i$, and the collection of all such embeddings covers $\prod_{i \to \mathcal{U}} N_i$. Since T is $\aleph_0$-categorical, an ultraproduct of countable models is just an ultrapower of the unique countable model.

So, it remains to check that the diagram

$$\begin{array}{ccc} X\left(M^{\mathcal{U}}\right) & \xrightarrow{\Phi_{(M)}} & X(M)^{\mathcal{U}} \\ & {\scriptstyle \iota_{M^{\mathcal{U}}}} \nwarrow \quad \nearrow {\scriptstyle \prod_{i \to \mathcal{U}} \iota_i} & \\ & A\left(M^{\mathcal{U}}\right) & \end{array}$$

commutes. Each component ι_N of the ultratransformation is determined by filtered colimits of the countable model M, with ι_M determined by sending the support $a_x \in A(M)$ to x. Since $\Delta_M : M \to M^{\mathcal{U}}$ is part of the filtered diagram of countable submodels of $M^{\mathcal{U}}$, $\iota_{M^{\mathcal{U}}}$ of $\Delta_M(a_x) = X(\Delta_M)(x)$, and since (X, Φ) was a Δ-functor, $\Phi_{(M)} \circ X(\Delta_M)(x) = \Delta_{X(M)}(x)$.

On the other hand,

$$\prod_{i \to \mathcal{U}} \iota_i \left(\Delta(a_x)\right) = [\iota_M(a_x)]_{i \to \mathcal{U}} = \Delta_{X(M)}(x).$$

So the diagram commutes, and now we are done by the direction **??** $\implies$ **??** of the theorem.

□

6 Counterexamples in the non-$\aleph_0$-categorical case

In this section, we will show that the (strengthened) conclusion of the main theorem 5.18 fails when the assumption that T is $\aleph_0$-categorical is removed. In fact, we will work with the simplest non-$\aleph_0$-categorical theory—the theory of an infinite set, expanded by countably many distinct constants—and construct an example of a pre-ultrafunctor which is not a Δ-functor, and an example of a Δ-functor which fails to preserve the generalized diagonal embeddings 3.15.

For the rest of this section, T will mean the theory of an infinite set with countable many distinct constants $\{c_i\}_{i\in\omega}$. In a single variable, T has a unique non-isolated type $p(x)$, whose realizations are those elements which are not any constants.

Definition 6.1. The underlying functor X for the pre-ultrafunctors we will construct will be given on the objects of $\mathbf{Mod}(T)$ by:

$$X(M) \stackrel{\text{df}}{=} p(M) \cup \left\{c_k^M \,\middle|\, k \text{ is even}\right\}.$$

On elementary embeddings $f : M \to N$, we set $X(f)$ to just be the restriction of f to $X(M)$.

There is an obvious map which compares $\prod_{i\to\mathcal{U}} X(M_i)$ with $X\left(\prod_{i\to\mathcal{U}} M_i\right)$, namely the inclusion of the former in the latter. However, by 2.9, this cannot be an isomorphism. To complete the construction of the counterexamples, it remains to construct transition isomorphisms for X.

For our convenience, we record an analysis of the automorphisms of the functor X which will be useful in the construction of the exotic Δ-functor 6.1.

Lemma 6.2. *Any automorphism $\eta : X \to X$ of X satisfies the following property: for every $M \models T$, $\eta_M : X(M) \to X(M)$ permutes the constants and fixes the nonconstants.*

Proof. Fix an arbitrary model M, let $\Delta_M : M \to M^U$ be the diagonal embedding into some ultrapower $M^{\mathcal{U}}$, and consider the naturality diagram which must be satisfied by the components $\{\eta_M\}_{M\in\mathbf{Mod}(T)}$ of η:

$$\begin{array}{ccccc} M & \quad & X(M) & \xrightarrow{\ \eta_M\ } & X(M) \\ \big\downarrow{\scriptstyle \Delta_M} & & \big\downarrow{\scriptstyle X(\Delta_M)} & & \big\downarrow{\scriptstyle X(\Delta_M)} \\ M^{\mathcal{U}} & & X\left(M^{\mathcal{U}}\right) & \xrightarrow[\ \eta_{M^{\mathcal{U}}}\ ]{} & X\left(M^{\mathcal{U}}\right) \end{array}$$

