MATH2R3 - TUTORIAL 1

Problems:

(1) Let $S = \{s_1, \ldots, s_n\}$ be independent in V, and $x \notin \text{span}(S)$. Show that $S \cup \{x\}$ is independent.

Proof. Suppose that

$$ax + a_1 s_1 + \ldots + a_n s_n = 0.$$

First we will show that the coefficient a is 0. If it were not, then we would have

$$\begin{aligned} ax &= -(a_1 s_1 + \ldots + a_n s_n) \\ \implies x &= -\frac{1}{a}(a_1 s_1 + \ldots a_n s_n) \\ \implies x &= -\frac{a_1}{a} s_1 + \ldots + \frac{-a_n}{a} s_n, \end{aligned}$$

contradicting the fact that $x \notin \text{span}(S)$. So $a = 0$. Now we see that

$$0 = ax + a_1 s_1 + \ldots + a_n s_n = a_1 s_1 + \ldots + a_n s_n.$$

Since S is independent, and $a_1 s_1 + \ldots + a_n s_n = 0$, this shows that each coefficient $a_i = 0$. Hence we have shown that if a linear combination of the vectors in $S \cup \{x\}$ is zero, then all of the coefficients must be zero. This is the definition of independence.

***R*emark.** *Why can we not immediately conclude that $a_1, \ldots, a_n$ are all zero from the equation $ax + a_1 s_1 + \ldots + a_n s_n = 0$?*

□

(2) Let $F[0,1]$ be the vector space of functions $f\colon [0,1] \to \mathbb{R}$. Find an uncountable independent set in $F[0,1]$.

Proof. To construct an uncountable independent set, we need an uncountable set $S \subset F[0,1]$ for which every finite subset is linearly independent.

***R*emark.** *What does this mean in the space $F[0,1]$?*

Suppose that the set is indexed by $c \in [0,1]$. That is, $S = \{f_c \colon c \in [0,1]\}$, where each $f_c\colon [0,1] \to \mathbb{R}$. This means that for every finite subset of S, say $f_{c_1}, \ldots, f_{c_n}$, if

$$a_1 f_{c_1} + \ldots + a_n f_{c_n} = 0,$$

Then each coefficient $a_i = 0$. Remember that $F[0,1]$ is a space of functions, so 0 here is the zero function, i.e. $0\colon [0,1] \to \mathbb{R}$ defined by $0(x) = 0$. So when we write $a_1 f_{c_1} + \ldots + a_n f_{c_n} = 0$, what we mean is that for every $x \in [0,1]$, we have

$$a_1 f_{c_1}(x) + \ldots + a_n f_{c_n}(x) = 0(x) = 0.$$

Back to the proof:

For any $c \in [0,1]$, let f_c be the function defined by

$$f_c(x) = \begin{cases} 1 & \text{if } x = c \\ 0 & \text{if } x \neq c \end{cases},$$

and let

$$S = \{f_c\colon c \in [0,1]\}.$$

S is uncountable and independent. Indeed, suppose that $\{f_{c_1}, \ldots, f_{c_n}\}$ is a finite subset and

$$a_1 f_{c_1} + \ldots + a_n f_{c_n} = 0.$$

Then evaluating at $x = c_1$ shows that

$$a_1 f_{c_1}(c_1) + \ldots + a_n f_{c_n}(c_1) = 0(c_1) = 0.$$

By how these functions were defined, we have $f_{c_1}(c_1) = 1$, and for each $k \neq 1$, $f_{c_k}(c_1) = 0$. So we see that

$$a_1 f_{c_1}(c_1) + \ldots + a_n f_{c_n}(c_1) = a_1,$$

and hence

$$a_1 = 0.$$

We can apply the same argument to each function f_{c_k}. When we evaluate at c_k, we see that the coefficient $a_k = 0$.

This shows that all coefficients are 0, and so $\{f_{c_1}, \ldots, f_{c_n}\}$ is independent. Since this was an arbitrary finite subset, this shows that every finite subset is independent and hence S is independent.

Remark. *The vector space $F[0,1]$ is huge. Even though we have found an uncountable set of independent vectors, they don't even come close to spanning the whole space. Think about what the span of S is.*

When a vector space is infinite dimensional (we will see examples of important infinite dimensional spaces), the idea of a basis from finite dimensions is not useful. What we need to do is make sense of convergence of infinite sums in a vector space, so that we can try to come up with a countable set of vectors that we can use to represent each vector.

$\square$

(3) Let P be the vector space of all polynomials. For each $n \in \mathbb{N}$, show that there is an independent set of size n.

Proof. Let $n \in \mathbb{N}$ and let $S = \{1, x, x^2, \ldots, x^{n-1}\}$. Suppose that

$$a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} = 0.$$

Again, as in the previous question this means that the polynomial $f(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1}$ is constantly zero. If one of the coefficients $a_1, \ldots, a_{n-1}$ is non zero, then the polynomial $f(x)$ has some positive degree, and we know that a polynomial of degree k has at most k roots. Since f has infinitely many roots, each coefficient $a_1, \ldots, a_{n-1}$ is zero. So $f(x) = a_0$. Since $f(x)$ is constantly zero, this shows that $a_0 = f(x) = 0$, and hence every coefficient is zero. Thus $\{1, x, \ldots, x^{n-1}\}$ is an independent set of size n.

$\square$