Math 1A03 C01, C02, C03 Fall 2011
McMaster University Practice Final Exam
Duration: 3 hours

Instructors: M. Bays, L. Barto, D. Haskell

Given Name: ______________________ **Family Name:** __________________________

Student ID Number: _______________

Instructions
– Give full justification for your answers.
– Only the McMaster standard calculator, the Casio fx 991, is permitted.

Problem	Points	Problem	Points
1		7	
2		8	
3		9	
4		10	
5		11	
6			
		Total	

1) Find the derivatives of the following functions. No partial credit will be given on this problem.

a) $f(x) = \arctan(x^2)$

$$f'(x) = \frac{1}{1+(x^2)^2} 2x = \frac{2x}{1+x^4}$$

b) $\dfrac{x+\sin(x)}{\ln(x)} = (x+\sin x)(\ln x)^{-1}$

$$\frac{d}{dx}(\uparrow) = (1+\cos x)(\ln x)^{-1} + (x+\sin x)(-1)(\ln x)^{-2} \cdot \frac{1}{x}$$

c) $e^{2x}\tan(x)$

$$\frac{d}{dx}(\uparrow) = e^{2x}\, 2\tan x + e^{2x} \sec^2 x$$
$$= e^{2x}(2\tan x + \sec^2 x)$$

2) Evaluate the following limits. No partial credit will be given on this problem.

a) $\lim_{x \to 0} \frac{\sin(x)}{x}$

$= 1$

b) $\lim_{x \to \infty} (2x^2 - x^3)$

$= -\infty$

c) $\lim_{x \to 2^+} f(x)$ and $\lim_{x \to 2^-} f(x)$, where $f(x) = x^2 + x + 5$, for $x < 2$ and $f(x) = 3x - 2$ for $x > 2$. Is f continuous at 2?

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 3x - 2 = 6 - 4 = 2$

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 + x + 5 = 4 + 2 + 5 = 11$

<u>not</u> continous at $x = 2$

3) Find the general forms of the following indefinite integrals. No partial credit will be given on this problem.

a) $\int \frac{1}{x^{1/3}}\, dx$

$= \frac{3}{2} x^{\frac{2}{3}} + C$

b) $\int (2x+13)\ln(x^2+13x+1)\, dx$

$u = x^2 + 13x + 1$

$du = (2x+13)\, dx$

$= \int \ln u \, du$

$= u \ln u - u + C$

$= (x^2+13x+1)\ln(x^2+13x+1) - (x^2+13x+1) + C$

c) $\int (e^{2x}+5)\, dx$

$= \frac{1}{2} e^{2x} + 5x + C$

4) The speed $v(t)$ of a runner in m/s during the first 5 seconds is shown in the table. Use Simpson's Rule to estimate the distance D (in meters) the runner covered during those 5 seconds.

Given that $-100 \le v^{(4)}(t) \le 67$, find an interval $[a, b]$ such that you can say with confidence that D lies in $[a, b]$.

time in seconds	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
speed in m/s	0	4.67	7.34	8.86	9.73	10.22	10.51	10.67	10.76	10.81	10.81

$n = 10$

$$S_{10} = \frac{0.5}{3}\left[f(0) + 4f(0.5) + 2f(1) + \dots + 4f(4.5) + f(5)\right]$$

$$= \frac{0.5}{3}\left[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81\right]$$

$$= 44.735$$

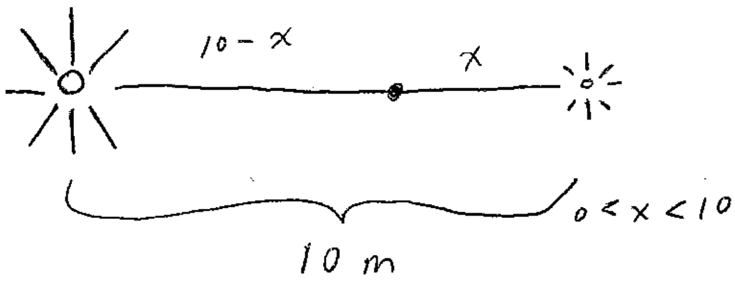
$$D = \int_0^5 v(t)\,dt \approx S_{10} = 44.735 \text{ m}$$

$$|E_S| \le \frac{K(b-a)^5}{180n^4} = \frac{100(5-0)^5}{180(10)^4} = \frac{25}{144}$$