Suppose η_M sends a constant c to a nonconstant $\eta_M(c)$. Then the commutativity of the naturality diagram tells us $\eta_{M^{\mathcal{U}}}$ sends $X(\Delta_M)(c) = \Delta_M(c)$ to $X(\Delta_M)(\eta_M(c)) = \Delta_M(\eta_M(c))$. However, any injection $M \to M^U$ which identifies constants with constants and sends nonconstants to nonconstants is an elementary embedding, and we can certainly find an embedding $f : M \to M^U$ which does not send the nonconstant $\eta_M(c)$ to $\Delta_M(\eta_M(c))$. Then, since elementary embeddings fix constants, the naturality diagram

$$\begin{array}{ccc} M & \quad & X(M) \xrightarrow{\eta_M} X(M) \\ \Big\downarrow f & & X(f)\Big\downarrow \qquad\qquad \Big\downarrow X(f) \\ M^{\mathcal{U}} & & X\left(M^{\mathcal{U}}\right) \xrightarrow[\eta_{M^{\mathcal{U}}}]{} X\left(M^{\mathcal{U}}\right) \end{array}$$

would not commute. So, η_M must send constants to constants. Since η is an isomorphism and hence invertible, η_M cannot send nonconstants to constants either.

Now suppose that η_M does not fix the nonconstants, so that for some nonconstant d, $d \neq \eta_M(d)$, with $\eta_M(d)$ a nonconstant. Consider again the naturality diagram for $\Delta_M : M \to M^{\mathcal{U}}$:

$$\begin{array}{ccc} M & \quad & X(M) \xrightarrow{\eta_M} X(M) \\ \Big\downarrow \Delta_M & & X(\Delta_M)\Big\downarrow \qquad\qquad \Big\downarrow X(\Delta_M) \\ M^{\mathcal{U}} & & X\left(M^{\mathcal{U}}\right) \xrightarrow[\eta_{M^{\mathcal{U}}}]{} X\left(M^{\mathcal{U}}\right) \end{array}$$

This tells us that $\eta_{M^{\mathcal{U}}}(\Delta_M(d)) = \Delta_M(\eta_M(d))$.

Let d' stand for $\Delta_M(\eta_M(d))$, and let e be another nonconstant in $M^{\mathcal{U}}$, distinct from $\Delta_M(d)$ and d'. Since d' and e are nonconstants, we can find an automorphism $\sigma : M^{\mathcal{U}} \to M^{\mathcal{U}}$ which fixes $\Delta_M(d)$ but which moves d' to e. Then the naturality diagram for σ

$$\begin{array}{ccc} M^{\mathcal{U}} & \quad & X\left(M^{\mathcal{U}}\right) \xrightarrow{\eta_{M^{\mathcal{U}}}} X\left(M^{\mathcal{U}}\right) \\ \Big\downarrow \sigma & & X(\sigma)\Big\downarrow \qquad\qquad \Big\downarrow X(\sigma) \\ M^{\mathcal{U}} & & X\left(M^{\mathcal{U}}\right) \xrightarrow[\eta_{M^{\mathcal{U}}}]{} X\left(M^{\mathcal{U}}\right) \end{array}$$

tells us that

$$\begin{aligned} \sigma \circ \eta_{M^{\mathcal{U}}}(\Delta_M(d)) &= \eta_{M^{\mathcal{U}}} \circ \sigma(\Delta_M(d)) \\ &= \sigma(d') = \eta_M^{\mathcal{U}}(\Delta_M(d)) \\ &= e = d', \end{aligned}$$

a contradiction. Therefore, η_M fixes the nonconstants. □

Finally, we remark that *any* permutation of the constants can be realized in an automorphism $\eta : X \to X$, and in fact $\mathrm{Aut}(X) \simeq \mathrm{Sym}(\omega)$.

6.1 The exotic Δ-functor

Now we will construct a transition isomorphism Φ for X such that (X, Φ) is a Δ-functor which is not an ultrafunctor (and, in fact, which fails to preserve the generalized diagonal embeddings 3.15).

Fix I and a non-principal ultrafilter $\mathcal{U}$. Let $(M_i)_{i \in I}$ be an I-indexed sequence of models. Consider $X\left(\prod_{i \to \mathcal{U}} M_i\right)$, in which we can canonically identify $\prod_{i \to \mathcal{U}} X(M_i)$ as a subset.

