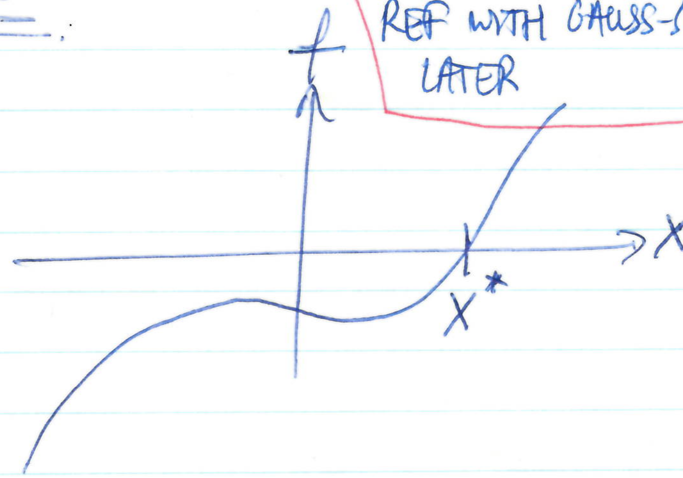
LECTURE 3

NONLINEAR SCALAR EQUATIONS

SHOULD ADD FIXED POINT FOR REF WITH GAUSS-SIEDEL LATER

Methods for treating $f(x)=0$

Supposing $f(x)$ is a continuous function.



~~BISE~~ EXHASTIVE SEARCH

1) Select an initial guess x_0 Q (Should believe $x_0 < x^*$)
2) Find $f(x_0)$

~~3) ... Find $f(x_0+h)$ for some h (How should we choose h?)~~

~~4) ... if a sign change is observed~~

3) Find $f(x_0+h)$, $f(x_0+2h)$, $f(x_0+3h)$ ---- until a sign change observed
4) Set $x_1 = x_0 + nh$, ~~...~~ Decrease h Q (How to choose h?)
5) Repeat

NOTE - Inefficient but robust.

Q (How much to decrease h)?

BISECTION

1) Find a, b s.t $f(a)f(b) < 0$
2) Decide if $f(a)<0$ or $f(b)<0$
3) Set $m = \frac{a+b}{2}$ and compute $f(m)$
4) If $f(m)<0$ and $f(a)<0$ $m=a$ etc
5) Repeat.

Note: Faster than exhaustive search but still quite slow.

Note: if $b-a=\omega$ interval length halves each iteration. Therefore

$$|x_k - x^*| < \frac{b-a}{2^{k-1}}$$

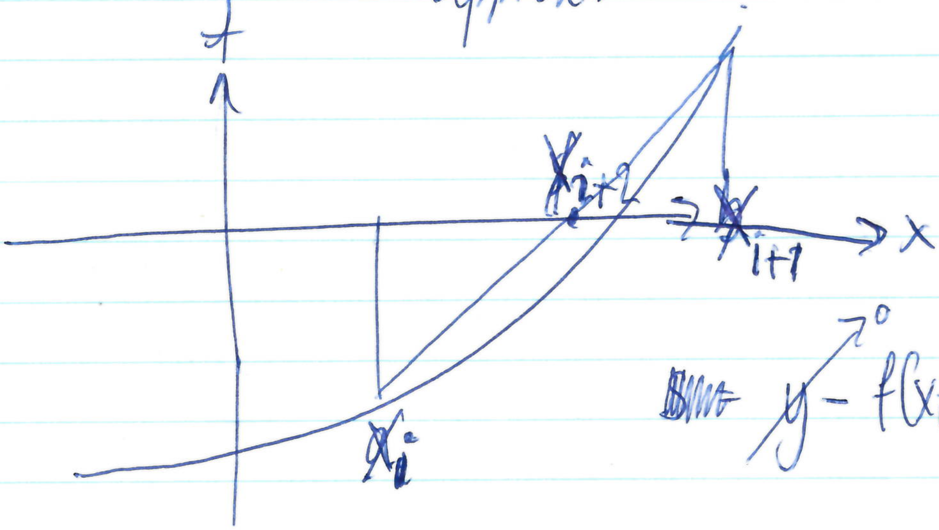
Q How many iterations before $|x-x^*| < \epsilon_{mach}$? When $b-a=1$

$k = \quad ?$

SECANT METHOD

Previous methods don't depend on 'shape' of f. Can we do better by using it?

What if we use a straight line approximation?



$$\cancel{y} - f(x_{i+1}) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x_{i+2} - x_{i+1})$$

$$\Rightarrow x_{i+2} = x_{i+1} - f(x_{i+1})\frac{x_{i+1} - x_i}{f(x_{i+1}) - f(x_i)}.$$

Convergence
p68

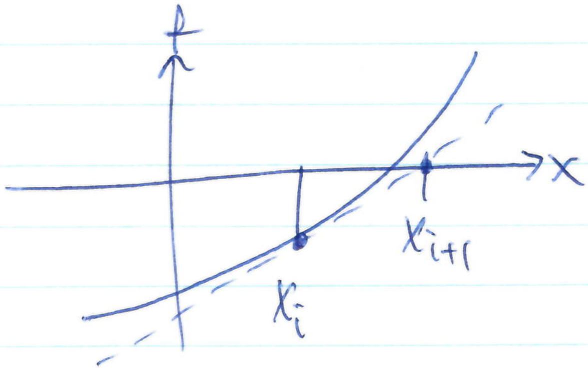
stopping conditions
p62.

