

Math 1A03 C01, C02, C03 Fall 2011
McMaster University Practice Final Exam
Duration: 3 hours

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Given Name: _____ **Family Name:** _____

Student ID Number: _____

Instructions

- Give full justification for your answers.
- Only the McMaster standard calculator, the Casio fx 991, is permitted.

Problem	Points	Problem	Points
1		7	
2		8	
3		9	
4		10	
5		11	
6			
		Total	

1) Find the derivatives of the following functions. No partial credit will be given on this problem.

a) $f(x) = \arctan(x^2)$

b) $\frac{x + \sin(x)}{\ln(x)}$

c) $e^{2x} \tan(x)$

2) Evaluate the following limits. No partial credit will be given on this problem.

a) $\lim_{x \rightarrow 0} \frac{\sin(x)}{x}$

b) $\lim_{x \rightarrow \infty} (2x^2 - x^3)$

c) $\lim_{x \rightarrow 2^+} f(x)$ and $\lim_{x \rightarrow 2^-} f(x)$, where $f(x) = x^2 + x + 5$, for $x < 2$ and $f(x) = 3x - 2$ for $x > 2$.
Is f continuous at 2?

3) Find the general forms of the following indefinite integrals. No partial credit will be given on this problem.

a) $\int \frac{1}{x^{1/3}} dx$

b) $\int (2x + 13) \ln(x^2 + 13x + 1) dx$

c) $\int (e^{2x} + 5) dx$

4) The speed $v(t)$ of a runner in m/s during the first 5 seconds is shown in the table. Use Simpson's Rule to estimate the distance D (in meters) the runner covered during those 5 seconds.

Given that $-100 \leq v^{(4)}(t) \leq 67$, find an interval $[a, b]$ such that you can say with confidence that D lies in $[a, b]$.

time in seconds	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
speed in m/s	0	4.67	7.34	8.86	9.73	10.22	10.51	10.67	10.76	10.81	10.81

5) The illumination of an object by a light source is directly proportional to the strength of the source and inversely proportional to the square of the distance from the source. If two light sources, one three times as strong as the other, are placed 10 m apart, where should an object be placed on the line between the sources so as to receive the least illumination?

6) Let $f(x) = \frac{x-1}{x^2}$. Analyze the function f ; that is, find the domain, intercepts, asymptotes, intervals of increase/decrease, local and absolute extrema, intervals where the curve is concave up/down, inflection points. Finally, sketch the graph of f .

7) a) Sketch the graph of the function $f(x) = \sqrt[3]{x}$. On this sketch illustrate why Newton's method fails when applied to the equation $\sqrt[3]{x} = 0$ with $x_1 = 1$.

b) Sketch the graph of a function f which is continuous everywhere, $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 2$, $f''(x) > 0$ for all $x \neq 3$ and $f(x)$ is not differentiable at 3.

c) Does there exist a function f such that $f''(x)$ is defined and positive for all real x , and $f(x) = 0$ has precisely 3 solutions? Explain your answer.

8) Evaluate the integral

$$\int_4^5 \frac{1}{t^2 \sqrt{t^2 - 9}} dt.$$

9) Find the general form of the following indefinite integrals.

a) $\int \frac{10}{4 - x^2} dx$

b) $\int \ln(3x + 1) dx$

10) Find the length of the curve $y = \sqrt{2 - x^2}$, $0 \leq x \leq 1$.

- 11) Consider the region R enclosed by the curves $y = x^3 + 1$ and $x = \frac{1}{4}(y - 1)$, $x \geq 0$.
- a) Find the area of R .

b) Find the volume of the solid obtained by rotating R around the line $x = -2$.

Derivatives

$$(x^n)' = nx^{n-1}, \quad n \neq 0$$

$$(a)' = 0$$

$$(e^x)' = e^x$$

$$(a^x)' = a^x \ln a$$

$$(\ln x)' = \frac{1}{x}$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(\sin x)' = \cos x$$

$$(\cos x)' = -\sin x$$

$$(\tan x)' = \sec^2 x$$

$$(\cot x)' = -\csc^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(\csc x)' = -\csc x \cot x$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$

Integrals (constants of integration are omitted)

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \quad n \neq -1$$

$$\int \frac{1}{x} dx = \ln |x|$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\ln a}$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \tan x dx = -\ln |\cos x|$$

$$\int \cot x dx = \ln |\sin x|$$

$$\int \sec x dx = \ln |\sec x + \tan x|$$

$$\int \csc x dx = -\ln |\csc x + \cot x|$$

$$\int \sec^2 x dx = \tan x$$

$$\int \csc^2 x dx = -\cot x$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \csc x \cot x dx = -\csc x$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \left(\frac{x}{a} \right)$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \left(\frac{x}{a} \right)$$

Trigonometry

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x \quad 1 + \cot^2 x = \csc^2 x$$

$$\sin(2x) = 2 \sin x \cos x \quad \cos(2x) = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1$$

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \quad \cos^2 x = \frac{1 + \cos(2x)}{2}$$

Approximate integration

$$\int_a^b f(x) dx \approx M_n = \Delta x \left[f\left(\frac{x_0 + x_1}{2}\right) + f\left(\frac{x_1 + x_2}{2}\right) + \cdots + f\left(\frac{x_{n-1} + x_n}{2}\right) \right]$$

$$\int_a^b f(x) dx \approx T_n = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$\int_a^b f(x) dx \approx S_n = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)] \quad n \text{ is even}$$

Error bounds:

If $|f''(x)| \leq K$ on $[a, b]$, then $|E_T| \leq K(b-a)^3/(12n^2)$, $|E_M| \leq K(b-a)^3/(24n^2)$.

If $|f^{(4)}(x)| \leq K$ on $[a, b]$, then $|E_S| \leq K(b-a)^5/(180n^4)$.

Newton's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Arc length

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

Volume

$$V = \int_a^b A(x) dx \quad (A(x) = \text{area of cross-section through } x)$$

$$V = \int_a^b 2\pi x f(x) dx \quad (\text{rotating } y = f(x) \text{ about } y\text{-axis})$$