McMaster University Math 1A03
Fall 2010 Midterm 2 November 11 2010
Duration: 90 minutes

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Name: Solutions

Student ID Number: ______________

Instructions

– This test paper is printed on both sides of the page. There are 7 questions on pages 2 through 8, and a table of formulas on pages 9 and 10. You are responsible for ensuring that your copy of this test is complete. Bring any discrepancies to the attention of the invigilator.

– Give full justification for your answers.

– Only the McMaster standard calculator, the Casio fx * * * , is permitted.

– Answers must be written in pen.

Problem	Points
1 [10]	
2 [5]	
3 [8]	
4 [7]	
5 [8]	
6 [6]	
7 [6]	
Total [50]	

1) [10 marks] Consider $\int_4^8 \frac{1}{x}\,dx$.

a) Use the trapezoidal rule with $n = 4$ to approximate the value of this integral.

$\Delta x = \frac{8-4}{4} = 1 \qquad x_0 = 4,\ x_1 = 5,\ x_2 = 6,\ x_3 = 7,\ x_4 = 8$

$T_4 = \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right) =$

$= \frac{1}{2}\left(\frac{1}{4} + \frac{2}{5} + \frac{2}{6} + \frac{2}{7} + \frac{1}{8}\right) =$

$= \frac{1171}{1680} \doteq 0.697$

b) What value of n will ensure that the approximation to this integral given by Simpson's rule S_n has an error less than 10^{-4}?

upper bound for $|f^{(4)}(x)|$

$f(x) = x^{-1},\ f'(x) = -x^{-2},\ f''(x) = 2x^{-3},\ f^{(3)}(x) = -6x^{-4},\ f^{(4)}(x) = 24x^{-5}$

$f^{(4)}(x)$ decreasing on $[4,8]$ and positive $\rightarrow$ $|f^{(4)}(x)| = |24x^{-5}| \leq 24\cdot 4^{-5} = K$

error estimate

$|E_S| \leq \frac{K\cdot(b-a)^5}{180n^4}$ we want $\frac{K\cdot(b-a)^5}{180n^4} \leq 10^{-4}$

$\frac{24\cdot 4^{-5}\cdot 4^5}{180n^4} \leq 10^{-4}$

$\frac{24\cdot 10^4}{180} \leq n^4$

if we choose $n \geq 8$, $\leftarrow$ $6.043 \leq n$

then $|E_S| < 10^{-4}$

2) [5 marks] Find the definite integral $\int_{\pi/3}^{\pi} \sin^2(x)\, dx$.

$$\int \sin^2 x \, dx = \int \left[\frac{1}{2} - \frac{\cos(2x)}{2}\right] dx =$$

$$= \int \frac{1}{2} dx - \int \frac{\cos(2x)}{2}\, dx =$$

subst. $u = 2x$
$du = 2\,dx$

$$= \frac{x}{2} - \int \frac{\cos(u)}{4}\, du = \frac{x}{2} - \frac{\sin(u)}{4} + C$$

$$= \frac{x}{2} - \frac{\sin(2x)}{4} + C$$

$$\int_{\pi/3}^{\pi} \sin^2(x) = \left.\frac{x}{2} - \frac{\sin(2x)}{4}\right]_{\pi/3}^{\pi} =$$

$$= \left(\frac{\pi}{2} - \frac{\sin(2\pi)}{4}\right) - \left(\frac{\pi}{6} - \frac{\sin\left(\frac{2}{3}\pi\right)}{4}\right)$$

$$= \frac{\pi}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{8}\right) =$$

$$= \frac{\pi}{3} + \frac{\sqrt{3}}{8}$$

3) [8 marks] Find the general form of the following indefinite integrals.