Definition 6.3. Let $A_{(M_i)}$ be the complement of $\prod_{i \to \mathcal{U}} X(M_i)$ inside $X\left(\prod_{i \to \mathcal{U}} M_i\right)$. $A_{(M_i)}$ consists of those elements $[x_i]_{i \to \mathcal{U}}$ of $\prod_{i \to \mathcal{U}} M_i$ which:

1. realize the non-isolated type $p(x)$, i.e. are not constants, and
2. such that any representative sequence $(x_i)_{i \to \mathcal{U}}$ is $\mathcal{U}$-often an odd constant (equivalently, can be represented by a sequence made up entirely of odd constants).

Let $B_{(M_i)}$ be the subset of $\prod_{i \to \mathcal{U}} X(M_i)$ which consists of those elements $[x_i]_{i \to \mathcal{U}}$ of $\prod_{i \to \mathcal{U}}$ which:

1. realize the non-isolated type $p(x)$, i.e. are not constants, and
2. such that any representative sequence $(x_i)_{i \to \mathcal{U}}$ is $\mathcal{U}$-often an even constant (equivalently, can be represented by a sequence made up entirely of even constants).

Finally, let $C_{(M_i)}$ be the complement of $B_{(M_i)}$ inside $\prod_{i \to \mathcal{U}} X(M_i)$.

Note that $C_{(M_i)}$ consists precisely of those elements of $X\left(\prod_{i \to \mathcal{U}} M_i\right)$ which are either constants or which are nonconstants $[x_i]_{i \to \mathcal{U}}$ for which any representative sequence $(x_i)_{i \to \mathcal{U}}$ is $\mathcal{U}$-often a nonconstant.

Since elementary embeddings preserve the property of a tuple being constant or nonconstant, for any sequence of elementary embeddings $(f_i : M_i \to N_i)_{i \to \mathcal{U}}$, we have that $[f_i]_{i \to \mathcal{U}}$ restricts to a map $C_{(M_i)} \to C_{(N_i)}$, and furthermore because elementary embeddings fix constants, $[f_i]_{i \to \mathcal{U}}$ restricts to bijections $A_{(M_i)} \to A_{(N_i)}$ and $B_{(M_i)} \to B_{(N_i)}$.

Now, we have disjoint unions

$$X\left(\prod_{i \to \mathcal{U}} M_i\right) = A_{(M_i)} \sqcup B_{(M_i)} \sqcup C_{(M_i)} \quad \text{and} \quad \prod_{i \to \mathcal{U}} X(M_i) = B_{(M_i)} \sqcup C_{(M_i)},$$

and our task is to find a transition isomorphism $\Phi_{(M_i)} : A_{(M_i)} \sqcup B_{(M_i)} \sqcup C_{(M_i)} \xrightarrow{\sim} B_{(M_i)} \sqcup C_{(M_i)}$.

We define $\Phi_{(M_i)}$ to be the identity on $C_{(M_i)}$. It remains to specify a bijection $\sigma : A_{(M_i)} \sqcup B_{(M_i)} \simeq B_{(M_i)}$. Since any such σ only involves identifying certain ultraproducts of constants with other ultraproducts of constants, then after fixing a σ we can use σ to define $\Phi_{(N_i)}$ for arbitrary I-indexed sequences of models (N_i). With this setup, we will show that any choice of σ works.

While in general, transition isomorphisms depend on the three pieces of information $I, \mathcal{U}$ and (M_i), we have constructed candidate transition isomorphisms by making a choice σ which only depends on I and $\mathcal{U}$, so we make this explicit by writing $\sigma_{I,\mathcal{U}}$.

Now, fix $\sigma_{I,\mathcal{U}}$ and let $(M_i \xrightarrow{f_i} N_i)_{i\in I}$ be an I-indexed sequence of elementary embeddings, and consider the pre-ultrafunctor diagram

$$\begin{array}{ccc} X\left(\prod_{i\to\mathcal{U}} M_i\right) & \xrightarrow{\Phi_{(M_i)}} & \prod_{i\to\mathcal{U}} X(M_i) \\ {\scriptstyle X([f_i]_{i\to\mathcal{U}})}\downarrow & & \downarrow{\scriptstyle [X(f_i)]_{i\to\mathcal{U}}} \\ X\left(\prod_{i\to\mathcal{U}} N_i\right) & \xrightarrow[\Phi_{(N_i)}]{} & \prod_{i\to\mathcal{U}} X(N_i). \end{array}$$

To show it commutes, consider an arbitrary element $[x_i]_{i\to\mathcal{U}}$ of the top left corner $X\left(\prod_{i\to\mathcal{U}} M_i\right)$. There are three cases:

