~~LECTURE 7~~ EIGENVALUE PROBS

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \quad \text{—— (1)} \qquad \underline{\underline{A}} \text{ - square.}$$

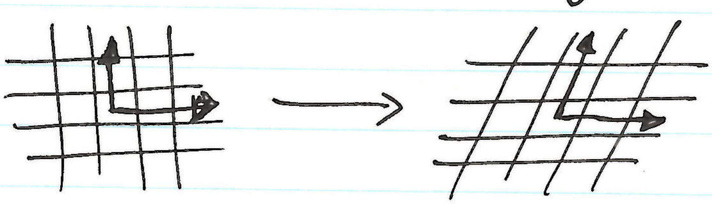
(1) is an eigenvalue problem if there are nonzero solutions, $\underline{x}$, for specific values of λ.

1) ~~The set of~~ Vals. of λ - eigenvalues
2) Corresponding $\underline{x}$ - eigenvectors.

3) The set of values of λ - the spectrum of $\underline{\underline{A}}$
4) $\rho(\underline{\underline{A}}) = \max\{|\lambda| : \lambda \in \lambda(\underline{\underline{A}})\}$ - spectral radius of $\underline{\underline{A}}$

Geometrical interpretation:

If $\underline{\underline{A}}$ is square, it can be thought of as a linear transformation. e.g., the shear trans $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$



$\underline{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\lambda = 1$

An eigenvector is a vector, $\underline{x}$, whose direction is unaffected by the transformation.
The change in its length under transformation is the associated eigenvalue.

An engineering/physical example: Sturm-Liouville problem.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = XT, \quad XT' = X''T$$

$$u\big|_{x=0,1} = 0$$

$$\Rightarrow \frac{X''}{X} = \frac{T'}{T} = -\lambda^2$$

$$\Rightarrow \boxed{X'' = -\lambda^2 X}, \quad T' = -\lambda^2 T$$

with $\boxed{X(0) = X(1) = 0}$

$X = 0$

$X = A\sin(\lambda x) + \cancel{B}^{\,0}\cos(\lambda x)$

$\lambda = n\pi$

$X = A\sin(\lambda x)$ if $\lambda = n\pi$ $\Leftarrow$ Only discrete values of λ have a non-trivial soln.

$X = 0$ otherwise

Some propertoes of eigs:

1) $\sum_i \lambda_i = \text{trace}(\underline{\underline{A}})$

2) $\prod_i \lambda_i = \det(\underline{\underline{A}})$

3) num of eigs $= n$.

4) If $\underline{\underline{A}}$ is triangular, the eigenvalues of $\underline{\underline{A}}$ are its diagonal entries.

~~EX/ Find eigenvals and vectors of~~

$$\cancel{\underline{\underline{A}} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \Rightarrow (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{x} = \underline{0}}$$

$$\cancel{\Rightarrow \det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0}$$

$$\cancel{\Rightarrow (3-\lambda)^2 - 1 = 0}$$

$$\cancel{\Rightarrow \lambda = 3 \pm 1 = 2, 4}$$

$$\cancel{\lambda = 2 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underline{0}}$$

Diagonalisation

If $\underline{\underline{A}}$ has n distinct eigenvalues it is diagonalisable. i.e. there is a diagonal matrix, $\underline{\underline{D}}$, that is similar to $\underline{\underline{A}}$, s.t.,

$$\underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1} \quad \text{for some } \underline{\underline{P}}.$$

Then $\underline{\underline{D}}$'s diagonal entries are the eigs of $\underline{\underline{A}}$. $\underline{\underline{D}} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$
and $\underline{\underline{P}}$'s columns are eigen vecs of $\underline{\underline{A}}$

Solution using characteristic polynomial:

$$\underline{\underline{A}}\,\underline{x} = \lambda \underline{x} \implies \left(\underline{\underline{A}} - \underline{\underline{I}}\lambda\right)\underline{x} = 0$$

Has soln. only if $\underline{\underline{A}} - \lambda \underline{\underline{I}}$ is singular. $\iff$ det $= 0$

$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 \longleftarrow$ polynomial of degree n in λ. THE CHARACTERISTIC POLY. Its roots are eigenvals of $\underline{\underline{A}}$.

<u>EX</u> $\underline{\underline{A}} = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad \underline{\underline{A}} - \lambda \underline{\underline{I}} = \begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = (3-\lambda)^2 - 1 = 0 \implies (3-\lambda)^2 = 1$$

$$\implies 3 - \lambda = \pm 1$$

$$\implies \lambda = 3 \mp 1$$

$$\implies \lambda = 2, 4.$$

~~NOTE: $\sum \lambda_i = \text{trace}\, A$~~

~~NOTE ALSO: num of eigs $= n$~~

Issues with this approach:

1) For large n have to solve poly of deg. n $\leftarrow$ hard or at least intensive.
2) Co.effs of CP highly sensitive to perturbations in $\underline{A}$ (can result from roundoff).