So

$$a = 44.735 - \frac{25}{144} \le D \le 44.735 + \frac{25}{144} = b.$$

5) The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 m apart, where should an object be placed on the line between the sources so as to receive the least illumination?



Let I be the illumination.

$$I = \frac{3}{(10-x)^2} + \frac{1}{x^2}$$

$$I' = \frac{6}{(10-x)^3} - \frac{2}{x^3}$$

Critical points: $I' = 0$

$$\frac{6}{(10-x)^3} = \frac{2}{x^3}$$

$$3 = \left(\frac{10-x}{x}\right)^3$$

$$\sqrt[3]{3} = \frac{10-x}{x}$$

$$1+\sqrt[3]{3} = \frac{10}{x}$$

$$x = \frac{10}{1+\sqrt[3]{3}}$$

So, placing the object $\frac{10}{1+\sqrt[3]{3}}$ meters from the weaker light source minimizes illumination.

6) Let $f(x) = \dfrac{x-1}{x^2}$. Analyze the function f; that is, find the domain, intercepts, asymptotes, intervals of increase/decrease, local and absolute extrema, intervals where the curve is concave up/down, inflection points. Finally, sketch the graph of f.

domain: $x \neq 0$

intercepts: $f(x)=0 \Rightarrow x=1$. x-int: $(1,0)$

asymptotes: vertical: $\lim\limits_{x\to 0^+} \dfrac{x-1}{x^2} = -\infty$, $\lim\limits_{x\to 0^-} \dfrac{x-1}{x^2} = -\infty$

horizontal: $\lim\limits_{x\to\infty} \dfrac{x-1}{x^2} = \lim\limits_{x\to\infty} \dfrac{\frac{1}{x}-\frac{1}{x^2}}{1} = 0$

$\lim\limits_{x\to-\infty} \dfrac{x-1}{x^2} = 0$

increase / decrease:

$f'(x) = (x-1)(-2)x^{-3} + x^{-2} = x^{-3}[-2x+2+x]$

$= x^{-3}(2-x)$

critical points: $x=2$

	x^{-3}	$2-x$	$f'(x)$
$(-\infty,0)$	$-$	$+$	$-$
$(0,2)$	$+$	$+$	$+$
$(2,\infty)$	$+$	$-$	$-$

f decreases on $(-\infty,0)$ and $(2,\infty)$, increases on $(0,2)$

local max at $x=2$, value $f(2)=\frac{1}{4}$.

concave up / down:

