## MAT2R3 TUTORIAL 6

- Before we start, I would like feedback on the tutorials. Please take the time to write on a piece of paper (1) something you want me to continue doing, (2) something you want me to start doing, and (3) something you want me to stop doing. Make it anonymous and be honest please!! I spend a lot of time preparing for the tutorials so I would like to make it a good use of my time.
- If there are any questions about the test, we can discuss these after you have handed in feedback.

**Problems on function approximation:**

Recall:

If $\{u_1, \ldots, u_n\}$ is an orthogonal basis for a subspace $U$, then the projection of a vector $v$ onto $U$ is given by the formula

$$\sum_{k=1}^{n} \frac{\langle v, u_k \rangle}{\|u_k\|^2} u_k.$$

The least square approximation to a continuous function $f$ from a finite dimensional subspace $W$ of $C[a, b]$ is the orthogonal projection of $f$ onto $W$.

A trigonometric polynomial is a function of the form

$$g(x) = a_0 + a_1 \sin(x) + a_2 \sin(2x) + \ldots + a_n \sin(nx) + b_1 \cos(x) + \ldots + b_n \cos(nx).$$

If one of $a_n$ or $b_n$ is non zero, then the order of $g$ is $n$.

**Problem 1**

Find the least squares approximation of $f(x) = x^2$ over the interval $[0, 2\pi]$ by (1) a trigonometric polynomial of order 2, and (2) a trigonometric polynomial of order $n$.

**Solution:**

(1) We need to calculate the projection of $x^2$ onto the space $\{1, \sin(x), \sin(2x), \cos(x), \cos(2x)\}$. These vectors are orthogonal but not orthonormal. In general, $\|1\| = \sqrt{2\pi}$ and $\|\sin(nx)\| = \sqrt{\pi} = \|\cos(nx)\|$. We calculate the coefficients for the approximation by

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} x^2 dx \qquad a_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \sin(kx) dx, \qquad b_k = \frac{1}{\pi} \int_0^{2\pi} x^2 \cos(kx) dx.$$

So we have,

$$a_0 = \frac{1}{2\pi}\left[\frac{1}{3}x^3\right]_0^{2\pi} = \frac{(2\pi)^2}{3}.$$

Now we must integrate by parts with $f = x^2$ and $dg = \sin(x)$. First calculating the indefinite integral, we have

$$\frac{1}{\pi}\int x^2\sin(x)dx = \frac{1}{\pi}\left(-x^2\cos(x) + \int 2x\cos(x)dx\right)$$

Again, we must integrate by parts and we get that

$$\int 2x\cos(x) = 2x\sin(x) - 2\int \sin(x)dx = 2x\sin(x) + 2\cos(x).$$

This gives

$$\frac{1}{\pi}\left[-x^2\cos(x) + 2x\sin(x) - \cos(x) = 2x\sin(x) + \cos(x)(2-x^2)\right].$$

Now evaluating at 0 and $2\pi$ we get

$$a_1 = \frac{1}{\pi}\left[2(2\pi)\sin(2\pi) + \cos(2\pi)(2-(2\pi)^2) - 2(0)\sin(0) + \cos(0)(2-0^2)\right] = -4\pi.$$

Similar integral calculations show that

$$\begin{aligned} a_2 &= -2\pi \\ b_1 &= 4 \\ b_2 &= 1 \end{aligned}$$

This gives the approximation

$$g(x) = \frac{4\pi^2}{3} - 4\pi\sin(x) - 2\pi\sin(2x) + 4\cos(x) + \cos(2x).$$

(2) Now, for each $k \leq n$, we again integrate by parts to find

$$\begin{aligned} a_k = \frac{1}{\pi}\left[\int x^2\sin(kx)\right] &= \frac{1}{\pi}\left[-\frac{x^2\cos(kx)}{k} + \frac{2}{k}\int x\cos(kx)\right] \\ &= \frac{1}{\pi}\left[-\frac{x^2\cos(kx)}{k} + \frac{2x\sin(kx)}{k^2} - \frac{2}{k^2}\int \sin(kx)\right] \\ &= \frac{1}{\pi}\left[-\frac{x^2\cos(kx)}{k} + \frac{2x\sin(kx)}{k^2} + \frac{2}{k^3}\cos(kx)\right]_0^{2\pi} \\ &= \frac{1}{\pi}\left[\frac{-(2\pi)^2}{k} + 0 + \frac{2}{k^3} - 0 - 0 - \frac{2}{k^3}\right] \\ &= \frac{-(4\pi)}{k} \end{aligned}$$

Similarly, we get

$$b_k = \frac{4}{k^2}.$$

This gives the approximation

$$g(x) = \frac{4\pi^2}{3} + \sum_{k=1}^{n} \frac{4}{k^2}\cos(kx) - \frac{4\pi}{k}\sin(kx).$$

**Problem 2**

Find the Fourier series of $f(x) = \pi - x$ over the interval $(0, 2\pi)$.

**Solution:**

We calculate the Fourier coefficients.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} \pi - x dx = -\frac{1}{2\pi}\left|\frac{(\pi - x)^2}{2}\right|_0^{2\pi} = 0.$$

and using integration by parts,

$$a_n = \frac{1}{\pi}\int (\pi - x)\sin(nx)dx = (\pi - x)\frac{-\cos(nx)}{n} - \int \frac{\cos(nx)}{n} = (\pi - x)\frac{-\cos(nx)}{n} - \frac{\sin(nx)}{n^2},$$

evaluating from 0 to $2\pi$ we get

$$a_n = \frac{1}{\pi}\left[(\pi - 2\pi)\frac{-\cos(n2\pi)}{n} - \frac{\sin(n2\pi)}{n^2} - \left((\pi - 0)\frac{-\cos(n0)}{n} - \frac{\sin(n0)}{n^2}\right)\right] = \frac{2}{n}.$$

A similar calculation shows that

$$b_n = 0$$

so we get that

$$\pi - x = \sum_{n=1}^{\infty} \frac{2}{n}\sin(nx)$$

**Remark 1.** *Note that by equality, we mean equality in terms of the integral norm. In particular, the functions aren't equal at every value of $x$ (for instance $x = 0$), but they are equal at "most" values of $x$. There is a technical definition for "most" but it's not relevant to this course.*