Q: What can go wrong?

1) $f(x_{i+1}) = f(x_i)$ or even $f(x_{i+1}) \approx f(x_i)$

NEWTON RAPHSON

f - continuous and differentiable



$$y - f(x_i) = f'(x_i)(x - x_i)$$

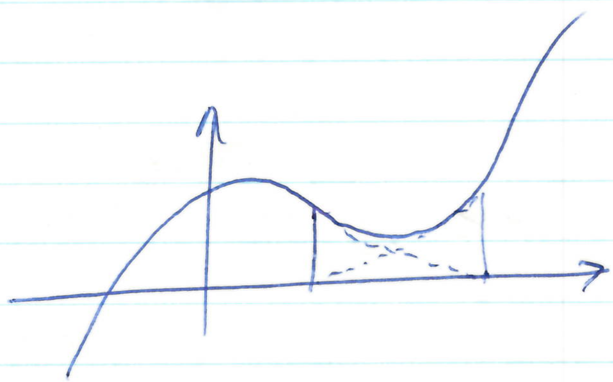
$y = 0 \quad x = x_{i+1}$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Q: What can go wrong?

1) $f'(x_i) \approx 0$.

2) Cycling, e.g.,



3) Divergence $f = |x|^{\frac{1}{3}}$

4) Can't compute f' - or it's hard/expensive.

NOTE: Limit of secant!

NEWTON VARIANTS (Avoiding derivatives).

1) Constant Slope Newton:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_0)}$$

PRO: No need to keep finding f'
CON: Slower if f' varies much.

2) Damped Newton

$$x_{i+1} = x_i + \mu_i \frac{f(x_i)}{f'(x_i)}$$

PRO: Can converge where Newton diverges.
CON: Slower

STOPPING CONDITIONS How do we decide to terminate an iterative method?

1) Num iterations
2) $f(x_k) < errtol$
3) $|x_k - x_{k+1}| < tol$

* A warning sign $|x_k - x_{k+1}| > \frac{1}{\delta}$

CONVERGENCE

$$\lim_{k \to \infty} \frac{|x_{k+1} - x^*|}{|x_k - x^*|^p} = \lambda.$$

p - order of convergence

λ - asymptotic error constant.

For bisection:
$$\left.\begin{aligned} |x_{k+1} - x^*| &< \frac{(b-a)}{2^k} \\ |x_k - x^*| &< \frac{(b-a)}{2^{k-1}} \end{aligned}\right\} \Rightarrow \frac{|x_{k+1} - x^*|}{|x_k - x^*|} < \frac{1}{2}$$

Linear, with err. const $= \frac{1}{2}$.

~~For Newton: $f(x^*) = f(x_n) + f'(x_n)(x - x_n) + O((x - x_n)^2)$~~

Set $x = x^*$

For Newton: $f(x_k) = \cancel{f(x^*)}^{\,0} + f'(x^*)\,\epsilon_k + \frac{f''(x^*)}{2}\,\epsilon_k^2 + \cdots$ (1)

$\frac{1}{2} f''$

$\Rightarrow f'(x_k) = f'(x^*) + f''(x^*)\,\epsilon_k$ (2)

$$\epsilon_{k+1} = \epsilon_k + (x_{k+1} - x_k)$$

$$= \epsilon_k - \frac{f(x_k)}{f'(x_k)}$$

$$\epsilon_{k+1} = \epsilon_k - \frac{f'(x^*)\epsilon_k + \frac{1}{2}f''(x^*)\epsilon_k^2}{f'(x^*) + f''(x^*)\epsilon_k}$$

for small ϵ_k

$$\epsilon_{k+1} \approx \epsilon_k\left(1 - \frac{f'(x^*) + \frac{1}{2}f''(x^*)\epsilon_k}{f'(x^*) + f''(x^*)\epsilon_k}\right)$$

$$\epsilon_{k+1} = \epsilon_k\left(1 - \frac{f'(x^*)}{f'(x^*)} + \frac{1}{2}\frac{f''(x^*)\epsilon_k}{f'(x^*)} + \cdots\right)$$

$$\epsilon_{k+1} \approx \frac{1}{2}\frac{f''(x^*)}{f'(x^*)}\epsilon_k^2$$

Quadratic convergence with asymp err const

$$= \frac{1}{2}\frac{f''(x^*)}{f'(x^*)}$$

$\boxed{\text{Q: What if } f'(x^*) = 0\,?}$

Do EX of $f = x^2$ with Newton

$x_{i+1} = x_i/2 \Rightarrow$ linear convergence

since $f'(x^*) = 0$.