a) $\int x\cos(x^2)\,dx = \int \frac{1}{2}\cos u\,du = \frac{1}{2}\sin u + C$

subst. $u = x^2$

$du = 2x\,dx$

$= \frac{1}{2}\sin(x^2) + C$

b) $\int x^2\cos(x)\,dx = x^2\sin(x) - \int 2x\sin(x)\,dx =$

by parts $u = x^2$ $v' = \cos(x)$

$u' = 2x$ $v = \sin(x)$

$= x^2\sin x - 2\int x\sin(x)\,dx = x^2\sin x - 2\left(x\cdot(-\cos x) - \int -\cos x\,dx\right) =$

by parts $u = x$ $v' = \sin(x)$

$u' = 1$ $v = -\cos(x)$

$= x^2\sin x + 2x\cos x - 2\int \cos x\,dx =$

$= x^2\sin x + 2x\cos x - 2\sin x + C$

4) [7 marks] Find the general form of the indefinite integral $\int \frac{1}{(9+x^2)^{3/2}}\, dx$.

subst. $x = 3\tan\theta$, $\theta \in (-\pi/2; \pi/2)$

$dx = 3\sec^2\theta\, d\theta$

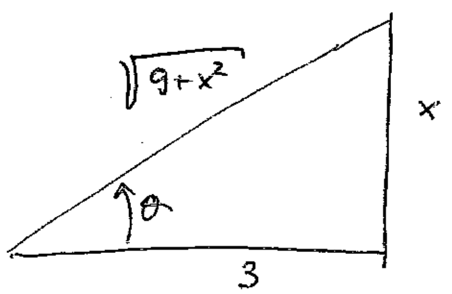
$$\int \frac{1}{(9+x^2)^{3/2}}\, dx = \int \frac{1}{\left(\sqrt{9+x^2}\right)^3}\, dx =$$

$$= \int \frac{1}{\left(\sqrt{9+9\tan^2\theta}\right)^3}\, 3\sec^2\theta\, d\theta =$$

$$= \int \frac{1}{\left(\sqrt{9\cdot\sec^2\theta}\right)^3}\, 3\sec^2\theta\, d\theta = \int \frac{1}{\left(3\,|\sec\theta|\right)^3}\cdot 3\sec^2\theta\, d\theta =$$

$$= \int \frac{3\sec^2\theta}{27\sec^3\theta}\, d\theta = \frac{1}{9}\int \frac{1}{\sec\theta}\, d\theta = \frac{1}{9}\int \cos\theta\, d\theta =$$

$$= \frac{1}{9}\sin\theta + C = \frac{1}{9}\cdot\frac{x}{\sqrt{9+x^2}} + C$$



5) [8 marks] Find the general form of the indefinite integral $\int \frac{-2x^3 - x^2 - 2x + 2}{x^3 - 3x + 2}\, dx$.

long division

$$\begin{array}{r|l} & \quad -2 \\ x^3 - 3x + 2 & -2x^3 - x^2 - 2x + 2 \\ & -2x^3 \quad\; + 6x - 4 \\ \hline & \quad -x^2 - 8x + 6 \end{array}$$

$$\frac{-2x^3 - x^2 - 2x + 2}{x^3 - 3x + 2} = -2 + \frac{-x^2 - 8x + 6}{x^3 - 3x + 2}$$

$x = 1$ is a root of $x^3 - 3x + 2$

$$x^3 - 3x + 2 = (x-1)(x^2 + x - 2) =$$
$$= (x-1)(x-1)(x+2) =$$
$$= (x-1)^2(x+2)$$

$$\begin{array}{r|l} & x^2 + x - 2 \\ x - 1 & x^3 - 3x + 2 \\ & x^3 - x^2 \\ \hline & x^2 - 3x + 2 \\ & x^2 - x \\ \hline & -2x + 2 \\ & -2x + 2 \\ \hline & 0 \end{array}$$

$$\frac{-x^2 - 8x + 6}{x^3 - 3x + 2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2} \quad \Big/ \cdot (x-1)^2(x+2)$$