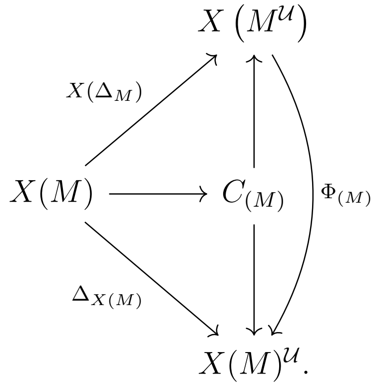
1. $[x_i]_{i\to\mathcal{U}}$ is in C_{M_i}. Recall that $\Phi_{(M_i)}$ and $\Phi_{(N_i)}$ were defined to be the identities on $C_{(M_i)}, C_{N_i}$, and that $[f_i]_{i\to\mathcal{U}}$ restricts to a map $C_{(M_i)} \to C_{(N_i)}$. Chasing $[x_i]_{i\to\mathcal{U}}$ through the diagram, we get
$$\begin{array}{ccc} [x_i]_{i\to\mathcal{U}} & \longmapsto & [x_i]_{i\to\mathcal{U}} \\ \downarrow & & \downarrow \\ [f_i x_i]_{i\to\mathcal{U}} & \longmapsto & [f_i x_i]_{i\to\mathcal{U}}. \end{array}$$

2. $[x_i]_{i\to\mathcal{U}}$ is in $A_{(M_i)}$. Recall that $[f_i]_{i\to\mathcal{U}}$ restricts to bijections $A_{(M_i)} \to A_{(N_i)}$ and $B_{(M_i)} \to B_{(N_i)}$. Chasing $[x_i]_{i\to\mathcal{U}}$ through the diagram, we get
$$\begin{array}{ccc} [x_i]_{i\to\mathcal{U}} & \longmapsto & [\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}} \\ \downarrow & & \downarrow \\ [x_i]_{i\to\mathcal{U}} & \longmapsto & [\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}}. \end{array}$$

3. $[x_i]_{i\to\mathcal{U}}$ is in $B_{(M_i)}$. Recall that $[f_i]_{i\to\mathcal{U}}$ restricts to bijections $A_{(M_i)} \to A_{(N_i)}$ and $B_{(M_i)} \to B_{(N_i)}$. Chasing $[x_i]_{i\to\mathcal{U}}$ through the diagram, we get
$$\begin{array}{ccc} [x_i]_{i\to\mathcal{U}} & \longmapsto & [\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}} \\ \downarrow & & \downarrow \\ [x_i]_{i\to\mathcal{U}} & \longmapsto & [\sigma_{I,\mathcal{U}} x_i]_{i\to\mathcal{U}}. \end{array}$$

Therefore, after making choices of bijections $\sigma_{I,\mathcal{U}}$ for every I and $\mathcal{U}$, we obtain a transition isomorphism Φ such that (X, Φ) is a pre-ultrafunctor.

(X, Φ) is also a Δ-functor: for any ultrapower $M^{\mathcal{U}}$, recall that the subset $C_{(M_i)}$ 6.3 of $X\left(M^{\mathcal{U}}\right)$ contains all those elements which are constants or nonconstants that are ultraproducts of nonconstants. In particular, if $a \in M$, then $\Delta_M(a) = [a]_{i\to\mathcal{U}}$ is a constant if a is a constant or a nonconstant which is an ultraproduct of nonconstants if a is a nonconstant, so the image of $\Delta_{X(M)}$ is contained inside $C_{(M_i)} \subseteq X(M)^{\mathcal{U}}$. $X(\Delta_M)$ is just the restriction of Δ_M to $X(M)$, so the image of $X(\Delta_M)$ also lies in $C_{(M)}$ and agrees with the image of $\Delta_{X(M)}$. This means in the below diagram, the upper-left and lower-left triangles commute:



Furthermore, $\Phi_{(M)}$ was defined to be the identity on $C_{(M)}$, so the curved subdiagram on the right commutes. Therefore, the entire diagram commutes; in particular, the outer triangle from the definition 3.8 of a Δ-functor commutes, so (X, Φ) is a Δ-functor.

The theory T is countable, and by strong conceptual completeness there as many isomorphism classes of ultrafunctors as there are definable sets of T. But for any I and $\mathcal{U}$, *any* choice of a bijection $\sigma_{I,\mathcal{U}}$ worked. We will show that there are at least uncountably many isomorphism classes of Δ-functors (X, Φ) that arise from our construction. This will imply that there is some choice of Φ such that (X, Φ) is not an ultrafunctor.

