MAT2R3 TUTORIAL 10

Review:

Recall from last time,

A square matrix A is called unitary if $A^* = A^{-1}$.

- A is unitary if and only if the columns / rows of A form an orthonormal basis of $\mathbb{C}^n$.
- The inverse of a unitary matrix is unitary, the product of unitary matrices are unitary, and the determinant of a unitary matrix has length 1.
- $\|Ax\| = \|x\|$ and $Ax \cdot Ay = x \cdot y$.

We call a matrix normal if $A^*A = AA^*$. Then we have the following equivalences

- A is unitarily diagonalizable,
- A has an orthonormal set of eigenvectors,
- A is normal.

*E*xample:

Let $A = \begin{pmatrix} 0 & 2i \\ -2i & 2 \end{pmatrix}$. The characteristic polynomial is $\lambda(\lambda - 2) - 4 = (\lambda - 1)^2 - 5$. So the eigenvalues are $1 \pm \sqrt{5}$.

For $\lambda = 1 + \sqrt{5}$, we have

$$\lambda I - A = \begin{pmatrix} 1+\sqrt{5} & -2i \\ 2i & 1+\sqrt{5}-2 \end{pmatrix} \to \begin{pmatrix} 1 & \frac{-2i}{\sqrt{5}+1} \\ 0 & 0 \end{pmatrix},$$

which gives $x = \frac{2i}{\sqrt{5}+1} y$, giving the eigenvector $\begin{pmatrix} \frac{2i}{\sqrt{5}+1} \\ 1 \end{pmatrix}$.

For $\lambda = 1 - \sqrt{5}$, we get the eigenvector $\begin{pmatrix} \frac{2i}{1-\sqrt{5}} \\ 1 \end{pmatrix}$.

Then we have

$$\begin{pmatrix} \frac{2i}{\sqrt{5}+1} \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{2i}{1-\sqrt{5}} \\ 1 \end{pmatrix} = \frac{4}{(1+\sqrt{5})(1-\sqrt{5})} + 1 = \frac{4}{-4} + 1 = 0.$$

We see that the eigenvectors are orthogonal, so we could normalize to find an orthonoraml set of eigenvectors. By the above equivalences, we conclude that A is unitarily diagonalizable, and that it is normal.

Problems:

Problem 1
From the example above, verify that A is unitarily diagonalizable and that it is normal.

Solution:

$A^* = \begin{pmatrix} 0 & 2i \\ -2i & 2 \end{pmatrix} = A$, so $A^*A = A^2 = AA^*$.

The first eigenvector has norm $\sqrt{\frac{10+2\sqrt{5}}{6+2\sqrt{5}}}$. So we take $v_1 = \frac{1}{\sqrt{\frac{10+2\sqrt{5}}{6+2\sqrt{5}}}} \begin{pmatrix} \frac{2i}{\sqrt{5}+1} \\ 1 \end{pmatrix}$. Similarly, $v_2 = \frac{1}{\sqrt{\frac{10-2\sqrt{5}}{6-2\sqrt{5}}}} \begin{pmatrix} \frac{2i}{1-\sqrt{5}} \\ 1 \end{pmatrix}$.

Then the matrix $P = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ is a unitary matrix (since the columns are orthonormal) which diagonalizes A.

Problem 2
Prove that if A is a unitary matrix, then so is A^*.

Solution:

Since A is unitary, $A^* = A^{-1}$. Then we have $A^*A = AA^* = I$, and $(A^*)^* = A$, so we see that $(A^*)^* = (A^*)^{-1}$, showing that A^* is unitary.

Problem 3
Prove that the eigenvalues of a unitary matrix have modulus 1.

Solution:

let λ be an eigenvalue of the unitary matrix A with associated eigenvector x. Then $\|Ax\| = \|x\|$, and $Ax = \lambda x$, so $\|\lambda x\| = \|x\|$, which implies $|\lambda|\,\|x\| = \|x\|$. Since x is an eigenvector, it is non zero, so $\|x\| \neq 0$. Then dividing by $\|x\|$ we get that $|\lambda| = 1$.

Problem 4
Let $A = \begin{pmatrix} 3 & -i \\ i & 3 \end{pmatrix}$, find a unitary matrix P that diagonalizes A

Solution:

The eigenvalues are $4, 2$ with eigenvectors $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ and $\begin{pmatrix} i \\ 1 \end{pmatrix}$. We see that the complex dot product is 0, so the eigenvectors are orthogonal. Then the norm of each eigenvector is $\sqrt{2}$ so we take

$$P = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}.$$