MAT2R3 TUTORIAL 11

Review:

(1) *quadratic forms*

A quadratic form is a linear combination of degree two terms in multiple variables. It is a sum of terms like $a_{i,j}x_ix_j$ where $a_{i,j}$ is a coefficient and x_i, x_j are variables. For example,

$$ax^2 + bxy + cy^2,$$

is a quadratic form in two variables. We can express this in terms of a matrix / vector equation, x^TAx. The diagonal entries of the matrix A will be the coefficients of the squared terms, and the off diagonal entries will be half the coefficient of the mixed terms.

$$A = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}.$$

Then we see that

$$(x, y)\begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x, y)\begin{pmatrix} ax + b/2y \\ b/2x + cy \end{pmatrix} = ax^2 + b/2xy + b/2xy + cy^2 = ax^2 + bxy + cx^2.$$

(2) *Principal axis theorem*

If A is symmetric and P orthogonally diagonalizes A, then making the change of variables $x = Py$, we can change the quadratic form x^TAx to y^TDy. That is, we can change variables so that the quadratic form contains no cross terms.

(3) *positive/negative definite matrices*

A quadratic form is called positive definite if $x^TAx > 0$ for all $x \neq 0$. If a matrix A is symmetric, then the quadratic form x^TAx is positive definite if and only if all the eigenvalues of A are positive, and in this case x^TAy defines an inner product on $\mathbb{R}^n$.

A symmetric matrix is positive definite if and only if the determinant of all the principal minors is positive.

Similarly, A is negative definite if $x^TAx < 0$ for all $x \neq 0$. It is negative definite if and only if the determinants of the principal minors alternate between negative and positive, starting with a negative for the first principal sub matrix.

It is indefinite if there are values for which $x^T Ax > 0$ and there are values for which $x^T Ax < 0$. It is indefinite if and only if it is neither positive nor negative definite, and there is at least one principal sub matrix with a positive determinant and there is at least one principal sub matrix with a negative determinant.

If A is 2×2, then

- $x^T Ax = 1$ is an ellipse if A is positive definite,
- $x^T Ax = 1$ has no graph if A is negative definite,
- $x^T Ax = 1$ is a hyperbola if A is indefinite.

Problems:

Problem 1

Find the matrix of the quadratic form $x_1^2 + 2x_1x_2 + 3x_1x_3 + x_2^2 + x_3^2$. Determine if the form is positive definite, negative definite, or indefinite.

Solution:

The matrix is

$$A = \begin{pmatrix} 1 & 1 & 3/2 \\ 1 & 1 & 0 \\ 3/2 & 0 & 1 \end{pmatrix}.$$

The first principal sub matrix has determinant 1, so it is not negative definite. The second principal sub matrix has determinant 0 so it is not positive definite. The third principal sub matrix has determinant $-9/4$. Since there are principal sub matrices with both positive and negative determinant, the quadratic form is indefinite.

Problem 2

Express the quadratic form $2x^2 + 2xy + 2y^2$ in terms of new variables so that there are no cross terms. What is the graph of the equation $x^T Ax = 1$?

Solution:

The matrix representation of this quadratic form is $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic polynomial is $(x-3)(x-1)$, so the eigenvalues are 1 and 3. Calculating the eigenvectors, we get

$$A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

and

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Normalizing gives $\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ and $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$.

Hence, the matrix P is

$$\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

Solving the variable equation $x = Py$, we need to calculate $y = P^{-1}x$. But $P = P^{-1}$, so we calculate the new variables as $y = Px$.

The original variables were $\begin{pmatrix} x \\ y \end{pmatrix}$, so we have the new variables,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 1/\sqrt{2} \begin{pmatrix} y - x \\ y + x \end{pmatrix}.$$

Then the quadratic form can be represented by

$$\begin{aligned} u^2 + 3v^2 = 1/2[(y - x)^2 + 3(y + x)^2] &= 1/2[y^2 - 2xy + x^2 + 3y^2 + 6xy + 3x^2] \\ &= 1/2[4y^2 + 4xy + 4x^2] \\ &= 2x^2 + 2xy + 2y^2. \end{aligned}$$

Since the eigenvalues are positive, the graph is an ellipse, and we have a specific formula in terms of the variables u, v, that is $u^2 + 3v^2 = 1$.