$$-x^2 - 8x + 6 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$$

$x = 1 \ldots \quad -3 = B \cdot 3 \rightarrow B = -1$

$x = -2 \ldots \quad 18 = C \cdot 9 \rightarrow C = 2$

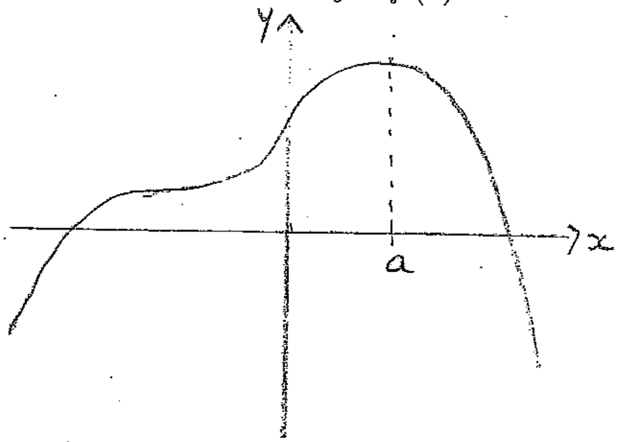
$x = 0 \quad 6 = A \cdot (-2) + (-1) \cdot 2 + 2 \cdot 1 \rightarrow A = -3$

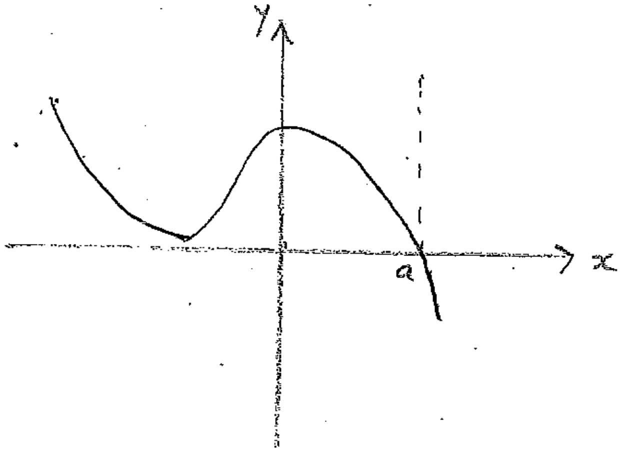
$$\int \frac{-2x^3 - x^2 - 2x + 2}{x^3 - 3x + 2}\, dx = \int \left(-2 + \frac{-3}{x-1} + \frac{-1}{(x-1)^2} + \frac{2}{x+2}\right) dx =$$

$$= -2x - 3\ln|x-1| + \frac{1}{x-1} + 2\ln|x+2| + C$$

6 [6 marks] On this question, you do not need to justify your answers.

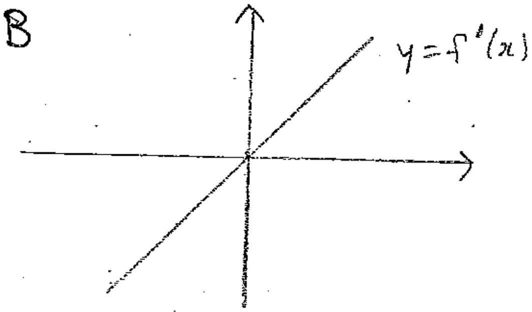
a) The graph of a function $y = f(x)$ is given below. On the blank coordinate axes, sketch the graph of the derivative $y = f'(x)$.



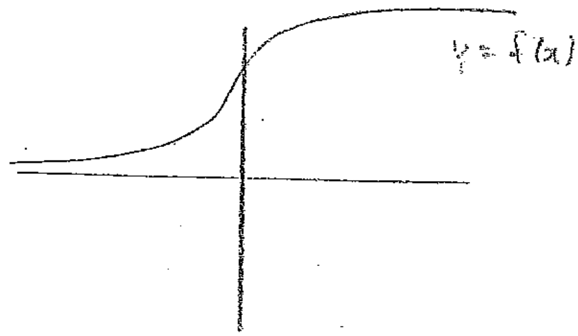


b) The graphs on the left show the derivatives of some functions. The graphs on the right show the functions. Next to the number of the graph on the left, write the letter of the graph on the right which shows the corresponding function.

