of LECTURE 4

SYSTEMS, LINEAR EQNS

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right\} \equiv \underline{A}\,\underline{x} = \underline{b}$$

For now assume $\underline{A} \in \mathbb{R}^{n \times n} \leftarrow \underline{A}$ is square.

~~SENSITIVITY AND CONDITIONING~~ $\underline{A}$ has full rank.

~~VECTOR NORMS~~ VECTOR NORMS

p-norm of $\underline{x}$ with entries x_i

$$\|\underline{x}\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$$

1-norm $\|\underline{x}\|_1 = \sum_{i=1}^{n} |x_i|$

2-norm $\|\underline{x}_2\| =$

∞-norm $\|x_\infty\| = \max_{1 \leq i \leq n} (x_i)$

EX: $\underline{x} = \begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$ $\|\underline{x}\|_1 =$

$\|\underline{x}\|_2 =$

$\|\underline{x}\|_\infty =$

Properties:

~~Note~~. 1) $\|x_1\| \geq \|x_2\| \geq \|x_\infty\|$

2) $\|\underline{x}\| > 0$ if $\underline{x} \neq \underline{0}$

3) $\|r\underline{x}\| = |r| \cdot \|\underline{x}\| \quad \forall r \in \mathbb{R}$

4) $\|\underline{x}\underline{y}\| = \|\underline{x}\| \cdot \|\underline{y}\|$

MATRIX NORMS

$$\|\underline{\underline{A}}\|_p = \max_{\underline{x} \neq 0} \frac{\|\underline{\underline{A}}\underline{x}\|_p}{\|\underline{x}\|_p}$$

Usually use

$$\|\underline{\underline{A}}_1\| = \max_j \sum_{i=1}^{m} |a_{ij}|$$

$$\|\underline{\underline{A}}_\infty\| = \max_i \sum_{j=1}^{m} |a_{ij}|$$

EX
$$\underline{\underline{A}} = \begin{pmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 3 & -1 & 4 \end{pmatrix}$$

$\|\underline{\underline{A}}\|_1 = 6$

$\|\underline{\underline{A}}\|_\infty = 8$

Properties 1. $\|\underline{\underline{A}}\| > 0$ if $\underline{\underline{A}} \neq 0$

2. $\|r\underline{\underline{A}}\| = |r| \cdot \|\underline{\underline{A}}\|$

3. $\|\underline{\underline{A}} + \underline{\underline{B}}\| \leq \|A\| + \|B\|$

4. $\| \underline{\underline{A}}\, \underline{\underline{B}} \| \leq \| \underline{\underline{A}} \| \| \underline{\underline{B}} \|$

5. $\| \underline{\underline{A}}\, \underline{x} \| \leq \| \underline{\underline{A}} \| \| \underline{x} \|$

CONDITION NUMBER : $C_{\underline{\underline{A}}} = \| \underline{\underline{A}} \| \cdot \| \underline{\underline{A}} \|^{-1}$

$C_{\underline{\underline{A}}} = \infty$ iff $\underline{\underline{A}}$ is singular

i.e. $\det(\underline{\underline{A}}) = 0$.

TRIANGULAR SYSTEMS

An upper triangular matrix ~~is~~ ~~each~~ has the form

$$\underline{\underline{A}} = \begin{pmatrix} x & x & x & x & \cdots \\ 0 & x & x & x & \cdots \\ 0 & 0 & x & x & \cdots \\ 0 & 0 & 0 & x & \ddots \\ 0 & 0 & 0 & 0 & \cdots \end{pmatrix}$$

i.e. a_{ij} with $a_{ij} \neq$ $j \geq i$

$a_{ij} = 0$ $j < i$

Normally indicate this property by denoting matrix $\underline{\underline{U}}$

$\underline{\underline{U}}\, \underline{x} = \underline{b}$ can be solved directly

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{33} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$$

Write down x_1 x_2 x_3 x_4.

In general $x_n = b_n / u_{nn}$

$$x_{n-1} = \frac{b_{n-1} - u_{n-1,n}\, x_n}{u_{n-1,n-1}}$$

$$x_{n-2} = \frac{b_{n-2} - (u_{n-2,n}\, x_n + u_{n-2,n-1}\, x_{n-1})}{u_{n-2,n-2}}$$

$$\vdots$$

$$x_i = \frac{b_i - \sum_{j=i+1}^{n} u_{ij}\, x_j}{u_{ii}} \qquad i = n-1, n-2, \ldots 2, 1.$$

Lower triangular denoted $\underline{\underline{L}}$

$$x_1 = b_1 / l_{11}$$

$$x_2 = \frac{b_2 - l_{21} x_1}{l_{22}}$$

$$x_3 = \frac{b_3 - (l_{31} x_1 + l_{32} x_2)}{l_{33}}$$

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} l_{ij} x_j}{l_{ii}} \qquad i = 1, 2, 3, \ldots n$$

GAUSS ELIMINATION AND LU FACTORISATION.

Want to solve $\underline{\underline{A}}\,\underline{x} = \underline{b}$

If we could rewrite as $(\underline{\underline{L}}\,\underline{\underline{U}})\,\underline{x} = \underline{b}$

Could solve by:

1. Call $\underline{\underline{U}}\,\underline{x} = \underline{y}$
2. Solve $\underline{\underline{L}}\,\underline{y} = \underline{b}$ for $\underline{y}$
3. Solve $\underline{\underline{U}}\,\underline{x} = \underline{y}$ for $\underline{x}$

How do we write $\underline{\underline{A}}$ as $\underline{\underline{L}}\,\underline{\underline{U}}$?

Apply a some of transformation to $\underline{\underline{A}}\,\underline{x} = \underline{b}$

$$\underline{\underline{M}}_1 \underline{\underline{A}}\,\underline{x} = \underline{\underline{M}}_1 \underline{b}$$

$$\underline{\underline{M}}_2 \underline{\underline{M}}_1 \underline{\underline{A}}\,\underline{x} = \underline{\underline{M}}_2 \underline{\underline{M}}_1 \underline{b}$$

$$\vdots$$

$$\underbrace{\underline{\underline{M}}_{n-1} \cdots \underline{\underline{M}}_2 \underline{\underline{M}}_1 \underline{\underline{A}}}_{\underline{\underline{U}}}\,\underline{x} = \underbrace{\underline{\underline{M}}_{n-1} \cdots \underline{\underline{M}}_2 \underline{\underline{M}}_1}_{\underline{\underline{L}}^{-1}} \underline{b}$$

$$\Rightarrow \underline{\underline{L}}\,\underline{\underline{U}}\,\underline{x} = \underline{b}$$