Let I now be countable, and let Φ and Φ' be two different transition isomorphisms which arise from making the choices of $\sigma_{I,\mathcal{U}}$ and $\sigma'_{I,\mathcal{U}}$ during our construction. An isomorphism of pre-ultrafunctors $(X, \Phi) \to (X, \Phi')$ is an automorphism $\eta : X \to X$ such that, additionally, all diagrams of the form

$$\begin{array}{ccc} X\left(\prod_{i\to\mathcal{U}} M_i\right) & \xrightarrow{\Phi} & \prod_{i\to\mathcal{U}} X(M_i) \\ {\scriptstyle \eta_{\prod_{i\to\mathcal{U}} M_i}}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{i\to\mathcal{U}} \eta_{M_i}} \\ X\left(\prod_{i\to\mathcal{U}} M_i\right) & \xrightarrow[\Phi']{} & \prod_{i\to\mathcal{U}} X(M_i) \end{array}$$

commute.

By our earlier analysis 6.2 of the automorphisms of X, it is easy to see that when restricted to $C_{(M_i)}$, the above diagram commutes.

However, if we restrict to $A_{(M_i)} \sqcup B_{(M_i)}$, then chasing an element around the diagram

$$\begin{array}{ccc} A_{(M_i)} \sqcup B_{(M_i)} & \xrightarrow{\sigma_{I,\mathcal{U}}} & B_{(M_i)} \\ {\scriptstyle \eta_{\prod_{i\to\mathcal{U}} M_i}}\big\downarrow & & \big\downarrow{\scriptstyle \prod_{i\to\mathcal{U}} \eta_{M_i}} \\ A_{(M_i)} \sqcup B_{(M_i)} & \xrightarrow[\sigma'_{I,\mathcal{U}}]{} & B_{(M_i)} \end{array}$$

yields the tentative equality

$$\begin{array}{ccc}
[x_i]_{i\to\mathcal{U}} & \longmapsto & \sigma_{I,\mathcal{U}}([x_i]_{i\to\mathcal{U}} \\
\downarrow & & \downarrow \\
[x_i]_{i\to\mathcal{U}} & \longmapsto & \sigma'_{I,\mathcal{U}}([x_i]) \stackrel{?}{=} \prod_{i\to\mathcal{U}} \eta_{M_i}\left(\sigma_{I,\mathcal{U}}\left([x_i]_{i\to\mathcal{U}}\right)\right)
\end{array}$$

so we see that if the transition isomorphisms Φ and Φ' induced by $\sigma_{I,\mathcal{U}}$ and $\sigma'_{I,\mathcal{U}}$ are isomorphic, then there is an automorphism $\eta : X \to X$ such that $\prod_{i\to\mathcal{U}} \eta_{M_i} \circ \sigma_{I,\mathcal{U}} = \sigma'_{I,\mathcal{U}}$. Therefore, defining G to consist of all ultraproducts $\prod_{i\to\mathcal{U}} \eta_{(M_i)}$ admissible in the above diagram (so only those which restrict to a permutation on $B_{(M_i)}$), the number of isomorphism classes among the (X, Φ) is bounded from below by the number of orbits of the action by composition

$$G \rightharpoonup \text{Bijections}\left(A_{(M_i)} \sqcup B_{(M_i)}, B_{(M_i)}\right).$$

However, G can be identified with a subgroup of $\text{Sym}(\omega)^{\mathcal{U}}$. Since I was countable, $\text{Sym}(\omega)^{\mathcal{U}}$ has size $\leqslant \mathfrak{c}$ the size of the continuum.

On the other hand, the set on which G acts has the same cardinality as $|\,\text{Sym}\left(B_{(M_i)}\right)| \geqslant 2^{\mathfrak{c}}$.

Therefore, this action has uncountably many orbits, and so there are uncountably many isomorphism classes of (X, Φ) arising from our construction. So, one of them cannot be an ultrafunctor.

We can also see that Φ can be chosen to violate a generalized diagonal embedding 3.15. Fix indexing sets I and J such that $|I| > |J|$, a surjection $g : I \twoheadrightarrow J$, and $\mathcal{U}$ an ultrafilter on I with $\mathcal{V}$ its pushforward $g_*\mathcal{U}$. Let $(M_j)_{j\in J}$ be a J-indexed sequence of models.

