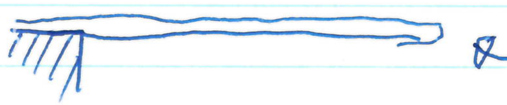
LECTURES 15 + 16: ODEs (continued)

EX Shooting for a higher order system.

$$\underbrace{\frac{d^4 w}{dx^4}}_{\text{Equation for deflection of a beam.}} = f = \underbrace{(1 - x^2)}_{\text{The load applied to the beam}}$$

Boundary conditions: $\quad w|_{x=0} = \left.\frac{dw}{dx}\right|_{x=0} = 0 \longleftarrow$ fixed end

$$\left.\frac{d^2 w}{dx^2}\right|_{x=1} = \left.\frac{d^3 w}{dx^3}\right|_{x=1} = 0 \longleftarrow \text{free end}$$



This is a model for the deflection of a ruler hanging off a table.

1) Invent pseudo-conditions to pose an IVP. In this case we choose

$$\left.\frac{d^2 w}{dx^2}\right|_{x=0} = \alpha \qquad \left.\frac{d^3 w}{dx^3}\right|_{x=0} = \beta .$$

2) Define the 'target'. In this case there are 2 BCs to be shot for. We want to choose α and β such that.

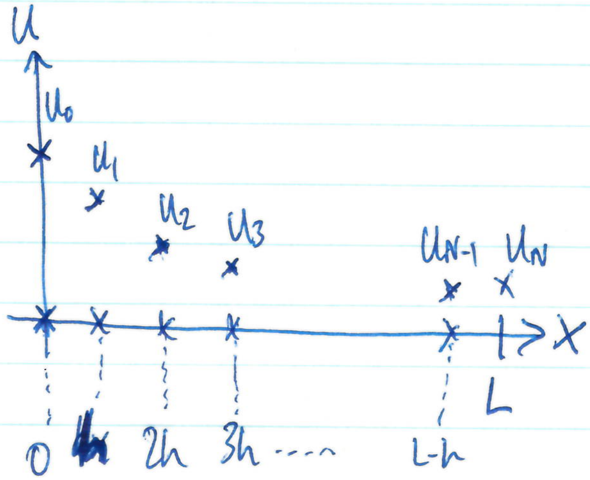
$$\underline{S}(\alpha,\beta) = 0 \quad \text{where} \quad \begin{bmatrix} S_1(\alpha,\beta) \\ S_2(\alpha,\beta) \end{bmatrix} = \begin{bmatrix} \left.\frac{d^2w}{dx^2}\right|_{x=1} \\ \left.\frac{d^3w}{dx^3}\right|_{x=1} \end{bmatrix}$$

3) If we can find α, β s.t. $\underline{S} = \underline{0}$ we have solved the BVP. This is now a root finding prob (which we can solve)!

See MATLAB CODE FOR IMPLEMENTATION.

BOUNDARY VALUE PROBLEMS BY FINITE DIFFERENCES.

Rather than converting a BVP to an IVP, finite differences allows us to solve BVPs directly.



General idea: Convert the BVP into a system of algebraic equations (which we know how to solve), by replacing derivatives with finite differences. E.g.

$$\left.\frac{d^2u}{dx^2}\right|_{x=mh} \approx \frac{u_{m+1} - 2u_m + u_{m-1}}{h^2}.$$

Lets first understand how to construct these FD approximations. We use our old friend, the Taylor Series:

EX 1st order Forward approximation of $\frac{d}{dx}$.

Taylor expansion for u_{m+1} about u_m.

$$u_{m+1} = u_m + h \cdot \left.\frac{du}{dx}\right|_{x=mh} + O(h^2)$$

$$\Rightarrow \boxed{\left.\frac{du}{dx}\right|_{x=mh} = \frac{u_{m+1} - u_m}{h} + O(h)}$$

This is first order because error term is $O(h)$ and it is a forward approximation because the derivative at a point is approximated using information from points to the right.

EX A 1st order backward approx. of $\frac{d}{dx}$ follows from expanding u_{m-1} about u_m. We find

$$\left.\frac{du}{dx}\right|_{x=mh} = \frac{u_m - u_{m-1}}{h} + O(h).$$

EX A second order, central difference approximation of the first derivative can be derived by combining the two previous ideas.

