ODEs (cont): LECTURES 13+14

Stability of Backward Euler

$$\boxed{y_{i+1} = y_i + h_i f(t_{i+1}, y_{i+1})}$$

Consider the ODE $y' = \lambda y$ with $y|_{t=0} = y_0$

Applying a backward Euler step (with a fixed stepsize, h) we find

$$y_{i+1} = y_i + h \lambda y_{i+1}$$

$$\Longrightarrow (1-\lambda h) y_{i+1} = y_i$$

$$\Longrightarrow y_{i+1} = \frac{1}{1-\lambda h} y_i$$

By induction $$y_{i+1} = \left(\frac{1}{1-\lambda h}\right)^k y_0$$

Thus, method is stable if $$\frac{1}{|1-\lambda h|} \leq 1$$

$$\Longrightarrow |1-\lambda h| \geq 1$$

$$\Rightarrow \lambda h < 0 \quad \text{or} \quad \lambda h > 2$$

SOME 'RULES OF THUMB' (Explicit vs Implicit methods).

Explicit	Implicit.
A single time step is cheap (evaluate the RHS).	A single time step is more expensive (Solve a system).
Limited stability regions	(Typically) larger stability regions.

Explicit.

Runge-Kutta Methods

Both schemes we have seen so far are locally $O(h^2)$ (globally $O(h)$), i.e. Forward and Backward Euler. They are both called 'first order' techniques.

The RK family are <u>higher order explicit</u> methods.

They judiciously use information about the 'slope' (i.e. $f(y,t)$) at more than one point to extrapolate to the future.

Returning to Euler methods for a moment we see that Forward (Backward) Euler uses only information at the previous (current) time to predict the future.

In general, if we use n function evaluations we can derive an 'n-th order' scheme

EX 2nd order RK/Midpoint method

$$y' = f(t,y). \quad \text{——— (1)}$$

Imagine we want to find y_{i+1} given y_i in the following fashion:

$$k_1 = h f(t_i, y_i)$$

$$k_2 = h f(t_i + \alpha h, y_i + \beta k_1) \quad \text{——— (3)}$$

$$y_{i+1} = y_i + a k_1 + b k_2. \quad \text{——— (5)}$$

How do we choose α, β, a, b so that the local error is $O(h^3)$?

Consider:

$$\text{(2) ———} \quad y(t_{i+1}) = y(t_i) + h \left.\frac{dy}{dt}\right|_{t=t_i} + \frac{h^2}{2} \left.\frac{d^2y}{dt^2}\right|_{t=t_i} + O(h^3).$$

From (1) we have $\frac{d^2y}{dt^2} = \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f.$

Substituting back into 2

4 —
$$y(t_{i+1}) = y(t_i) + h\,f(t_i, y_i) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial y} f\right)\Bigg|_{y=y_i,\,t=t_i} + O(h^3).$$

Return to equation 3 and expand

$$k_2 = h\left(f(t_i, y_i) + \alpha h \frac{\partial f}{\partial t}\bigg|_{y=y_i,\,t=t_i} + \beta k_1 \frac{\partial f}{\partial t}\bigg|_{y=y_i,\,t=t_i}\right) + O(h^3).$$

Substituting into 5 $\Rightarrow$

$$y_{i+1} = y_i + (a+b)h\,f(t_i, y_i) + b h^2\left(\alpha \frac{\partial f}{\partial t} + \beta f \frac{\partial f}{\partial t}\right)\Bigg|_{y=y_i,\,t=t_i} + O(h^3).$$

Comparing the above to equation to 4 $\Rightarrow$

$$\left.\begin{aligned} a+b &= 1 \\ \alpha b &= \tfrac{1}{2} \\ \beta b &= \tfrac{1}{2}. \end{aligned}\right\} \Rightarrow$$

Infinitely many solns here. The choice $\alpha = 1,\ \beta = 1,\ a = 1/2,\ b = 1/2.$ gives the classical RK2 scheme as follows.

$$\boxed{\begin{aligned} k_1 &= h\,f(t_i, y_i) \\ k_2 &= h\,f(t_i + h,\ y_i + k_1) \\ y_{i+1} &= y_i + k_1/2 + k_2/2. \end{aligned}}$$

This is locally correct to $O(h^3)$. Thus it is 'second-order'.

