MATH2R3 - TUTORIAL 2

Problems:

(1) Let $u = (i, 2i, 3)$, $v = (4, -2i, 1+i)$, and $w = (2-i, 2i, 5+3i)$. Compute $u \cdot v$, $u \cdot w$, and $v \cdot w$.

Proof. This exercise is to check that you understand how the complex dot product works. I recall the definition, and leave it to you to check that your work is correct.

If $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ are vectors in $\mathbb{C}^n$, then we define their dot product as

$$z \cdot w = \sum_{i=1}^{n} z_i \overline{w_i}.$$

This is similar to the regular dot product between real vectors. The dot product satisfies a bunch of nice properties and is an example of something called an *inner product*. In order for these same properties to be satisfied with the complex dot product, we need to slightly change the definition. One of the properties is that $x \cdot x > 0$ for every $x \neq 0$. If we took the same definition for the complex dot product as the real dot product then this would fail. For example $(i, 0) \cdot (i, 0) = i^2 = -1$. By conjugating the second vector, this fixes the problem. □

(2) Let C be a complex matrix. Show that there exists real matrices A and B such that $C = A + iB$.

Proof. Let C be an $n \times m$ complex matrix, with components $[C]_{k,j} = a_{k,j} + ib_{k,j}$. Take the matrix A to be the matrix with components $[A]_{k,j} = a_{k,j}$, and B to be the matrix with components $[B]_{k,j} = b_{k,j}$. Then the matrix $A + iB$ has components $[A]_{k,j} + i[B]_{k,j} = a_{k,j} + ib_{k,j} = [C]_{k,j}$. Hence C and $A+iB$ have the same components, so they are equal. Here is an example:

$$\begin{pmatrix} i & 1 \\ 2+i & 5+2i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 5 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

□

(3) Suppose that V is an n dimensional complex vector space. Show that V is a $2n$ dimensional real vector space. (Hint: Use a basis $\{v_1, \ldots, v_n\}$ for V as a complex vector space to come up with a basis for V as a real vector space).

Proof. Let $\{v_1, \ldots, v_n\}$ be a basis for V as a vector space over $\mathbb{C}$. We wish to find a basis for V as a vector space over $\mathbb{R}$. That is, we need a set of vectors in V which is independent and spans V when we only consider coefficients from $\mathbb{R}$.

We show that the set $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ does the job.

First we show that the set is independent. Suppose that we have a linear combination,

$$a_1 v_1 + \ldots + a_n v_n + b_1(iv_1) + \ldots + b_n(iv_n) = 0$$

Then re arranging and factoring out the vectors $v_1, \ldots, v_n$, we see that this is the same as

$$(a_1 + ib_1)v_1 + \ldots + (a_n + ib_n)v_n = 0.$$

Since the vectors $v_1, \ldots, v_n$ are independent when considered as vectors over $\mathbb{C}$, we conclude that each complex coefficient $(a_k + ib_k) = 0$. This is only possible when $a_k = 0$ and $b_k = 0$, so the set $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ is independent when thought of as vectors over $\mathbb{R}$.

Similarly, since $\{v_1, \ldots, v_n\}$ is a basis over $\mathbb{C}$, if v is any vector in V then there is a representation

$$v = (a_1 + ib_1)v_1 + \ldots + (a_n + ib_n)v_n.$$

Then we see that

$$v = a_1 v_1 + \ldots + a_n v_n + b_1(iv_1) + \ldots + b_n(iv_n),$$

so the vectors $\{v_1, \ldots, v_n, iv_1, \ldots, iv_n\}$ span V as a vector space over $\mathbb{R}$.

It may be useful to see a concrete example of this. Consider the vector space $\mathbb{C}^2$ over $\mathbb{C}$. There is a standard basis

$$\{(1, 0), (0, 1)\}.$$

Suppose that we have some vector $v = (a_1 + ib_1, a_2 + ib_2) \in \mathbb{C}^2$. Then we have the representation in terms of the standard basis,

$$v = (a_1 + ib_1)(1, 0) + (a_2 + ib_2)(0, 1).$$

However, if we want to restrict ourselves to only real coefficients, we have

$$
\begin{aligned}
v &= (a_1 + ib_1)(1,0) + (a_2 + ib_2)(0,1) \\
&= a_1(1,0) + ib_1(1,0) + a_2(0,1) + ib_2(0,1) \\
&= a_1(1,0) + b_1(i,0) + a_2(0,1) + b_2(0,i)
\end{aligned}
$$

When we restrict to only real coefficients, we see that we also need the vectors $\{(i,0),(0,i)\}$. So when we take $\mathbb{C}^2$ as a vector space over $\mathbb{R}$, the standard basis is

$$\{(1,0),(0,1),(i,0),(0,i)\} = \{(1,0),(0,1),i(1,0),i(0,1)\}.$$

□