$f''(x) = (-3)x^{-4}(2-x) + x^{-3}(-1)$

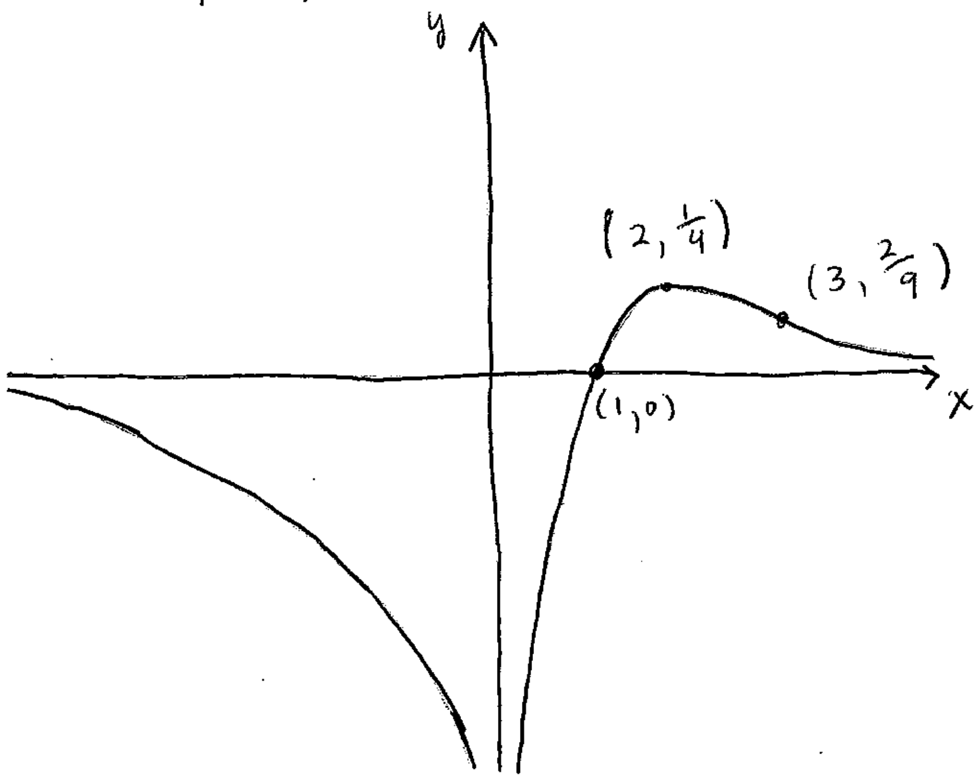
$= x^{-4}[-6+3x-x] = x^{-4}(2x-6)$

$f''(x) = 0 \Rightarrow 2x - 6 = 0, \quad x = 3$

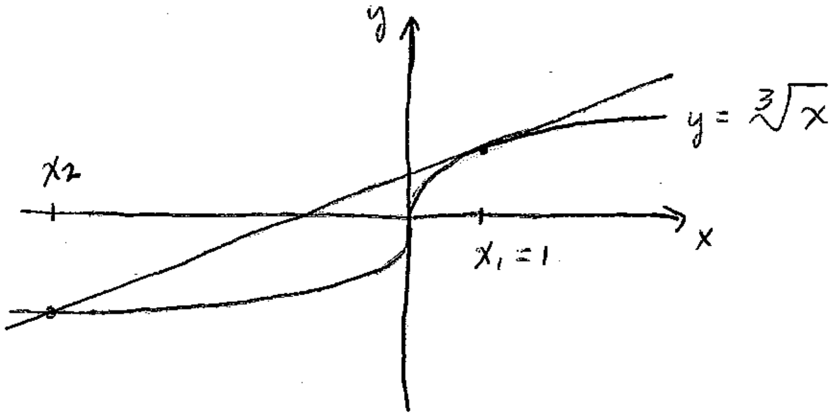
	x^{-4}	$2x-6$	$f''(x)$
$(-\infty, 0)$	$+$	$-$	$-$
$(0, 3)$	$+$	$-$	$-$
$(3, \infty)$	$+$	$+$	$+$

f is concave down on $(-\infty, 0)$ and $(0, 3)$ and concave up on $(3, \infty)$.

