BASIC HILBERT SPACE THEORY

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1. Basic definitions / results

The goal of these notes is to define the Fourier series in an arbitrary Hilbert space H, and to show that the Fourier series of an element in H converges to that element.

Recall from linear algebra the following:

1.1. **Definition.** An inner product on a complex vector space V is a function $\langle \cdot, \cdot \rangle : V^2 \to \mathbb{C}$ satisfying

(1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$
(2) $\langle \cdot, \cdot \rangle$ is linear in the first entry,
(3) $\langle \cdot, \cdot \rangle$ is positive definite.

1.2. **Lemma.** *Any inner product induces a norm,* $\|x\| = \sqrt{\langle x, x \rangle}$*, and satisfies:*

(1) $|\langle x, y \rangle| \leq \|x\| \, \|y\|$,
(2) $\|y\| \leq \|\lambda x + y\|$ *for every* $\lambda \in \mathbb{C}$ *iff* $\langle x, y \rangle = 0$,
(3) $\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$.

The proof is skipped for time.

1.3. **Lemma.** *The norm function* $\|\cdot\| : V \to \mathbb{R}$ *and the inner product* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ *are continuous functions.*

Proof. The norm is continuous by the triangle inequality,

$$| \, \|x\| - \|y\| \, | \leq \|x - y\| .$$

The inner product is continuous since

$$|\langle x_0, y_0 \rangle - \langle x, y \rangle| = |\langle x_0 - x, y_0 \rangle + \langle x, y_0 - y \rangle| \leq \|x_0 - x\| \, \|y_0\| + \|x_0\| \, \|y - y_0\|$$

by the triangle and Cuachy-Schwarz inequality. Then if $\|x - x_0\| < \varepsilon/(2\|y_0\|)$ and $\|y - y_0\| \leq \varepsilon/(2\|x_0\|)$, then $|\langle x_0, y_0 \rangle - \langle x, y \rangle| < \varepsilon$ which shows that $\langle \cdot, \cdot \rangle$ is continuous. $\square$

Also probably skip this for time.

1.4. **Definition.** A Hilbert space is an inner product space, $(H, \langle\cdot,\cdot\rangle)$, which is complete with respect to the norm induced by the inner product. i.e. It is a Banach space whose norm comes from an inner product.

In proposition 8 from the L^p spaces section, we saw that L^p is a complete metric space, and is hence a Banach space. A natural question is when is L^p a Hilbert space?

1.5. *Example.* If $p = 2$, then $\langle f, g\rangle = \int_X f\overline{g}d\mu$ is an inner product. All of the properties are easily seen to be satisfied, and $\|f\|_2^2 = \int_X |f|^2 d\mu = \int_X f\overline{f}d\mu = \langle f, f\rangle$

However, consider the space $L^p(X,\mu)$ where $p \neq 2$. If X contains two disjoint subsets with finite measure, then $L^p(X,\mu)$ is not a Hilbert space. To see this, we normalize the indicator functions of the two sets and show that the parallelogram law fails.

Let A, B be the two sets, and let $f = 1/(\mu(A))^{1/p}\chi_A$, $g = 1/(\mu(B))^{1/p}\chi_B$. Then,

$$\|f+g\|^2 = \left\{\int_X \left|\frac{1}{\mu(A)^{1/p}}\chi_A + \frac{1}{\mu(B)^{1/p}}\chi_B\right|^p\right\}^{2/p}.$$

$f + g$ is 0 outside of $A \cup B$, and since $A \cap B = \emptyset$ we calculate the integral to be

$$\int_X \left|\frac{1}{\mu(A)^{1/p}}\chi_A\right|^p + \int_X \left|\frac{1}{\mu(B)^{1/p}}\chi_B\right|^p = \frac{1}{\mu(A)}\mu(A) + \frac{1}{\mu(B)}\mu(B) = 2,$$

which gives $\|f+g\|^2 = 2^{2/p}$. Similarly we get that $\|f-g\|^2 = 2^{2/p}$.