Computational approaches:

General strategy is to transform the matrix into a simpler one whose eig vals and vecs can be found more easily: Some approaches include:

1. Powers
2. Similarity

(Von Mises iteration)

1. Powers If $\underline{\underline{A}}\,\underline{x} = \lambda\,\underline{x} \Rightarrow \underline{\underline{A}}^k\,\underline{x} = \lambda^k\,\underline{x}$

Suppose $\underline{\underline{A}}$ is diagonalisable with eigs.

$$|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$$

then $\underline{\underline{A}} = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1} \Longrightarrow \underline{\underline{A}}^k = \underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1}\cdot\underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1}\cdots\underline{\underline{P}}\,\underline{\underline{D}}\,\underline{\underline{P}}^{-1}$

$$\Longrightarrow \underline{\underline{A}}^k = \underline{\underline{P}}\,\underline{\underline{D}}^k\,\underline{\underline{P}}^{-1}$$

$$\Longrightarrow \underline{\underline{A}}^k\,\underline{\underline{P}} = \underline{\underline{P}}\,\underline{\underline{D}}^k.$$

Now choose any vector $\underline{x} = \underline{\underline{P}}\,\underline{\tilde{x}}$ and consider

$$\underline{\underline{A}}^k\,\underline{x} = \underline{\underline{A}}^k\,\underline{\underline{P}}\,\underline{\tilde{x}} = \underline{\underline{P}}\,\underline{\underline{D}}^k\,\underline{\tilde{x}} = \sum_{j=1}^{n} v_j\,\lambda_j^k\,\tilde{x}_j$$

$$\Longrightarrow \underline{\underline{A}}^k\,\underline{x} = \lambda_1^k\left(\sum_{j=1}^{n} v_j\left(\frac{\lambda_j}{\lambda_1}\right)^k\tilde{x}_j\right)$$

As $k \nearrow \infty$ $\left(\frac{\lambda_j}{\lambda_1}\right)^k \searrow 0$ for all $j > 1$

Thus as $k \nearrow \infty$

$$\underline{\underline{A}}^k \underline{x} \to M \underline{v}_1 \quad \text{for some scalar } M$$

This motivates an iteration to locate 'largest' eigenvector of $\underline{\underline{A}}$. Namely

$$\underline{x}^{k+1} = \frac{\underline{\underline{A}}\,\underline{x}^k}{\|\underline{\underline{A}}\,\underline{x}^k\|} \leftarrow \text{POWER ITERATION}$$

Issues

1. If $\frac{\lambda_1}{\lambda_2} \approx 1$, then we converge to an eigenvector comprised of a linear combination of v_1, v_2.
2. We only get 1 eigvec
3. We get no eig. vals.
4. Convergence is reasonably slow (linear).

- If we want smallest eigvector of $\underline{\underline{A}}$ use the fact that $\underline{\underline{A}}^{-1}$ has inverse eigvals of $\underline{\underline{A}}$.

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Choose an x_0
for k = 1:N
    Solve A y_k = x_{k-1}
    x_k = y_k / |y_k|
end
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$\leftarrow$ INVERSE POWER ITERATION.

- We can use a shift to get at the other, intermediate, eigenvecs. Write $\underline{\underline{B}} = \underline{\underline{A}} - \sigma \underline{\underline{I}}$.

If we use an inverse power iteration on $\underline{\underline{B}}$ we find the eigenvector of $\underline{\underline{A}}$ with eigenvalue closest to σ.

Can define SHIFTED INVERSE ITERATION.

- In principle, we can find all eigenvecs of $\underline{\underline{A}}$ using this technique.

- Can find corresponding eigvals by computing the Rayleigh-Quotient.

$$\lambda = \frac{\underline{x}^T \underline{\underline{A}}\, \underline{x}}{\underline{x}^T \underline{x}}$$

In fact the RQ can help us increase the rate of converge. in

Choose an x_0

for $i = 1:N$ ← RQ ITERATION.

$\sigma_k = \underline{x}_{k-1}^T \underline{\underline{A}}\, \underline{x}_{k-1} \,/\, (\underline{x}_{k-1}^T\, \underline{x}_{k-1})$

Solve $(\underline{\underline{A}} - \sigma_k \underline{\underline{I}})\, y_k = x_{k-1}$ for y_k

$\underline{x}_k = \dfrac{y_k}{|y_k|}$;

end

RQ exhibits quadratic (if $\underline{A}$ symmetric cubic) convergence.

~~Did the other kid wanna do SVD? If so, why is he interested in this. I.e. how deep do we need to go to make it useful for him? (Wiki has a decent exp and example).~~ *

~~- If he didn't want SVD, go on to non-linear systems.~~

~~- Then ODEs. There are some notes on this in Dbox. Also cover shooting prob.~~

* ~~As does MIT. Google 'singular value decomposition of a matrix'. 4th hit.~~