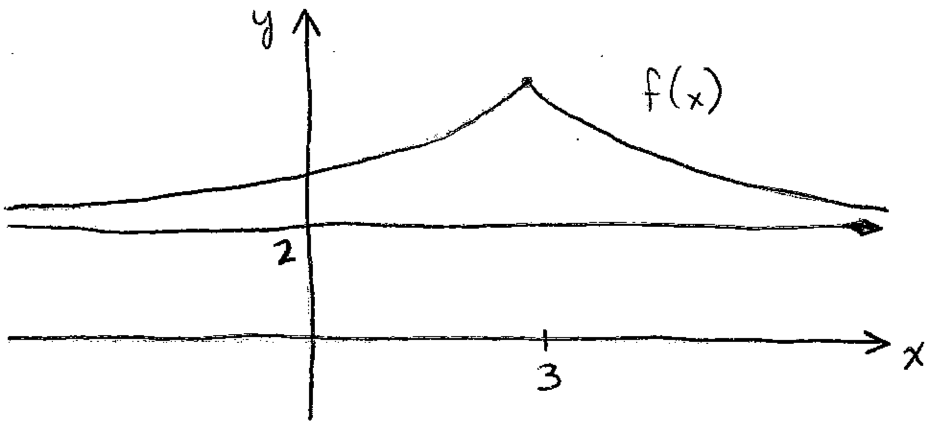
$f(3) = \frac{2}{9}$, $(3, \frac{2}{9})$ is an inflection point.



7) a) Sketch the graph of the function $f(x) = \sqrt[3]{x}$. On this sketch illustrate why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with $x_1 = 1$.



b) Sketch the graph of a function f which is continuous everywhere, $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 2$, $f''(x) > 0$ for all $x \neq 3$ and $f(x)$ is not differentiable at 3.



c) Does there exist a function f such that $f''(x)$ is defined and positive for all real x, and $f(x) = 0$ has precisely 3 solutions? Explain your answer.

$f''(x) > 0$ all $x \Rightarrow f'$ is strictly increasing.

If f has three distinct roots, say $a < b < c$, then by MVT there are points $p \in (a, b)$ and $q \in (b, c)$ where $f'(p) = 0$ and $f'(q) = 0$. But this is impossible since f' is strictly increasing.

8) Evaluate the integral

$$\int_4^5 \frac{1}{t^2\sqrt{t^2-9}}\,dt.$$

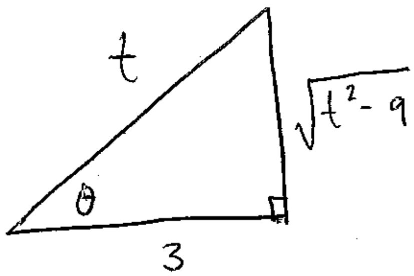
$t = 3\sec\theta \qquad 0 \le \theta < \pi/2$

$dt = 3\sec\theta\tan\theta\,d\theta$

$$= \int_?^? \frac{1}{(3\sec\theta)^2\sqrt{(3\sec\theta)^2-9}}\,3\sec\theta\tan\theta\,d\theta$$

$\sec^2\theta - 1 = \tan^2\theta$

$$= \int \frac{3\sec\theta\tan\theta}{9\sec^2\theta\,3\tan\theta}\,d\theta$$



$$= \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta$$

$$= \frac{1}{9}\int \cos\theta\,d\theta$$

$$= \frac{1}{9}\sin\theta\Big|_?^? = \frac{1}{9}\frac{\sqrt{t^2-9}}{t}\Big|_4^5 = \frac{1}{9}\frac{\sqrt{16}}{5} - \frac{1}{9}\frac{\sqrt{7}}{4}$$

$$= \frac{4}{45} - \frac{\sqrt{7}}{36}$$

9) Find the general form of the following indefinite integrals.

a) $\int \frac{10}{4-x^2}\,dx$

$$= \int \frac{10}{(2-x)(2+x)}\,dx$$

$$= \int \frac{10/4}{2-x} + \frac{10/4}{2+x}\,dx$$

$$= -\frac{10}{4}\ln|2-x| + \frac{10}{4}\ln|2+x| + C$$

b) $\int \ln(3x+1)\,dx$

$u = 3x+1$
$du = 3\,dx$

$\int f'g = fg - \int fg'$

$$= \frac{1}{3}\int \ln u\,du$$

$$= \frac{1}{3}\left[\int \underbrace{(1)}_{f'}\underbrace{(\ln u)}_{g}\,du\right]$$

$$= \frac{1}{3}\left[u\ln u - \int u\frac{1}{u}\,du\right]$$

$$= \frac{1}{3}\left[u\ln u - u\right] + C = \frac{1}{3}(3x+1)\ln(3x+1) - \frac{1}{3}(3x+1) + C$$