We have seen that

$$u_{m+1} = u_m + h \left.\frac{du}{dx}\right|_{x=mh} + h^2 \left.\frac{d^2u}{dx^2}\right|_{x=mh} + O(h^3)$$

$$u_{m-1} = u_m + (-h) \left.\frac{du}{dx}\right|_{x=mh} + (-h)^2 \left.\frac{d^2u}{dx^2}\right|_{x=mh} + O(h^3)$$

The ~~sum~~ difference of the equations above gives.

$$u_{m+1} - u_{m-1} = 2h \left.\frac{du}{dx}\right|_{x=mh} + O(h^3)$$

$$\Longrightarrow \left.\frac{du}{dx}\right|_{x=mh} = \frac{u_{m+1} - u_{m-1}}{2h} + O(h^2).$$

By generalising these ideas we can get either forward, backward or central approximations of arbitrary accuracy by using information from more neighbouring points. This is summarized in the following tables:

CENTRAL

Order of deriv.	Order of accuracy	U_{m-2}	U_{m-1}	U_m	U_{m+1}	U_{m+2}
1	2	0	$-\frac{1}{2h}$	0	$+\frac{1}{2h}$	0
1	4	$\frac{1}{12h}$	$-\frac{2}{3h}$	0	$\frac{2}{3h}$	$-\frac{1}{12h}$
2	2	0	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$	0
2	4	$-\frac{1}{12h^2}$	$\frac{4}{3h^2}$	$-\frac{5}{2h^2}$	$\frac{4}{3h^2}$	$-\frac{1}{12h^2}$
3	2	$-\frac{1}{2h^3}$	$\frac{1}{h^3}$	0	$-\frac{1}{h^3}$	$\frac{1}{2h^3}$
4	2.	$\frac{1}{h^4}$	$-\frac{4}{h^4}$	$\frac{6}{h^4}$	$-\frac{4}{h^4}$	$\frac{1}{h^4}$

FORWARD

1	1	0	0	$-\frac{1}{h}$	$\frac{1}{h}$	0
1	2.	0	0	$-\frac{3}{2h}$	$\frac{2}{h}$	$-\frac{1}{2h}$
2	1	0	0	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h^2}$.

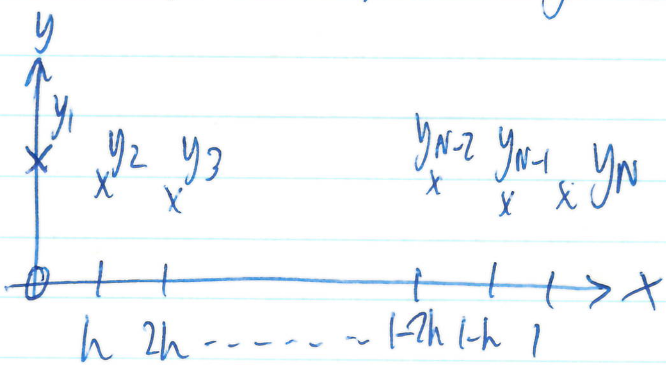
BACKWARD

1	1	0	$-\frac{1}{h}$	$\frac{1}{h}$	0	0
1	2	$\frac{1}{2h}$	$-\frac{2}{h}$	$\frac{3}{2h}$	0	0
2	1	$\frac{1}{h^2}$	$-\frac{2}{h^2}$	$\frac{1}{h}$.	0	0

When constructing a scheme using FDs the accuracy of the solution is limited by the least accurate FD approximation. Thus, we always try to <u>match</u> the order of approximation or effort is being wasted.

<u>EX</u>. Solve $y'' = -y$ with $y|_{x=0} = 0$ $y'|_{x=1} = 6$.
using 2nd order accurate finite differences

i) Split the domain into N grid points



$$h = \frac{1}{N-1}$$

2) At the internal points we want to solve the ODE. Replace $y'' = -y$ with its FD approximation. Use central diff. 2nd order $\Rightarrow$

$$① \text{ — } \quad \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = -y_i \quad \text{for } i = 2 \ldots N-1.$$

3) Build-in boundary condition at $x = 0$

$$y_1 = 0. \quad \text{— } ②$$

4) Build-in BC at $x=0$. Use 2^{nd} order backward difference of 1st derivative.

$$\frac{y_{N-2} - 4y_{N-1} + 3y_N}{2h} = 6. \quad (3)$$

5) (1), (2) and (3) is a system of (N) linear (because the original ODE and BCs was linear) equations in N unknowns.

Since linear can write the problem as

$$(\underline{\underline{A}}_1 + \underline{\underline{A}}_2)\underline{y} = \underline{0}$$

with

$$\underline{\underline{A}}_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & & & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 & \cdots & & 0 \\ 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \cdots & & 0 \\ 0 & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \frac{1}{2h} & -\frac{4}{2h} & \frac{3}{2h} \end{pmatrix} \qquad \underline{y} = \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_N \end{pmatrix}$$

$$\underline{\underline{A}}_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0\ 0 \cdots & 0 \end{pmatrix}.$$

For a non-linear problem the procedure is the same, except we have to think a bit harder about how to solve the resulting system of nonlinear equations.

EX $y'' = -xy^2$ with $y|_{x=0} = 0$ $y|_{x=1} = 1$.

Steps 1) is identical to before.

2). On the internal points.

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} = -(i-1)h\,y_i^2 \quad \text{—} \ 1$$

3) Left BC: $y_1 = 0$ — 2

4) Right BC $y_N = 1$ — 3

5) 1, 2 and 3 are a system of nonlinear equations which can be written as

$\underline{f}(\underline{y}) = 0$ ← can be solved by

- fixed point iter
- Secant
- Newton
- Broyden.

see notes on nonlinear systems.