Ultraproduct definability criteria for ω-categorical theories

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Overview

The first-order analogue of Birkoff's HSP theorem characterizing **equational** classes (those closed under **h**omomorphisms, **s**ubstructures, and **p**roducts) is Chang and Keisler's characterization of **elementary** classes as those closed under elementary embeddings, elementary substructures, and **ultraproducts**.

One of Lawvere's insights in his thesis was that the definable sets of an equational theory could be reconstructed from its models as precisely the ones which respected the extra structure stipulated by the HSP theorem: there is an equivalence of categories

$$\mathbf{Def}(T) \simeq \mathrm{Lex}(\mathbf{Mod}(T), \mathbf{Set})$$

between the definable sets of an equational theory T and the left-exact functors $\mathbf{Mod}(T) \to \mathbf{Set}$.

In his works around strong conceptual completeness in the '80s, Makkai established the analogous reconstruction result for **first-order theories** T, by finding the appropriate notion ("ultracategories") of a category with the extra structure of ultraproducts and canonical maps between them:

$$\mathbf{Def}(T) \simeq \mathrm{Ult}(\mathbf{Mod}(T), \mathbf{Set})$$

i.e. there is an equivalence of categories between the category of (eq)-definable sets of a first-order theory T and the category of "ultrafunctors" $\mathbf{Mod}(T) \to \mathbf{Set}$. ("Ultrafunctors are precisely the evaluation functors corresponding to (eq)-definable sets.")

We therefore think of this as a criterion for recognizing when a procedure to expand every model of a theory $M \models T$ by a new sort $X(M)$ is, in fact, identical to expanding by an imaginary sort of T. (We then say the functor $X : \mathbf{Mod}(T) \to \mathbf{Set}$ is **definable**.)

We simplify this criterion when T is ω-categorical.

ω-categorical theories

A theory T is **ω-categorical** if it has a unique countable model M up to isomorphism. This implies that M is "highly symmetric": T has only finitely many types and so the action $\mathrm{Aut}(M) \curvearrowright M$ is oligomorphis, i.e. has only finitely many orbits on each $M, M^2, M^3 \ldots$.

A crucial fact is

- the **Ryll-Nardzewski theorem**, which establishes the converse: if $\mathrm{Aut}(M) \curvearrowright M$ is oligomorphic, then $\mathrm{Th}(M)$ must be ω-categorical.

Thus ω-categorical theories are essentially the data of $\mathrm{Aut}(M) \curvearrowright M$, and in fact if you replace the data of the action with the pointwise convergence topology you can distinguish ω-categorical theories up to bi-interpretability.

Positive reconstruction results for ω-categorical theories

- The **Coquand-Ahlbrandt-Ziegler theorem** says that a bi-interpretation between ω-categorical theories $T_1 \simeq T_2$ can be reconstructed from just a topological isomorphism $\mathrm{Aut}(M_1) \simeq \mathrm{Aut}(M_2)$.
- An isomorphism of monoids $\mathrm{End}(M_1) \simeq \mathrm{End}(M_2)$ induces (by taking filtered colimits) an equivalence of categories $\mathbf{Mod}(T_1) \simeq \mathbf{Mod}(T_2)$. (Of course, this equivalence may not come from a bi-interpretation.)

Negative reconstruction results for ω-categorical theories

- In 1990 Evans and Hewitt exhibited a counterexample to being able to reconstruct an ω-categorical theory up to bi-interpretability from just an isomorphism $\mathrm{Aut}(M_1) \simeq \mathrm{Aut}(M_2)$ of groups.
- In 2015 Bodirsky, Evans, Kompatscher and Pinsker exhibited a counterexample to reconstruction up to bi-interpretability from an isomorphism $\mathrm{End}(M_1) \simeq \mathrm{End}(M_2)$ of monoids.

Main results

- Let T_1 and T_2 be ω-categorical theories with isomorphic endomorphism monoids $\mathrm{End}(M_1) \simeq \mathrm{End}(M_2)$ but which are not bi-interpretable.

  In light of **strong conceptual completeness**, the failure of the induced equivalence of categories $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ to come from a bi-interpretation of T with T' should be witnessed **at the level of ultracategories**, because $T \not\simeq T'$ if and only if $\mathbf{Mod}(T) \not\simeq \mathbf{Mod}(T')$ as ultracategories.

