NONLINEAR SYSTEMS

How do we solve ('square') problems like:

$$x_1^2 + x_2 + \gamma = 0$$
$$3x_1 + x_2^3 = 0 \quad ?$$

We cannot write the problem as $\underline{A}\,\underline{x} = \underline{b}$ and use the methods from LINEAR SYSTEMS.

What ideas can we borrow from NONLINEAR SCALARS?:

1) Exhaustive search
2) Bisection
3) Fixed point
3) Secant
4) Newton (plus variations).

FIXED POINT $\quad g(\underline{x}) : \mathbb{R}^n \mapsto \mathbb{R}^n.$

$$\underline{x}^{i+1} = g(\underline{x}^i)$$

NEWTON

$$\underline{x}^{i+1} = \underline{x}^{i} - \underline{\underline{J}}_f^{-1}(\underline{x}^{i})\, g(\underline{x}^{i})$$

where $\underline{\underline{J}}$ is the Jacobian. Defined by

$$J_{ij} = \frac{\partial g_i}{\partial x_j}$$

Apply this directly involves inverting $\underline{\underline{J}}$ which can be costly.
A 'trick' to avoid this is to solve

$$\underline{\underline{J}}_f(\underline{x}^{i})\, \underline{s}^{i} = -g(\underline{x}^{i})$$

and use the iteration

$$\underline{x}^{i+1} = \underline{x}^{i} + \underline{s}^{i}$$

Variants: CONSTANT SLOPE retain $\underline{\underline{J}}^{-1}(\underline{x}^{0})$

$$\underline{x}^{i+1} = \underline{x}^{i} - \underline{\underline{J}}_f^{-1}(\underline{x}^{0})\, g(\underline{x}^{i})$$

DAMPED

$$\underline{x}^{i+1} = \underline{x}^{i} - \mu^{i} \underline{\underline{J}}_g^{-1}(\underline{x}^{i})\, g(\underline{x}_i).$$

SECANT METHOD (Finite difference Newton).

Approximate the Jacobian by

$$J_{ij} = \frac{\partial g_i}{\partial x_j} \approx \frac{g(\underline{x}+\underline{h}) - g(\underline{x}-h)}{2h}$$

BROYDEN'S METHOD

Start with an approximation to the Jacobian and update at each iteration.

Compute $\underline{\underline{J}}_g^0(\underline{x}^0)$ somehow (SECANT if need be).

1. Solve $\underline{\underline{J}}^i \underline{s}^i = -f(\underline{x}_i)$
2. $\underline{x}^{i+1} = \underline{x}^i + \underline{s}^i$
3. $\underline{y}^i = f(\underline{x}^{i+1}) - f(\underline{x}^i)$
4. $\underline{\underline{J}}^{i+1} = \underline{\underline{J}}^i + \left((\underline{y}^i - \underline{\underline{J}}^i \underline{s}^i)\underline{s}^{iT}\right) / (\underline{s}^{iT}\underline{s}^i)$
5. $i = i+1$
6. Repeat