However, $\|f\|^2 = \|g\|^2 = 1$, so $2\|f\|^2 + 2\|g\|^2 = 4 \neq 2 \cdot 2^{2/p}$. Hence the parallelogram law fails, so $\|\cdot\|_p$ cannot come form an inner product.

If $p = \infty$, then $f = \chi_A$ and $g = \chi_B$ contradicts the parallelogram law since $\|f\|_\infty = \|g\|_\infty = \|f+g\|_\infty = \|f-g\|_\infty = 1$.

1.6. **Proposition.** *Let H be a Hilbert space, and $A \subset H$ be a non-empty closed convex subset. Then A contains a unique element of minimal norm.*

Proof. Let $d = \inf_{y\in E}\|y\|$. Since E is non empty and the norm is non-negative, we know that d is some finite number. We can find a sequence of points $x_n \in E$ with $\|x_n\|$ converging to d (if we could not, then d would not be the infimum).

Since E is convex, $(x_m + x_n)/2 \in E$, and so we have $\|(x_m + x_n)/2\| \geq d$. Then using the parallelogram law, we have

$$\|(x_m - x_n)/2\|^2 = (\|x_m\|^2 + \|x_n\|^2)/2 - \|(x_m + x_n)/2\|^2 \leq (\|x_m\|^2 + \|x_n\|^2)/2 - d^2.$$

Then as $n, m \to \infty$, we have

$$\|(x_m - x_n)/2\| \to (d^2 + d^2)/2 - d^2 = 0,$$

showing that x_n is a Cauchy sequence. Since E is closed and H is complete, we have $x_n \to x \in E$.

Since $\|\cdot\|$ is continuous, $\|x\| = d$. If there were some other point $x' \in E$ with $\|x'\| = d$, then again using the parallelogram law and the fact that $(x + x')/2 \in E$, we have

$$\|(x - x')/2\|^2 = (\|x\|^2 + \|x'\|^2)/2 - \|(x + x')/2\|^2 \leq (d^2 + d^2)/2 - d^2 = 0,$$

showing that $x = x'$. □

We define the *orthogonal compliment* of a subspace $M \subset H$ to be

$$M^\perp = \{x \in H : \langle x, y\rangle = 0 \text{ for every } y \in M\}.$$

If M is closed, then it is itself a Hilbert space, and so is $M^\perp$. The fact that $M^\perp$ is a subspace follows easily from the linearity of the inner product. The fact that $M^\perp$ is closed follows from the continuity of the inner product.

If H_1 and H_2 are Hilbert spaces with $H_1 \cap H_2 = \{0\}$, then we define the direct sum to be

$$H_1 \oplus H_2 = \{h_1 + h_2 : h_1 \in H_1, h_2 \in H_2\}.$$

1.7. **Proposition.** *Let H be a Hilbert space and $M \subset H$ a closed subspace. Then M is a direct summand of H.*

Proof. Since M and $M^\perp$ are Hilbert spaces, we show that $H = M \oplus M^\perp$.

$M \cap M^\perp = \{0\}$ since if $x \in M \cap M^\perp$, then $\langle x, x\rangle = 0$ so $x = 0$.

For any $y \in H$, let $E = y - M$. Then E is a convex closed subset of H. Indeed, if $y - m_1$ and $y - m_2$ are in E, then the line between these two points is

$$t(m_2 - m_1) + (y - m_2) = y - (m_2 - t(m_2 - m_1)) \in E,$$

since $m_2 - t(m_2 - m_1) \in M$, so E is convex. Moreover its closed since it is the translate of a closed set. Then by proposition (1.6) there is a unique element with minimum norm. Let that element be $y - m$. Then for any $x \in M$, and $\lambda \in \mathbb{C}$ we have

$$\|y - m\| \leq \|y - m + \lambda x\|.$$

Then by Lemma (1.2), we have $\langle y - m, x\rangle = 0$, so $y - m \in M^\perp$. Moreover $y = m + (y - m)$ showing that $H = M \oplus M^\perp$. □

2. Orthonormal Bases

If M is a closed subspace of a Hilbert space H, then for $x \in H$, we can define the orthogonal projection onto M as $P_M(x) = m$, where $x = m + m' \in M \oplus M^{\perp}$. This is well defined since there is a unique such representation for x.