  This is indeed true: the equivalence $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ **fails to preserve** either a diagonal embedding or an ultraproduct of a family of elementary self-maps $M \preceq M$.
- This implies that ω-categorical T and T' are **bi-interpretable** if and only if there is an equivalence $\mathbf{Mod}(T) \simeq \mathbf{Mod}(T')$ which **preserves ultraproducts and diagonal maps**.
- Actually, the proof of this gives us a bit more. If T and T' are ω-categorical, then any functor $X : \mathbf{Mod}(T) \to \mathbf{Mod}(T')$ which preserves ultraproducts and diagonal maps induces for each model $N \models T$ a continuous monoid homomorphism $X_N : \mathrm{End}(N) \to \mathrm{End}(X(N))$.
- Finally, with this we can show: when T is ω-categorical,

  $X : \mathbf{Mod}(T) \to \mathbf{Set}$ **is definable if and only if** X **preserves ultraproducts and diagonal maps.**

Ultracategories, ultramorphisms, ultrafunctors

An **ultracategory** is a category $\mathbf{K}$ with the following extra structure:

- For each small I and ultrafilter $\mathcal{U}$ on I, an **ultraproduct functor** $[\mathcal{U}] : \mathbf{K}^I \to \mathbf{K}$ which picks out $\mathcal{U}$-ultraproducts, and
- collections of specified morphisms, called **ultramorphisms**, which are "canonically defined maps between ultraproducts", e.g. the collection of diagonal maps into ultrapowers.
- Formally, an ultramorphism δ is a natural transformation $(k) \to (\ell)$ between evaluation functors

$$\mathrm{Hom}(\Gamma, \mathbf{K}) \underset{(\ell)}{\overset{(k)}{\rightrightarrows}} \mathbf{K}, \;\; k, \ell \in \Gamma, \;\text{ where:}$$

  - Γ is an **ultragraph**, which is a directed graph whose vertices are partitioned into two sets Γ^b and Γ^f such that for each $\beta \in \Gamma^b$ we assign a triple $(I_\beta, \mathcal{U}_\beta, g_\beta)$ where U_β is an ultrafilter on I_β and g_β is a map $I \to \Gamma^f$,
  - $\mathrm{Hom}(\Gamma, \mathbf{K})$ is the category of all **ultradiagrams** $\Gamma \to \mathbf{K}$, where the objects are functors $A : \Gamma \to \mathbf{K}$ which additionally satisfy:
  $$\text{for all } \beta \in \Gamma^b,\ A(\beta) \simeq \textstyle\prod_{i \in I_\beta} A\left(g_\beta(i)\right) \Big/ U_\beta,$$
  and the morphisms are natural transformations $\Phi : A \to B$ between functors $A, B : \Gamma \to \mathbf{K}$ which analogously satisfy:
  $$\Phi_\beta \simeq \textstyle\prod_{i \in I_\beta} \Phi_{g_\beta(i)} \Big/ U_\beta.$$

An **ultrafunctor** $\mathbf{K} \to \mathbf{K}'$ is a functor which preserves all ultramorphisms and which preserves the specified ultraproduct functors up to natural "transition" isomorphisms.

Strong conceptual completeness

Pretoposes are essentially categories of the form $\mathbf{Def}(T^{\mathrm{eq}})$. **Conceptual completeness**, proved by Makkai and Reyes, says that the functor

$$\mathbf{Mod}(-) : \mathbf{Pretop}^{\mathrm{op}} \to \mathbf{Cat}$$

reflects equivalences: if an interpretation $F : T \to T'$ induces an equivalence of categores by taking reducts $\mathbf{Mod}(F) : \mathbf{Mod}(T') \to \mathbf{Mod}(T)$, then F was part of a bi-interpretation $T \simeq T'$.

Strong conceptual completeness, proved by Makkai, generalizes this to the statement that there is a dual adjunction ("Stone adjunction") between the category **Pretop** of pretoposes and the category **Ult** of ultracategories, with **Set** as the dualizing object, whose **counit is an equivalence**. That is, there is an adjunction

$$\mathbf{Ult}(-, \mathbf{Set}) : \mathbf{Ult} \overset{\dashv}{\leftrightarrows} \mathbf{Pretop} : \mathbf{Pretop}(-, \mathbf{Set})$$

whose counit transformation

$$1_{\mathbf{Ult}} \xrightarrow{\sim} \mathbf{Ult}(\mathbf{Pretop}(-, \mathbf{Set}), \mathbf{Set})$$

is an equivalence, and evaluates at any pretopos $\mathbf{Def}(T)$ to a bi-interpretation

$$\mathbf{Def}(T) \simeq \mathbf{Ult}(\mathbf{Pretop}(\mathbf{Def}(T), \mathbf{Set})).$$

So any theory T is bi-interpretable with the (theory presenting the) pretopos of ultrafunctors $\mathbf{Mod}(T) \to \mathbf{Set}$.

In this sense, **all definable sets are ultrafunctors**.

That this counit is an equivalence indeed generalizes conceptual completeness: if $\mathbf{Mod}(T) \overset{\mathbf{Mod}(F)}{\simeq} \mathbf{Mod}(T')$, then

$$T \simeq \mathbf{Ult}(\mathbf{Mod}(T), \mathbf{Set}) \simeq \mathbf{Ult}(\mathbf{Mod}(T'), \mathbf{Set}) \simeq T'.$$

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