10) Find the length of the curve $y = \sqrt{2 - x^2}$, $0 \leq x \leq 1$.

$$y' = \frac{1}{2}\left(2 - x^2\right)^{-1/2}(-2x) = \frac{-x}{\sqrt{2 - x^2}}$$

$$L = \int_0^1 \sqrt{1 + [y']^2}\, dx$$

$$= \int_0^1 \sqrt{1 + \frac{x^2}{2 - x^2}}\, dx$$

$$= \int_0^1 \sqrt{\frac{2}{2 - x^2}}\, dx$$

$$= \int_0^1 \sqrt{\frac{1}{1 - (x/\sqrt{2})^2}}\, dx$$

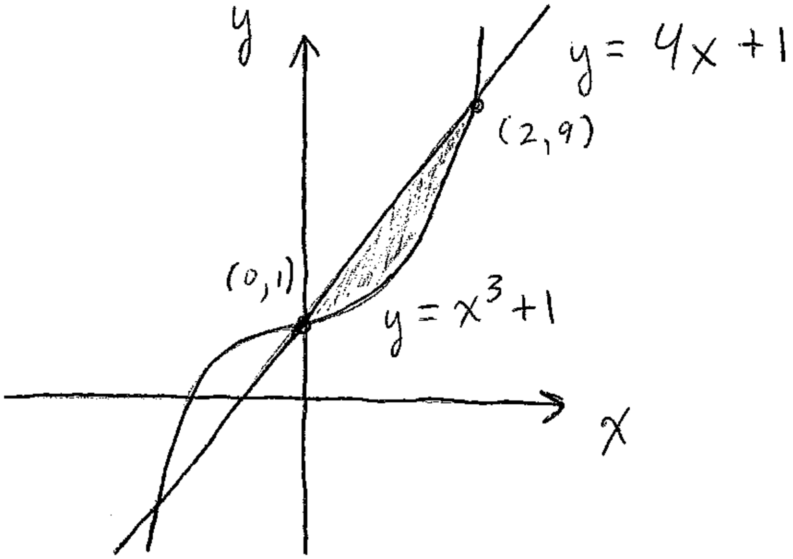
$t = x/\sqrt{2}$

$dt = dx/\sqrt{2}$

$$= \sqrt{2}\int_0^{1/\sqrt{2}} \frac{1}{\sqrt{1 - t^2}}\, dt = \sqrt{2} \arcsin t \Big|_0^{1/\sqrt{2}}$$

$$= \sqrt{2}\, \frac{\pi}{4}$$

11) Consider the region R enclosed by the curves $y = x^3 + 1$ and $x = \frac{1}{4}(y-1)$, $x \geq 0$.
a) Find the area of R.



intersections: $x^3 + 1 = 4x + 1$

$x^3 - 4x = 0$

$x(x^2 - 4) = 0$

$x(x-2)(x+2) = 0$

$x = 0,\ y = 1$
$x = -2,\ y = -7$
$x = 2,\ y = 9$

$$\text{area of } R = \int_0^2 (4x+1) - (x^3+1)\, dx$$

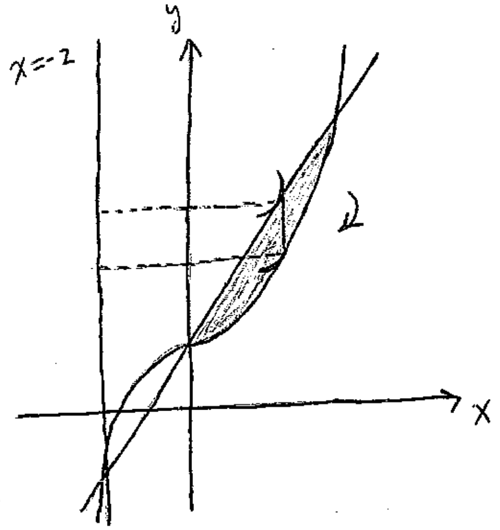
$$= \int_0^2 4x - x^3\, dx = \left(2x^2 - \tfrac{1}{4}x^4\right)\Big|_0^2$$