Just like in finite dimensional linear algebra, we use the inner products and orthogonal projections to define the best approximation to a vector x in a finite dimensional subspace. This is done by projecting x onto an orthonormal basis for the subspace.

A subset $U \subset H$ is called *orthonormal* if $\langle u_\alpha, u_\beta \rangle = \delta_\alpha^\beta$ for all $u_\alpha, u_\beta \in A$.

We define the *Fourier coefficients* of x with respect to U as $x_\alpha = \langle x, u_\alpha \rangle$.

2.1. **Theorem.** *Let $U = \{u_\alpha : \alpha \in A\}$ be an orthonormal set in a Hilbert space H and let $\{\alpha_1, \dots, \alpha_n\}$ be a finite subset of A. Then,*

(1) *If $x = \sum_{i=1}^n c_i u_{\alpha_i}$, then $c_i = x_{\alpha_i}$ and $\|x\|^2 = \sum_{i=1} |x_{\alpha_i}|^2$.*

(2) *For any $x \in H$ and any scalars λ_i, we have*

$$\left\| x - \sum_{i=1}^n x_{\alpha_i} u_{\alpha_i} \right\| \leq \left\| x - \sum_{i=1}^n \lambda_i u_{\alpha_i} \right\|$$

with equality if and only if $\lambda_i = x_{\alpha_i}$.

(3) *The vector $\sum_{i=1}^n x_{\alpha_i} u_{\alpha_i}$ is the orthogonal projection of x onto the subspace spanned by $\{u_{\alpha_i}\}$, $i = 1, \dots, n$.*

Proof. (1) This follows from orthonormality. $x_{\alpha_i} = \langle x, u_{\alpha_i} \rangle = c_i$.

$$\|x\|^2 = \langle \sum x_{\alpha_i u_{\alpha_i}}, \sum x_{\alpha_i u_{\alpha_i}} \rangle = \sum_i \sum_j x_{\alpha_i} \overline{x_{\alpha_j}} \langle u_{\alpha_i}, u_{\alpha_j} \rangle = \sum |x_{\alpha_i}|^2.$$

(2) Squaring and expanding the norm, we get

$$\|x\|^2 - \sum |x_{\alpha_i}|^2,$$

for the left hand side and

$$\|x\|^2 - 2 \operatorname{Re} \sum_{i=1}^n x_{\alpha_i} \overline{\lambda_i} + \sum_{i=1}^n |\lambda_i|^2.$$

for the right hand side. This is equivalent to

$$2 \operatorname{Re} \sum_{i=1}^n x_{\alpha_i} \overline{\lambda_i} \leq \sum_{i=1}^n |x_{\alpha_i}|^2 + \sum_{i=1}^n |\lambda_i|^2.$$

This follows from the Cauchy-Schwarz inequality,

$$|\sum_{i=1}^{n} x_{\alpha_i}\overline{\lambda_i}| \leq \sqrt{\sum_{i=1}^{n}|x_{\alpha_i}|^2}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2},$$

and the AGM inequality,

$$\sqrt{\sum_{i=1}^{n}|x_{\alpha_i}|^2}\sqrt{\sum_{i=1}^{n}|\lambda_i|^2} \leq \frac{\sum_{i=1}^{n}|x_{\alpha_i}|^2 + \sum_{i=1}^{n}|\lambda_i|^2}{2}$$

Since we have

$$\text{Re}\sum_{i=1}^{n} x_{\alpha_i}\overline{\lambda_i} \leq |\sum_{i=1}^{n} x_{\alpha_i}\overline{\lambda_i}| \leq \frac{\sum_{i=1}^{n}|x_{\alpha_i}|^2 + \sum_{i=1}^{n}|\lambda_i|^2}{2}.$$

(3) This follows from the definition of the orthogonal projection. We constructed the decomposition as $x = m + x - m$ where $x - m$ was the element of minimal norm in $x - M$, and then defined the projection to be m. (2) shows that the element of minimal norm in $x - M$ is $x - \sum_{i=1}^{n} x_{\alpha_i} u_{\alpha_i}$.