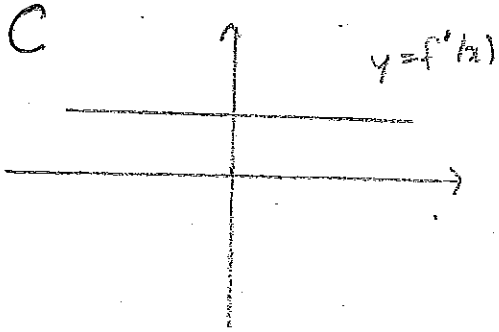
(i) B



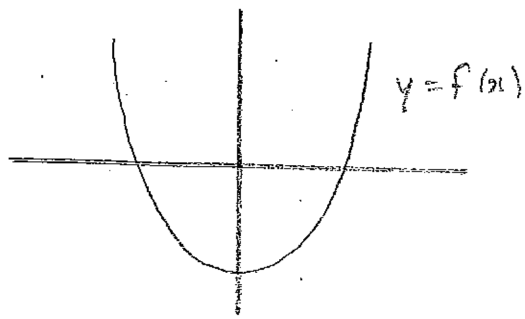
(A)



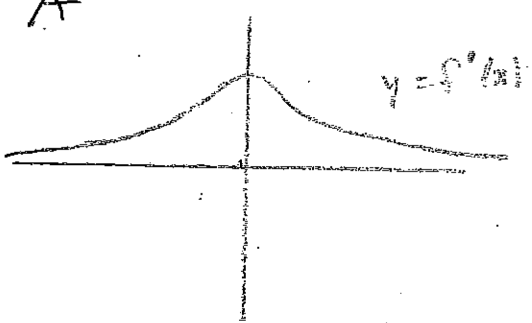
(ii) C



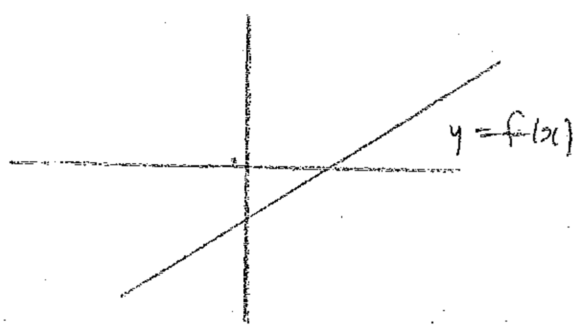
(B)



(iii) A



(C)



7) [6 marks] Find the absolute maximum and minimum values of the function $f(x) = \cos(x)e^{-x}$ on the interval $[0, 2\pi]$. Give a sketch of the graph of $y = f(x)$ on $[0, 2\pi]$ indicating where the function is increasing and where it is decreasing (you do not need to test for concavity).

$$f'(x) = (-\sin x)e^{-x} + \cos x \cdot (-e^{-x}) =$$
$$= -e^{-x}(\sin x + \cos x)$$

critical points

$$-e^{-x}(\sin x + \cos x) = 0$$
$$\sin x + \cos x = 0$$
$$\sin x = -\cos x$$
$$\frac{\sin x}{\cos x} = -1$$
$$\tan x = -1 \qquad x = \frac{3}{4}\pi,\ \frac{7}{4}\pi$$

abs. extremes

$$f(0) = \cos(0)\cdot e^{-0} = 1$$
$$f(2\pi) = \cos(2\pi)\cdot e^{-2\pi} = e^{-2\pi}$$
$$f\left(\tfrac{3}{4}\pi\right) = \cos\left(\tfrac{3}{4}\pi\right)\cdot e^{-\frac{3}{4}\pi} = -\frac{\sqrt{2}}{2}e^{-\frac{3}{4}\pi}$$
$$f\left(\tfrac{7}{4}\pi\right) = \cos\left(\tfrac{7}{4}\pi\right)\cdot e^{-\frac{7}{4}\pi} = \frac{\sqrt{2}}{2}e^{-\frac{7}{4}\pi}$$

abs. max 1 at 0

abs. min $-\frac{\sqrt{2}}{2}e^{-\frac{3}{4}\pi}$ at $\frac{3}{4}\pi$

incr / decr

	$f'(x)$	
$(0, \frac{3}{4}\pi)$	< 0	$\searrow$
$(\frac{3}{4}\pi, \frac{7}{4}\pi)$	> 0	$\nearrow$
$(\frac{7}{4}\pi, 2\pi)$	> 0	$\searrow$