MAT2R3 TUTORIAL 4

- Next week I will have extra office hours in preparation for the test (Tuesday Oct 8 from 2-4 in math cafe, or by appointment).
- There will be no tutorial on Friday Oct 11.
- There is a practice test posted on the course website. The only way to get good at mathematics is to practice a lot, and to struggle with questions. Don't leave it until the last minute!

More problems on inner products:

Problem 1

Let $C[0,1]$ have the integral inner product, $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. let $p(x) = 1$, and $q(x) = \frac{1}{2} - x$. Show that p and q are orthogonal and that these vectors satisfy the Pythagorean theorem ($\|p+q\|^2 = \|p\|^2 + \|q\|^2$).

Solution:

$$\int_0^1 1 * (1/2 - x)dx = \left[\frac{x}{2} - \frac{x^2}{2}\right]_{x=0}^{x=1} = (\frac{1}{2} - \frac{1}{2}) - (0) = 0,$$

so p and q are orthogonal.

Now,

$$\int_0^1 (1)^2 dx = 1, \quad \int_0^1 (\frac{1}{2} - x)^2 dx = \frac{1}{12},$$

so we have

$$\|p\|^2 + \|q\|^2 = 1 + \frac{1}{12} = \frac{13}{12},$$

and

$$\int_0^1 (1 + \frac{1}{2} - x)^2 = \frac{13}{12}.$$

so we also have

$$\|p+q\|^2 = \frac{13}{12}.$$

I am leaving the details of the integrals to you, but this shows that

$$\|p+q\|^2 = \|p\|^2 + \|q\|^2.$$

Problem 2

Do there exist scalars k and l such that the vectors

$$p_1(x) = 2 + kx + 6x^2, \quad p_2(x) = l + 5x + 3x^2, \quad p_3(x) = 1 + 2x + 3x^2,$$

are all mutually orthogonal with respect to the standard inner product? Recall that the standard inner product on polynomials of degree at most n is, for $p = a_0 + a_1x + \ldots + a_nx^n$ and $q = b_0 + b_1x + \ldots + b_nx^n$,

$$\langle p, q\rangle = a_0b_0 + \ldots + a_nb_n.$$

Solution:

The condition of these vectors all being mutually orthogonal gives the equations

$$l + 10 + 9 = 0, \quad 2 + 2k + 18 = 0, \quad 2l + 5k + 18 = 0.$$

The first two equations give

$$l = -19, \quad k = -10.$$

plugging these into the third shows

$$2l + 5k + 18 = 2(-19) + 5(-10) + 18 \neq 0.$$

This shows that there are no such scalars which make these polynomials mutually orthogonal.

Problem 3

Show that

$$|a\cos(\theta) + b\sin(\theta)|^2 \leq a^2 + b^2,$$

for any $a, b, \theta \in \mathbb{R}$. [hint: Think about Cauchy-Schwarz]

Solution:

Apply the Cauchy-Schwarz inequality to the vectors (a, b) and $(\cos(\theta), \sin(\theta))$ using the dot product on $\mathbb{R}^2$. This gives

$$|a\cos(\theta) + b\sin(\theta)|^2 \leq (a^2 + b^2)(\cos^2(\theta) + \sin^2(\theta)) = a^2 + b^2.$$

Problem 4

Show that if u is orthogonal to the vectors $\{v_1, \ldots, v_n\}$ in an inner product space V, then u is orthogonal to every vector in the span of $\{v_1, \ldots, v_n\}$.

Solution:

Let $v = \sum_{i=1}^{n} a_i v_i$ be in the span of these vectors. Then

$$\langle u, \sum_{i=1}^{n} a_i v_i \rangle = \sum_{i=1}^{n} \langle u, a_i v_i \rangle = \sum_{i=1}^{n} a_i \langle u, v_i \rangle = \sum_{i=1}^{n} a_i(0) = 0.$$

Therefore u is orthogonal to v.

Problem 5

Show that if $\{v_1, \ldots, v_n\}$ is a basis for an inner product space V, then the only vector in V that is orthogonal to all of the vectors $\{v_1, \ldots, v_n\}$ is the zero vector, 0.

Solution:

If u is orthogonal to each of the v_i, then it is orthogonal to every vector in their span by the previous question. Since the v_i form a basis, u is in their span, so we have that u is orthogonal to u. In other words, we have $\langle u, u \rangle = \|u\|^2 = 0$. By the properties of inner products, this happens if and only if $u = 0$.