□

Let $l^2(A)$ be the Hilbert space of square summable sequences with $|A|$ terms in the sequences. That is, $l^2(A) = \{\phi \colon A \to \mathbb{C} : \sum_{\alpha\in A}|\phi(\alpha)|^2 < \infty\}$ with inner product, $\langle \phi, \psi\rangle = \sum_{\alpha\in A}\phi(\alpha)\overline{\psi(\alpha)}$.

The next theorem shows that every such ϕ arises as the Fourier coefficients from an element of a Hilbert space, if that Hilbert space has an orthonormal set of cardinality $|A|$.

2.2. **Theorem.** *If $U = \{u_\alpha : \alpha \in A\}$ is an orthonormal set in a Hilbert space H, and $\phi \in l^2(A)$, then there is an $x \in H$ such that ϕ is equal to the function $\hat{x} \colon A \to \mathbb{C}$, $\hat{x}(\alpha) = \langle x, u_\alpha\rangle$.*

Proof. Since ϕ is square summable, at most countable many terms in ϕ can be non-zero. Indeed, if we let $A_n = \{\alpha : |\phi(\alpha)|^2 > 1/n\}$, then we see

$$\sum_{\alpha\in A}|\phi(\alpha)|^2 \geq \sum_{\alpha\in A_n}|\phi(\alpha)|^2 \geq \sum_{\alpha\in A_n} 1/n.$$

Since the left hand side of this inequality is finite, we must have that A_n is finite. Then

$$\bigcup_{n\in\mathbb{N}} A_n = \{\alpha \in A : |\phi(\alpha)|^2 > 0\}$$

is a countable union of finite sets and hence is countable.

Let $E = \{\alpha_n\}$ a countable set for which $\phi = 0$ on $A \setminus E$. Then define

$$x_n = \sum_{i=1}^{n}\phi(\alpha_i)u_{\alpha_i}.$$

Notice that x_n is Cauchy in H. Indeed, if $n > m$, then

$$\|x_n - x_m\| = \sum_{i=m+1}^{n} |\phi(\alpha_i)|^2 \to 0,$$

since ϕ is square summable.

Hence $x_n \to x$ for some $x \in H$, $x = \sum_{n=1}^{\infty} \phi(\alpha_n) u_{\alpha_n}$, and

$$\hat{x}(\alpha) = \langle \sum_{n=1}^{\infty} \phi(\alpha_n) u_{\alpha_n}, u_\alpha \rangle = \sum \phi(\alpha_n) \langle u_{\alpha_n}, u_\alpha \rangle = \phi(\alpha),$$

since if $\alpha \in \{\alpha_n\}$, the sum collapses to $\phi(\alpha)$. Otherwise it collapses to 0 and $\phi(\alpha) = 0$ since $\alpha \notin \{\alpha_n\}$. □

We would like to be able to approximate the elements of a Hilbert space by their Fourier expansions, with the limit equaling the element. The next Theorem and its Corollary give equivalent conditions for this to happen.

2.3. **Theorem.** *Let $U = \{u_\alpha : \alpha \in A\}$ be an orthonormal set in a Hilbert space H. The following are equivalent*

(1) $\|x\|_H = \left\{\sum_{\alpha \in A} x_\alpha^2\right\}^{1/2} = \|\hat{x}\|_{l^2(A)}$,

(2) *The linear map $\Lambda \colon H \to l^2(A)$, $\Lambda(x) = \hat{x}$ is a Hilbert space isomorphism,*

(3) *U is a maximal orthonormal set in H,*

(4) *The linear span of U is dense in H,*

Proof. Suppose that 1 holds. The polarization identity shows that the inner product is preserved.

$$\begin{aligned} \langle x, y \rangle_H &= 1/4 \left\{ \|x+y\|_H^2 - \|x-y\|_H^2 + i \|x+iy\|_H^2 - i \|x-iy\|_H^2 \right\} \\ &= 1/4 \left\{ \|\hat{x}+\hat{y}\|_{l^2(A)}^2 - \|\hat{x}-\hat{y}\|_{l^2(A)}^2 + i \left\|\hat{+}i\hat{y}\right\|_{l^2(A)}^2 - i \|\hat{x}-i\hat{y}\|_{l^2(A)}^2 \right\} \\ &= \langle \hat{x}, \hat{y} \rangle_{l^2(A)} \end{aligned}$$

Λ is onto by Theorem (2.2). It is injective since if $\hat{x} = 0$, then $\|x\| = \|\hat{x}\| = 0$, so $x = 0$.