Then the associated generalized diagonal embedding $\Delta_g : \prod_{j\to\mathcal{V}} M_j \to \prod_{i\to\mathcal{U}\mathcal{V}} M_{g(i)}$ induces, informally speaking, a relationship between ultraproducts computed with respect to different indexing sets and ultrafilters: for it to be preserved, the diagram

$$\begin{array}{ccc}
X\left(\prod_{j\to\mathcal{V}} M_j\right) & \xrightarrow{X(\Delta_g)} & X\left(\prod_{i\to\mathcal{U}} M_{g(i)}\right) \\
{\scriptstyle \Phi_{(M_j)}}\downarrow & & \downarrow{\scriptstyle \Phi_{(M_{g(i)})}} \\
\prod_{j\to\mathcal{V}} X(M_j) & \xrightarrow[\Delta_{X(g)}]{} & \prod_{i\to\mathcal{U}} X(M_{g(i)})
\end{array}$$

must commute, for all choices of (M_j). However, our construction of Φ involved a specification of $\Phi_{(M_j)}$ based on a choice of $\sigma_{J,\mathcal{V}}$ which is independent of the choice of $\sigma'_{I,\mathcal{U}}$ used to specify $\Phi_{(M_{g(i)})}$. To make this concrete, if for a given Φ and (M_j) the diagram above happens to commute, then for any $a \in A_{(M_j)}$ in the upper-left corner which gets sent to some $b \in B_{(M_{g(i)})}$ in the lower-right corner, we can change our choice of $\Phi_{(M_j)}$ so that $\Delta_{X(g)} \circ \Phi_{(M_j)}$ sends a to a different $b' \neq b$ while keeping the rest of Φ the same, with the modified transition isomorphism Φ' still making (X, Φ') a Δ-functor.

6.2 The exotic pre-ultrafunctor

In the previous section, the transition isomorphisms Φ making (X, Φ) a Δ-functor were constructed to be the identity on $C_{(M)}$, and hence also restricted to the identity on the image of diagonal embeddings $\Delta_M : M \to M^{\mathcal{U}}$.

In general, $C_{(M_i)}$ splits into a disjoint union of even constants and nonconstants which are ultraproducts of nonconstants of M_i:

$$C_{(M_i)} = C^{c}_{(M_i)} \sqcup C^{nc}_{(M_i)}.$$

We can easily modify the construction of the transition isomorphism to *not* preserve the diagonal map, by requiring that Φ restricts to the identity only on $C^{nc}_{(M_i)}$, while on $C^{c}_{(M_i)}$, we now require that Φ restricts to any permutation $C_{(M_i)} \to C_{(M_i)}$, while keeping the rest of the construction the same.

Now we verify the pre-ultrafunctor condition. When we verified the pre-ultrafunctor condition during the construction of the exotic Delta-functor, we had three cases 6.1, according to whether an element $[x_i]_{i\to\mathcal{U}} \in X\left(\prod_{i\to\mathcal{U}} M_i\right)$ was in $A_{(M_i)}, B_{(M_i)}$ or $C_{(M_i)}$. With the new definition, the verification of the first two cases remains the same, but the case of $C_{(M_i)}$ splits into the two case of whether $[x_i]_{i\to\mathcal{U}} \in C_{(M_i)}$ is a constant or nonconstant. If $[x_i]_{i\to\mathcal{U}}$ is a nonconstant, then since Φ still acts as the identity on $[x_i]_{i\to\mathcal{U}}$, the diagram commutes. If $[x_i]_{i\to\mathcal{U}}$ is a constant, then if Φ restricts to a nontrivial permutation of $C_{(M_i)}$, then the diagram commutes because elementary embeddings preserve constants.

However, when Φ restricts to a nontrivial permutation on the even constants, the diagonal embedding $\Delta_M : M \to M^{\mathcal{U}}$ is *not* preserved, i.e. the triangle diagram in 3.8 does *not* commute. For any even constant c in $X(M)$ which is not fixed by Φ (and identifying $X(M)^{\mathcal{U}}$ as a subset of $X\left(M^{\mathcal{U}}\right)$, and this as a subset of $M^{\mathcal{U}}$, $X(\Delta_M)(c) = \Delta_{X(M)}(c) = \Delta_M(c)$, but $\Phi(\Delta_{X(M)}(c)) \neq \Delta_M(c)$.

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