Runge-Kutta 4 (RK4) is derived similarly by cancelling terms up to $O(h^5)$ to give the following 'fourth order' scheme.

$$k_1 = hf(t_i, y_i)$$
$$k_2 = hf(t_i + h/2, y_i + k_1/2)$$
$$k_3 = hf(t_i + h/2, y_i + k_2/2)$$
$$k_4 = hf(t_i + h, y_i + k_3)$$
$$y_{i+1} = y_i + (k_1 + 2k_2 + 2k_3 + k_4)/6.$$

Since these RK methods are explicit they suffer from limited stability.

TRAPEZOIDAL METHOD

A higher order (second order) implicit method.

$$y_{i+1} = y_i + \frac{h}{2}\left(f(y_i, t_i) + f(y_{i+1}, t_{i+1})\right).$$

SUMMARY

In practice RK4 is the explicit method of choice
TRAPZ —"— implicit —"—

NOTE: Higher order implicit methods do exist, but stability regions get smaller as order is increased.

BOUNDARY VALUE PROBLEMS

So far have looked at IVPs where ~~bound~~ conditions are all given at the initial time.
In contrast, BVPs have conditions that must be satisfied at more than one point.

EX $$\frac{d^2y}{dt^2} = y^2 \quad \text{with } y\Big|_{t=0} = 0 \quad \frac{dy}{dt}\Big|_{t=1} = 12.$$

These require different approaches to be solved.
One simple tactic is to convert into a series of IVPs.

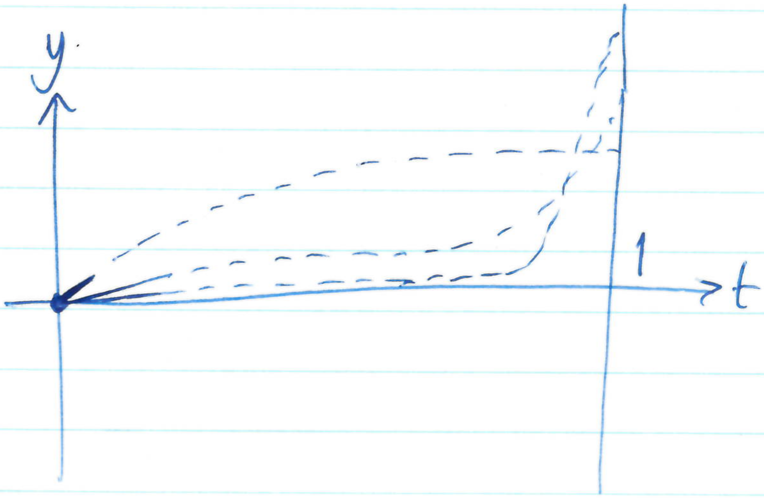
SHOOTING

- We ~~apply~~ fabricate some pseudo-initial conditions.
- Solve the resulting IVP
- Play with these pseudo-conditions until we 'hit' the correct boundary condition(s) at the other end.

For the example above:
Define a function $S(\alpha)$ such that

$$S(\alpha) = \frac{dy}{dt}\Big|_{t=1} - 12 \quad \text{where } y \text{ is solution to } y'' = y^2 \text{ with } y\Big|_{t=0} = 0 \quad y'\Big|_{t=0} = \alpha$$

We can solve this IVP for a given α. Then, we just need to find α s.t. $S(\alpha) = 0$ $\Leftarrow$ A root finding problem. We can solve these.



$\underline{\text{EX}}$ $\quad y'' = x - y \quad$ with $\quad y\big|_{x=0} = 0 \quad y'\big|_{x=1} = 6$

Solve this by shooting.

NOTE: If more than one, say n, BCs are to be imposed at the 'end time', the root finding problem becomes harder (in $\mathbb{R}^n$).

~~FINITE DIFFERENCE METHODS~~