MATH2R3 - TUTORIAL 3

Problems about eigenvalues/eigenvectors.

(1) Compute the eigenvalues, eigenvectors, and eigenspaces of the matrix operator

$$A = \begin{pmatrix} 0 & 1-i \\ -1 & 1 \end{pmatrix},$$

and find a matrix P which diagonalizes A.

Proof. The characteristic polynomial of A is

$$C_A(\lambda) = \lambda(\lambda - 1) + 1 - i = (\lambda + i)(\lambda - 1 - i).$$

So the eigenvalues are $\lambda = -i$ and $\lambda = 1 + i$.

For $\lambda = -i$, we get

$$A - \lambda I = \begin{pmatrix} i & 1-i \\ -1 & 1+i \end{pmatrix},$$

Then adding i times row 2 to row 1, we get

$$\begin{pmatrix} 0 & 0 \\ -1 & 1+i \end{pmatrix}.$$

The corresponding basis for the kernel of $A - \lambda I$ is then

$$\begin{pmatrix} 1+i \\ 1 \end{pmatrix}.$$

Similarly, for $\lambda = 1 + i$, we get the corresponding basis,

$$\begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Hence, the eigenvalues and corresponding eigenvectors are

$$\lambda = -i; \quad x_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \qquad \lambda = i + i; \quad x_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

The eigenspaces are the spans of these eigenvectors. The matrix P is

$$P = \begin{pmatrix} 1+i & -i \\ 1 & 1 \end{pmatrix}.$$

□

(2) Let $C^\infty(\mathbb{R})$ be the vector space of infinitely differentiable functions. Show that $D^2\colon C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ which takes a function f to its second derivative is a linear operator. Show that for any positive ω, the functions $\sin(\sqrt{\omega}x)$ and $\cos(\sqrt{\omega}x)$ are eigenvectors of D^2.

Proof. D^2 is linear because it is a composition of linear operators. $D^2 = D \circ D$ where D is the operator sending a function to its derivative. The derivative is a linear operator by first year calculus. Now we compute

$$D^2(\sin(\sqrt{\omega}x)) = D(\sqrt{\omega}\cos(\sqrt{\omega}x)) = \sqrt{\omega}\sqrt{\omega}\sin(\sqrt{\omega}x) = \omega\sin(\sqrt{\omega}x).$$

Hence $D^2(\sin(\sqrt{\omega}x)) = \omega\sin(\sqrt{\omega}x)$, so it is an eigenvector with corresponding eigenvalue ω. A similar calculation shows this for $\cos(\sqrt{\omega}x)$. □

(3) If A is an invertible matrix, λ an eigenvalue of A and x a corresponding eigenvector, Show that $\lambda \neq 0$ and $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with the same corresponding eigenvector x.

Proof. If A is invertible then $\det(A) \neq 0$, so $\det(A - 0I) \neq 0$. Hence 0 is not a root of the characteristic polynomial so all of the eigenvalues are non zero.

Now, if $Ax = \lambda x$, then $x = A^{-1}\lambda x$, and we can pull the eigenvalue λ out of the matrix product to get $x = \lambda A^{-1}x$ and hence $\frac{1}{\lambda}x = A^{-1}x$. This shows that x is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$. □

(4) Let A be a matrix operator and λ_1, λ_2 distinct eigenvalues of A. Let E_1 and E_2 be the corresponding eigenspaces for λ_1 and λ_2. Show that $E_1 \cap E_2 = \{0\}$.

Proof. Let $x \in E_1 \cap E_2$. Then $Ax = \lambda_1 x$ and $Ax = \lambda_2 x$. Putting these together we see that $\lambda_1 x = \lambda_2 x$, or equivalently $(\lambda_1 - \lambda_2)x = 0$. Since $\lambda_1 \neq \lambda_2$, we can divide by $\lambda_1 - \lambda_2$ and we see that $x = 0$. This shows that the only vector in the intersection is $x = 0$.

□

Problems about Inner products:

(1) Calculate the following inner products:

- Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Let $u = (1, 2)$, $v = (0, 3)$, $k = 2$. Compute $\langle u, v \rangle$, $\|v\|^2$, and $\|kv\|^2$, where the inner product is induced by A.
- Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$. Computer $\langle A, B \rangle$ where the inner product is the standard inner product on matrices.

- Let $f = \sin(x^2)$ and $g(x) = x$ in $C[0, 2\pi]$. Compute the inner product $\langle f, g\rangle$ where the inner product is the integral inner product. That is

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx.$$

Proof. We have

$$\langle u, v\rangle = v^t A^t A u = (0, 3)\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = (9, 6)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 9 + 12 = 21.$$

Similarly,

$$\|v\|^2 = (0, 3)\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = (9, 6)\begin{pmatrix} 0 \\ 3 \end{pmatrix} = 18,$$

and

$$\|kv\|^2 = (0, 6)\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 6 \end{pmatrix} = (18, 12)\begin{pmatrix} 0 \\ 6 \end{pmatrix} = 72 = 4 * 18.$$

Now

$$\langle A, B\rangle = \mathrm{tr}(A^t B) = \mathrm{tr}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} = \mathrm{tr}\begin{pmatrix} 15 & 5 \\ 6 & 2 \end{pmatrix} = 17.$$

Finally,

$$\langle f, g\rangle = \int_0^{2\pi} x\sin(x^2)dx = \frac{1}{2}\int_0^{4\pi^2} \sin(u)du = -\frac{1}{2}\cos(4\pi^2) + \frac{\cos(0)}{2} = \frac{1}{2}(1 - \cos(4\pi^2)).$$

□

(2) Suppose that f and g are continuous functions on $[0, 2\pi]$. Show that

$$\left[\int_0^{2\pi} f(x)g(x)dx\right]^2 \leq \left[\int_0^{2\pi} f(x)^2dx\right]\left[\int_0^{2\pi} g(x)^2dx\right].$$

Proof. Notice that the integral is an inner product, so it satisfies the Cauchy-Schwarz inequality.

$$\langle f, g\rangle^2 \leq \|f\|^2 \|g\|^2,$$

which is exactly the inequality we have in the question. □

(3) Let V be a finite dimensional real vector space with a basis $\{v_1, \ldots, v_n\}$. Construct an inner product on V.

Proof. Represent each vector in terms of the basis,

$$v = \sum_i a_i v_i; \quad w = \sum_i b_i v_i,$$

and define

$$\langle v, w\rangle = \sum_i a_i b_i.$$

Check that this satisfies the definition of an inner product.

□