Suppose 2. If U is not maximal, then there must be some $x \neq 0$, $x \notin U$ with $\langle x, u_\alpha \rangle = 0$ for every α. Then $\hat{x} = 0$, contradicting the fact that Λ is an isomorphism.

Suppose (3). If span(U) is not dense, then $\overline{\text{span}(U)} \neq H$, so there is some $x \in \overline{\text{span(U)}}^{\perp}$ with $x \neq 0$. Then $\langle x, u_\alpha \rangle = 0$ for every α which contradicts the maximality of U.

Suppose (4). Then $\Lambda\colon \operatorname{span}(U) \to l^2(A)$, $x \mapsto \hat{x}$ is continuous. Indeed, if $x = \sum_{\alpha\in F} c_\alpha u_\alpha$, then

$$\hat{x}(\alpha) = \begin{cases} c_\alpha & \alpha \in F \\ 0 & \text{else} \end{cases}.$$

So we see that $\|x\|_H = \|\hat{x}\|_{l^2(A)}$, and this gives continuity by the triangle inequality.

Next we extend Λ to H by taking limits. If $x \in H$, let $x_n \to x$ with $x_n \in \operatorname{span}(U)$, then we define $\Lambda(x) = \lim_{n\to\infty} \Lambda(x_n)$. $\Lambda(x)$ exists because $\Lambda(x_n)$ is a Cauchy sequence in $l^2(A)$. It is well defined since Λ is an isometry. Explicitly, if $x_n, y_n \to x$, then $\|\hat{x}_n - \hat{y}_n\| = \|x_n - y_n\| \to 0$, so $\hat{x}_n$ and $\hat{y}_n$ converge to the same point in $l^2(A)$.

Moreover, Λ is an isometry since $\|x\| = \lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|\hat{x}_n\| = \|\hat{x}\|$. This shows (1). □

This shows that there are at most countably many Fourier coefficients which are non-zero for any given $x \in H$ and orthonormal basis U.

2.4. **Corollary.** *If U is an orthonormal basis of H and $x \in H$, then there are at most countably many α with $x_\alpha \neq 0$. Moreover*

$$x = \sum_{\alpha\in F} x_\alpha u_\alpha,$$

where F is the set of α for which $x_\alpha \neq 0$, $F = \{\alpha_n\}_{n=1}^N$ where N is possibly infinite.

Proof. The at most countable part follows from $\|\hat{x}\| = \|x\| < \infty$, and the fact that if an uncountable sum converges then at most countably many terms can be non-zero (we showed this already).

If F is finite then the sum is a finite sum so it is an element in H. Suppose it is infinite. Then $x_n = \sum_{i=1}^n x_{\alpha_i} u_{\alpha_i}$ is a Cauchy sequence since

$$\|x_n - x_m\| = \left\| \sum_{i=m+1}^n x_{\alpha_i} u_{\alpha_i} \right\| = \sum_{i=m+1}^n |x_{\alpha_i}^2| \to 0,$$

since $\sum |x_{\alpha_i}|^2$ converges. Therefore, $\sum_{i=1}^\infty x_{\alpha_i} u_{\alpha_i}$ is in H.

Let $y = x - \sum_{i=1}^\infty x_{\alpha_i} u_{\alpha_i}$. Then for any α,

$$\hat{y}(\alpha) = \langle x - \sum_{i=1}^\infty x_{\alpha_i} u_{\alpha_i}, u_\alpha \rangle = \langle x, u_\alpha \rangle - \sum_{i=1}^\infty x_{\alpha_i} \langle u_{\alpha_i}, u_\alpha \rangle = x_\alpha - x_\alpha,$$

since

$$\sum_{i=1}^\infty x_{\alpha_i} \langle u_{\alpha_i}, u_\alpha \rangle = x_\alpha$$

if $\alpha = \alpha_j$ for some j. Otherwise it is 0, but so is x_α.