$$= 2\cdot 4 - \tfrac{1}{4}\cdot 16$$

$$= 8 - 4$$

$$= 4$$

b) Find the volume of the solid obtained by rotating R around the line $x = -2$.



using cylindrical shells....

$$\int_0^2 2\pi(x+2)(4x+1-(x^3+1))\,dx$$

$$= \int_0^2 2\pi(x+2)(4x-x^3)\,dx$$

$$= \int_0^2 2\pi(4x^2-x^4+8x-2x^3)\,dx$$

$$= \int_0^2 8\pi x^2 - 2\pi x^4 + 16\pi x - 4\pi x^3\,dx$$

$$= \frac{8\pi}{3}x^3 - \frac{2\pi}{5}x^5 + 8\pi x^2 - \pi x^4 \Big|_0^2$$

$$= \frac{64\pi}{3} - \frac{64\pi}{5} + 32\pi - 16\pi$$

$$= \frac{128}{15}\pi + 16\pi = \frac{368}{15}\pi$$

Derivatives

$$(x^n)' = nx^{n-1},\ n \neq 0 \qquad (a)' = 0$$

$$(e^x)' = e^x \qquad (a^x)' = a^x \ln a$$

$$(\ln x)' = \frac{1}{x} \qquad (\log_a x)' = \frac{1}{x \ln a}$$

$$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2} \qquad (\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

Integrals (constants of integration are omitted)

$$\int x^n\,dx = \frac{x^{n+1}}{n+1},\ n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x|$$

$$\int e^x\,dx = e^x \qquad \int a^x\,dx = \frac{a^x}{\ln a}$$

$$\int \sin x\,dx = -\cos x \qquad \int \cos x\,dx = \sin x$$

$$\int \tan x\,dx = -\ln|\cos x| \qquad \int \cot x\,dx = \ln|\sin x|$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| \qquad \int \csc x\,dx = -\ln|\csc x + \cot x|$$

$$\int \sec^2 x\,dx = \tan x \qquad \int \csc^2 x\,dx = -\cot x$$

$$\int \sec x \tan x\,dx = \sec x \qquad \int \csc x \cot x\,dx = -\csc x$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) \qquad \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)$$

Trigonometry

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x \qquad 1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2\sin x \cos x \quad \cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1$$

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

Approximate integration

$$\begin{aligned}
\int_a^b f(x)\,dx \approx M_n &= \Delta x \left[f\left(\frac{x_0+x_1}{2}\right) + f\left(\frac{x_1+x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1}+x_n}{2}\right) \right] \\
\int_a^b f(x)\,dx \approx T_n &= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \\
\int_a^b f(x)\,dx \approx S_n &= \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + \\
&\quad +4f(x_{n-1}) + f(x_n)] \qquad n \text{ is even}
\end{aligned}$$

Error bounds:

If $|f''(x)| \le K$ on $[a,b]$, then $|E_T| \le K(b-a)^3/(12n^2)$, $\quad |E_M| \le K(b-a)^3/(24n^2)$.
If $|f^{(4)}(x)| \le K$ on $[a,b]$, then $|E_S| \le K(b-a)^5/(180n^4)$.

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Arc length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\; dx$$

Volume

$$\begin{aligned}
V &= \int_a^b A(x)\,dx \quad (A(x) = \text{area of cross-section through } x) \\
V &= \int_a^b 2\pi x f(x)\,dx \quad (\text{rotating } y = f(x) \text{ about } y\text{-axis})
\end{aligned}$$

The End