Since U is a maximal orthonormal set, this shows that $y = 0$, and hence $x = \sum_{i=1}^{\infty} x_{\alpha_i} u_{\alpha_i}$.

□

2.5. *Example.* Add the standard example of $L^2(\mathbb{T})$.

3. Bounded Linear Operators

Let $L\colon X \to Y$ be a linear operator between normed vector spaces. L is said to be bounded if there exists a constant C such that

$$\|Lx\|_Y \leq C,$$

for every $x \in X$ with $\|x\|_X = 1$.

3.1. **Lemma.** *Let $L\colon X \to Y$ be a linear map between normed vector spaces. Then the following are equivalent.*

(1) *L is bounded,*

(2) *L is continuous,*

(3) *L is continuous at the origin.*

Proof. Suppose L is bounded with constant C. If $0 < \|x - y\|_X < \varepsilon/C$, then

$$x - y/(\|x - y\|_X)$$

has unit norm, so

$$\|L(x - y)\|_Y / \|x - y\|_X \leq C,$$

so

$$\|Lx - Ly\|_Y \leq C \|x - y\|_X = \varepsilon.$$

Clearly (2) implies (3).

If L is continuous at 0, then there is some $\delta > 0$ such that whenever $\|x\|_X < \delta$ we have $\|Lx\|_Y < 1$. Then for any x with $\|x\|_X = 1$, we have $\|(\delta/2)x\|_X = \delta/2 < \delta$, so

$$\|L(\delta/2)x\|_Y < 1,$$

and hence

$$\|Lx\|_Y < 2/\delta$$

so L is bounded. □

Define X^* to be the vector space of continuous linear functionals on X. Since we have just shown continuity is equivalent to boundedness, we define

$$\|\Lambda\| = \sup_{\|x\| \leq 1} |\Lambda x|.$$

This turns X^* into a Banach space. For a Hilbert space H, we have already shown that $\Lambda_y(x) = \langle x, y \rangle$ is a continuous linear functional. It turns out that this is the only type of linear functional on H.

3.2. **Theorem.** *Let H be a Hilbert space and $\Lambda \in H^*$. There is a unique $y \in H$ such that $\Lambda = \Lambda_y$. Moreover, $\|\Lambda\|_{H^*} = \|y\|_H$, and the map $y \mapsto \Lambda_y$ is a conjugate linear isometry.*

Proof. We have already shown that $\Lambda_y \in H^*$. The fact that $\|\Lambda_y\|_{H^*} = \|y\|_H$ follows from the Cauchy-Schwarz inequality. For $\|x\| \leq 1$ we have

$$|\Lambda_y(x)| \leq \|x\| \, \|y\| \leq y.$$

Moreover this bound is attained with $x = y/\|y\|$.

since $\Lambda_{\lambda y} = \overline{\lambda}\Lambda_y$, the map is a conjugate linear isometry.

If $\Lambda = 0$ then the statement holds with $y = 0$. Otherwise let $N = \ker(\Lambda) = \Lambda^{-1}(0)$. This is a proper closed subspace of H, so there is some $z \in N^\perp$ with $z \neq 0$. Then $(\Lambda x)z - (\Lambda z)x \in N$ for any x. Hence we get

$$0 = \langle (\Lambda x)z - (\Lambda z)x, z\rangle = \Lambda x \|z\|^2 - \Lambda z \langle x, z\rangle,$$

which gives

$$\Lambda x = \lambda_z \langle x, z\rangle / \|z\|^2 = \Lambda_y x$$

with $y = \overline{\Lambda z}/\|z\|^2$.

To prove uniqueness, suppose that

$$\langle x, y\rangle = \langle x, y'\rangle$$

for every $x \in H$. Then

$$\langle x, y - y'\rangle = 0$$

for all $x \in H$. In particular for $x = y - y'$ which gives

$$\|y - y'\|^2 = 0$$

and hence $y = y'$. □